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Abstract

Polynomials play a central role in algebra, analysis, number theory, and applied mathematics. This thesis presents a concise overview of several fundamental results that characterize their behavior and structure.

This thesis focuses on the study and analysis of special classes of polynomials, which are Bernoulli polynomials, Bell polynomials, p-Bell polynomials, r-Bell polynomials, partial Bell polynomials, and complete Bell polynomials. We present their generating functions, explicit expressions, recurrence relations, and their relationships with Stirling numbers.

Keywords: Generating function, explicit formula, recurrence relation, Stirling numbers, hypergeometric function, Bernoulli polynomials, Bell polynomials, p-Bell polynomials, r-Bell polynomials, Partial and Complete Bell polynomials.

الملخص

تؤدي كثيرات الحدود دورا أساسيا في الجبر، والتحليل الرياضي، ونظرية الأعداد، والرياضيات التطبيقية. تقدم هذه المذكرة عرضا موجزا لعدة نتائج أساسية تميز سلوكها وبنيتها الرياضية.

تركز هذه المذكرة على دراسة وتحليل أصناف خاصة من كثيرات الحدود، والمتمثلة في كثيرات حدود برنولي، وكثيرات حدود بيل، وكثيرات حدود بيل من الرتبة p، وكثيرات حدود بيل الجزئية، وكثيرات حدود بيل الكاملة. كما نستعرض دوالها المولدة، وصيغها الصريحة، وعلاقاتها التراجعية، إضافةً إلى ارتباطاتها بأعداد ستيرلينغ.

الكلمات المفتاحية: الدالة المولدة، الصيغة الصريحة، العلاقة التراجعية، أعداد ستيرلينغ، الدالة فوق الهندسية، كثيرات حدود برنولي، كثيرات حدود بيل المعممة، كثيرات حدود بيل من الرتبة p، كثيرات حدود بيل الجزئية والكاملة.

Notations

$\mathbb{N}$	The set of natural numbers.
$\mathbb{Z}$	The set of entire numbers.
$\mathbb{R}$	The field of real numbers.
$\mathbb{C}$	The field of complex numbers.
$[\mathrm{n}]$	The set of $\{1, 2, \ldots, n\}$.
$t^{\overline{n}}$	The rising factorial.
$t^{\underline{n}}$	The falling factorial.
$\binom{n}{k}$	The binomial coefficient.
$\Gamma(x)$	The Gamma function, extension of the factorial.
$\delta_{n,k}$	The Kronecker delta symbol.
${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; x\right)$	The generalized hypergeometric function.
$s(n,k)$	Signed Stirling numbers of the first kind.
$\lvert s(n,k)\rvert$	Unsigned Stirling numbers of the first kind .
$\left\{ {n \atop k} \right\}$	Stirling numbers of the second kind.
δ_{nm}	Kronecker delta symbol.
B_n	Bernoulli numbers.
$B_n(x)$	Bernoulli polynomials.
ϕ_n	The Bell numbers.
$\phi_n(x)$	The Bell polynomials.
$\mathcal{B}_{n,p}$	p-Bell numbers.
$\mathcal{B}_{n,p}(x)$	p-Bell polynomials.
$\mathbb{B}_n^{(p)}$	The poly-Bell numbers.
$\phi_{n,r}$	r-Bell numbers.
$\phi_{n,r}(x)$	r-Bell polynomials.
$B_{n,k}\left(x_1, x_2, \ldots, x_n\right)$	The (exponential) partial Bell partition polynomials.
$B_n\left(x_1, x_2, \ldots, x_n\right)$	The (exponential) complete Bell partition polynomials.

Introduction

Special polynomials are an important topic in mathematics, with applications in combinatorics, number theory, and analysis. In particular, Bell polynomials have attracted great attention because of their beautiful properties and their natural connections with many classical mathematical objects.

This thesis is dedicated to the study of Bell polynomials and their main generalizations. We consider several families, namely the p-Bell polynomials, the poly-Bell polynomials, the r-Bell polynomials, as well as the partial and complete Bell polynomials. Each of these families extends the classical Bell polynomials in a different and interesting direction.

For each family, we present the generating functions, explicit formulas, and recurrence relations. We also show how these polynomials are connected to Stirling numbers and binomial coefficients, which are key objects in combinatorics.

The results of this thesis show that all these families share common structures and properties, which makes their unified study both natural and fruitful.

The three tools studied in this thesis (the generating function, the explicit formula, and the recurrence relation) are not merely theoretical curiosities. They are essential instruments that have profoundly shaped the development of modern mathematics and continue to fuel numerous areas of active research.

The **generating function** has revolutionized the way mathematicians approach sequences and polynomials. By transforming a combinatorial or analytical problem into a manipulation of formal expressions, it allows one to obtain results that would otherwise be very difficult to establish. Its effectiveness lies in its ability to condense an infinite amount of information into a single compact identity. More precisely, given a sequence of polynomials $\{P_n(x)\}_{n\geq 0}$, its generating function is a formal power series of the form:

$$F(x,t) = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}$$

from which all the properties of the sequence can be systematically derived.

The **explicit formula**, on the other hand, represents the natural culmination of any study on a family of polynomials. Having a closed-form expression for the coefficients or values of a polynomial not only allows direct computations, but also reveals deep connections with other mathematical objects, in particular the Stirling numbers, binomial coefficients, and hypergeometric functions. For instance, many classical polynomials admit representations of the form:

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k\, x^{n-k}$$

where the coefficients a_k carry significant combinatorial meaning.

Finally, the **recurrence relation** plays a fundamental role both in theory and in practice. On the theoretical level, it reflects an internal structure intrinsic to each family of polynomials, often encoding differential or difference operators acting on the sequence. On the practical level, it provides a simple and efficient computational algorithm, particularly useful in numerical and computer science applications. A typical recurrence relation takes the form:

$$P_n(x) = F\big(n, P_{n-1}(x), P_{n-2}(x), \ldots, P_0(x)\big), \quad n \geq 1$$

where F is a function that determines how each term of the sequence is constructed from previous terms, and it characterizes the specific family of polynomials under consideration.

It is around these three axes that the study conducted in this thesis is organized, applied to the following families: **Bernoulli polynomials**, **Bell polynomials**, **p-Bell polynomials**, **partial Bell polynomials**, and **complete Bell polynomials**.

This work is divided into four chapters:

1. The **first chapter** provides the mathematical background and preliminaries needed throughout this work. We introduce fundamental combinatorial tools, including rising and falling factorials, binomial coefficients, and the Gamma function. We also present Stirling numbers of both kinds along with some of their generalizations, such as r-Stirling and Whitney numbers, as well as a brief treatment of hypergeometric functions and Bernoulli numbers and polynomials.

2. The **second chapter** is devoted to the study of **Bell numbers** and **Bell polynomials**. We begin with their combinatorial interpretation in counting set partitions, then present their main properties, generating functions, and their fundamental connections with Stirling numbers.

3. The **third chapter** deals with several important generalizations of the classical Bell polynomials. We study in particular **the p-Bell numbers**, **p-Bell polynomials**, **poly-Bell polynomials**, as well as **the r-Bell numbers** and **r-Bell polynomials**, presenting their definitions, main properties, and key identities.

4. The **fourth chapter** is dedicated to **the partial Bell partition polynomials** $B_{n,k}(x_1, x_2, \ldots, x_n)$. Introduced by Bell as an effective tool for expressing higher-order derivatives of composite functions, we study their main properties, explicit formulas, and their role in combinatorial analysis. We then introduce the **complete Bell polynomials** $B_n(x_1, x_2, \ldots, x_n)$, defined as the sum of all partial Bell polynomials over all

   possible degrees, and we present their exponential generating function, their explicit formula, as well as the fundamental relation connecting them to the partial Bell polynomials.

Finally, this thesis opens new perspectives for future research, whether by considering further generalizations of the polynomial families studied here, or by exploring their deeper connections with hypergeometric functions and their potential applications in combinatorics and mathematical analysis.

1

Basic Notions

In this chapter, we introduce the fundamental notions that will be used throughout this work. We present basic combinatorial tools such as rising and falling factorials and binomial coefficients, together with their main properties. We also study the Gamma function, Stirling numbers of both kinds and some of their generalizations, including r-Stirling and Whitney numbers. In addition, we briefly discuss hypergeometric functions and Bernoulli numbers and polynomials. For more precise and detailed concepts, the reader can refer to [2, 3, 6, 7, 8, 10, 14, 17, 18, 19, 22, 23, 29, 30, 31].

1.1 Rising Factorial and Falling Factorial

Definition 1.1. *Let t be a real or complex number and n a positive integer, we define the rising factorial of t of order n, denoted as $t^{\overline{n}}$, by:*

$$t^{\overline{n}} = t(t+1)\dots(t+n-1), \text{ if } n \geqslant 1 \text{ with } t^{\overline{0}} = 1,$$

and the falling factorial of t of order n, denoted as $t^{\underline{n}}$, by:

$$t^{\underline{n}} = t(t-1)\dots(t-n+1), \text{ if } n \geqslant 1 \text{ with } t^{\underline{0}} = 1.$$

We can give the relation between the rising factorial and the falling factorial by this expresion:

$$t^{\overline{n}} = (-1)^n(-t)^{\underline{n}}.$$

1.2 Binomial Coefficients

Definition 1.2. *The binomial coefficient $\binom{n}{k}$, represents the number of ways to choose a subset with k elements from a set with n elements.*

Definition 1.3. *Let n and k be two non-negative integers. The binomial coefficient of k, denoted as $\binom{n}{k}$, is given by:*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!} \quad (0 \leq k \leq n), \tag{1.1}$$

with $\binom{n}{k} = 0$ for $k > n$, and $\binom{n}{k} = 1$ for $k = 0$ or $k = n$.

Example 1.1.
Using the algebraic definition of the binomial coefficient, and substituting $n = 5$ and $k = 2$ in (1.1), we have:

$$\binom{5}{2} = \frac{5!}{(5-2)!\,2!} = \frac{5!}{3!\,2!} = \frac{120}{12} = 10.$$

In the set $[5] = \{1, 2, 3, 4, 5\}$, there are ten ways to choose two elements:

$$\{1,2\},\ \{1,3\},\ \{1,4\},\ \{1,5\},\ \{2,3\},\ \{2,4\},\ \{2,5\},\ \{3,4\},\ \{3,5\},\ \{4,5\}.$$

Thus, $\binom{5}{2} = 10$.

1.2.1 Newton's Binomial Formula

Let t and y be two real numbers and n a non-negative integer, so:

$$(t+y)^n = \sum_{k=0}^{n} \binom{n}{k} t^k y^{n-k}. \tag{1.2}$$

1.2.2 Pascal's Recurrence Relation

The binomial coefficients satisfy the following relation:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad \text{for every} \quad n \geq k \geq 1. \tag{1.3}$$

1.2.3 Pascal's Triangle

From the recurrence relation (1.3), we can construct Pascal's triangle.

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Table 1.1: Pascal's Triangle Numbers

1.2.4 Generalization of Binomial Coefficients

A generalization [18] of the binomial coefficient $\binom{n}{k}$, consists in replacing n by a complex number t, and to define $\binom{t}{k}$ by posing, $t \in \mathbb{C}$ and $k \in \mathbb{N}$ by:

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!} = \frac{t^{\underline{k}}}{k!}, \tag{1.4}$$

and having almost the same properties as $\binom{n}{k}$.
We have

$$\binom{-k}{n} = \binom{k+n-1}{n}(-1)^n. \tag{1.5}$$

1.3 The Kronecker Symbol

The Kronecker Symbol is given by:

$$\delta_{n,k} = \begin{cases} 1 & \text{if} \quad n = k \\ 0 & \text{otherwise.} \end{cases}$$

1.4 The Difference Operator Δ

For any function f from $\mathbb{R}$ to $\mathbb{R}$ (or from $\mathbb{C}$ to $\mathbb{C}$), the difference operator Δf at a point x is defined [18] by:

$$\Delta f(x) = f(x+1) - f(x).$$

The explicit formula for the n-th difference operator Δ^n is given by:

$$\Delta^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k).$$

1.5 Permutations, Cycles, and Partitions

1.5.1 Permutations

Let $n \in \mathbb{N}$. A permutation is defined [18] as any bijection σ from a finite set $[n] = \{1, 2, \ldots, n\}$ to it selve. The number of permutations of $[n]$ is $n!$.

A permutation σ can be represented in matrix form as:

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}.$$

The set $\mathfrak{S}_n$ of permutations, equipped with the composition law of two permutations σ and τ, denoted by $\sigma \circ \tau$, forms a noncommutative group called the *symmetric group* of $[n]$. The identity element of this group is the identity permutation, denoted by ε, which leaves every element of $[n]$ unchanged. A fixed point of a permutation $\sigma \in \mathfrak{S}_n$ is an element $i \in [n]$ such that $\sigma(i) = i$. An element that is not fixed by σ is called a mobile point.

1.5.2 Cycle

A cycle [18] of length k ($k \geq 2$), also called a k-cycle, is a permutation $\gamma \neq \varepsilon$ whose mobile elements (that is, the non-fixed elements by γ), called the *support* of the cycle, are of the form

$$\{i, \sigma(i), \sigma^2(i), \ldots, \sigma^{k-1}(i), \sigma^k(i) = i\}.$$

The orbit of i under the permutation σ, denoted by O_i, is the set of images of i obtained by successive applications of the permutation σ on the element i, namely

$$O_i = \left(\sigma^j(i)\right)_{j \in \mathbb{N}}.$$

Example 1.2. *In the set* $[6]$*, the following permutation is given by:*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 4 & 2 & 5 \end{pmatrix}.$$

The element 1 *is a fixed point because* $\sigma(1) = 1$*, and similarly, the element* 4 *is a fixed point since* $\sigma(4) = 4$*. This means that for all* $k \in \mathbb{N}$*, we have* $\sigma^k(1) = 1$ *and* $\sigma^k(4) = 4$*.*
Now, observe that:

$$\sigma(2) = 3, \quad \sigma(3) = 6, \quad \sigma(6) = 5, \quad \sigma(5) = 2,$$

so after four applications, we return to the starting point:

$$\sigma^4(2) = 2.$$

The elements $(2,\ 3,\ 6,\ 5)$*, taken in this order, form a cycle of length* 4*. Therefore, we can express the permutation* σ *as a product of cycles. In this example:*

$$\sigma = (2\ 3\ 6\ 5)(1)(4).$$

1.5.3 Partition

A partition $\mathcal{P}$ of $[n]$ is a collection of non-empty, pairwise disjoint subsets of $[n]$, called the blocks of the partition, whose union covers $[n]$; that is:

$$\bigcup \mathcal{P} = [n].$$

In particular, a k-partition is a partition consisting of exactly k blocks.

Example 1.3. *Consider the following cases:*

1. *If $n = 1$, then $[n] = \{1\}$. The only possible partition of $[n]$ is the so-called coarse partition, whose unique block is $[n]$ itself.*
2. *If $n = 2$, then $[n] = \{1, 2\}$. There are two possible partitions: $\{\{1\}, \{2\}\}$ and $\{\{1, 2\}\}$.*
3. *If $n = 3$, then $[n] = \{1, 2, 3\}$. There are five possible partitions:*

$$\{\{1\}, \{2\}\,\{3\}\}; \quad \{\{1\}, \{2, 3\}\}; \quad \{\{2\}, \{1, 3\}\}; \quad \{\{3\}, \{1, 2\}\}; \quad \{1, 2, 3\}.$$

1.6 Generating Functions

Generating functions are used to represent sequences as power series. They simplify combinatorial problems by turning them into algebraic ones. This makes it easier to solve recurrence relations, find formulas, and count complex structures [18].

Throughout what follows, we denote by $\mathbb{K}$ a commutative ring with a unit element.

1.6.1 Ordinary Generating Function

Let n be a natural number, and let $(a_n)_{n\geq 0}$ be a sequence of complex numbers. The ordinary generating function of the sequence is defined as:

$$A(t) = \sum_{n=0}^{\infty} a_n t^n,$$

where the coefficients a_n belong to a ring $\mathbb{K}$, and is an indeterminate t. The addition and multiplication of two ordinary generating functions are defined as follows:

$$\sum_{n=0}^{\infty} a_n t^n + \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} (a_n + b_n)\, t^n$$

and

$$\left(\sum_{n=0}^{\infty} a_n t_n\right)\left(\sum_{n=0}^{\infty} b_n t_n\right) = \sum_{n=0}^{\infty} C_n t_n,$$

with

$$C_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

1.6.2 Exponential Generating Function

The exponential generating function of the sequence $(a_n)_{n\geq 0}$, where n is natural number and $(a_n)_{n\geq 0}$ is a sequence of complex or real number, is defined as:

$$B(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}.$$

The addition and multiplication of two exponential generating functions are defined as follows:

$$\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} + \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} (a_n + b_n) \frac{t^n}{n!}$$

and

$$\left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

with

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

1.7 The Gamma Function

Definition 1.4. *The Gamma function $\Gamma(x)$ is defined, for $x \in \mathbb{C}$, by the integral:*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt, \tag{1.6}$$

where

$$\Gamma(m+1) = m! \quad \text{for } m \in \mathbb{N}.$$

The definition of rising factorial can be extended to any integer $n \in \mathbb{Z}$ by the following relation:

$$t^{\overline{n}} = \frac{\Gamma(t+1)}{\Gamma(t-n+1)}.$$

Remark 1. *The lower incomplete Gamma function $\gamma(s, z)$ is defined by:*

$$\gamma(s, z) = \int_0^{z} t^{s-1} e^{-t}\, dt, \tag{1.7}$$

which represents a part of the Gamma function.

1.8 Hypergeometric Functions

The origin of hypergeometric functions dates back to the beginning of the 19$^{\text{th}}$ century, when Gauss studied the following differential equation [2, 6, 18, 31]:

$$t(1-t)y'' + (c - (a+b+1)t)y' - aby = 0, \tag{1.8}$$

where $a, b, c \in \mathbb{C}$. This equation is known as Gauss's differential equation.The generalized hypergeometric function has a series representation:

$$\sum_{n=0}^{\infty} C_n,$$

with

$$\frac{C_{n+1}}{C_n},$$

a rational function of n, defined by the following relation:

$$\frac{C_{n+1}}{C_n} = \frac{(n+a_1)(n+a_p)t}{(n+b_1)\cdots(n+b_q)(n+1)}.$$

If $C_0 = 1$, then:

$$C_n = \frac{a_1^{\overline{n}} a_2^{\overline{n}} \cdots a_p^{\overline{n}}}{b_1^{\overline{n}} b_2^{\overline{n}} \cdots b_q^{\overline{n}}} \cdot \frac{t^n}{n!},$$

and the hypergeometric function is then:

$$ {}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};t\right)=\sum_{n=0}^{\infty}\frac{a_1^{\overline{n}}a_2^{\overline{n}}\cdots a_p^{\overline{n}}}{b_1^{\overline{n}}b_2^{\overline{n}}\cdots b_q^{\overline{n}}}\cdot\frac{t^n}{n!}. $$

If $p=1$ and $q=1$, we obtain the confluent hypergeometric function of Kummer:

$$ {}_1F_1\left(\begin{array}{c} a \\ b \end{array};t\right), $$

which is defined by :

$$ \sum_{n=0}^{\infty}\frac{a^{\overline{n}}}{b^{\overline{n}}}\cdot\frac{x^n}{n!}. \tag{1.9} $$

If $p=2$ and $q=1$, we obtain the Gauss hypergeometric function:

$$ {}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};t\right)=\sum_{n=0}^{\infty}\frac{a^{\overline{n}}b^{\overline{n}}}{c^{\overline{n}}}\cdot\frac{t^n}{n!}, $$

where $|t|<1$, the solution of the differential equation (1.8).

Theorem 1.1. *The confluent hypergeometric function* ${}_1F_1\left(\begin{array}{c} a \\ b \end{array};z\right)$ *admits the following integral representation [2]:*

$$ {}_1F_1\left(\begin{array}{c} a \\ b \end{array};z\right)=\frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}\,dt, \tag{1.10} $$

where Γ *denotes the Gamma function* (1.6).

1.9 The Stirling Numbers

1.9.1 Stirling Numbers of The First Kind

Definition 1.5. *The signed Stirling numbers of the first kind, denoted as* $s(n,k)$*, are the coefficients of the falling factorial polynomials in the canonical basis* t^n*:*

$$ t^{\underline{n}}=\sum_{k=0}^{n}s(n,k)t^k. $$

Definition 1.6. *The unsigned Stirling numbers of the first kind, denoted as* $|s(n,k)|$*, are the coefficients of the rising factorial polynomials in the canonical basis* t^n*:*

$$ t^{\overline{n}}=\sum_{k=0}^{n}|s(n,k)|t^k. \tag{1.11} $$

We have :

$$ |s(n,k)|=(-1)^{n-k}s(n,k). $$

Example 1.4. *Write* $t^{\underline{4}}$ *as a polynomial.*

$$ \begin{aligned} t^{\underline{4}} &= t(t-1)(t-2)(t-3) \\ &= (t^{\underline{3}})(t-3) \\ &= (t^3-3t^2+2t)(t-3) \\ &= t^4-6t^3+11t^2-6t. \end{aligned} $$

From this example, we obtain the following coefficients:
$s(4,0)=0$.

$s(4,1) = -6.$
$s(4,2) = 11.$
$s(4,3) = -6.$
$s(4,4) = 1.$

Theorem 1.2. *The exponential generating function of the signed Stirling numbers of the first kind is given by:*

$$\sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!} = \frac{(\ln(1+t))^k}{k!}.$$

Theorem 1.3. *The explicit formula for the signed Stirling numbers of the first kind:*

$$\mid s(n,k) \mid = \sum_{0 \leqslant j \leqslant h \leqslant n-k}^{\infty} (-1)^{j+h} \binom{h}{j} \binom{n-1+h}{n-k+h}.$$

The explicit formula for the signed Stirling numbers of the first kind:

$$s(n,k) = \frac{n!}{k!} \sum_{i_1+\cdots+i_k=n} \frac{1}{i_1 \cdots i_k}.$$

Theorem 1.4. *The Stirling numbers of the first kind satisfy the following recurrence relation:*

$$\begin{cases} s(n,0) = s(0,k) = 0 \quad \textit{except} \quad s(0,0) = 1 \\ s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \quad (n \geqslant k \geqslant 1). \end{cases} \tag{1.12}$$

Table of the Stirling Numbers of the First Kind

Using the triangular recurrence relation (1.12), we can calculate the Stirling numbers of the first kind $s(n,k)$:

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	−1	1					
3	0	2	−3	1				
4	0	−6	11	−6	1			
5	0	24	−50	35	−10	1		
6	0	−120	274	−225	85	−15	1	
7	0	720	−1764	1624	−735	175	−21	1

Table 1.2: Stirling Numbres of The First Kind

1.9.2 Stirling Numbers of The Second kind

Definition 1.7. *The Stirling numbers of the second kind, denoted as* $\left\{ {n \atop k} \right\}$ *are the coefficients of the polynomial* t^n*:*

$$t^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} t^{\underline{k}}. \tag{1.13}$$

Definition 1.8. *The Stirling numbers of the second kind,* $\left\{ {n \atop k} \right\}$ *counts the number of partitions of the set* $\{1, 2, \ldots, n\}$ *into* k *non-empty subsets* $(0 \leq k \leq n)$*.*

Example 1.5. *The set* $[4]$ *contain* 7 *partitions into two non-empty subsets* $\left\{{4 \atop 2}\right\}$ *represent the partitions into* 2 *subsets:*

$$\{\{1\}\{2,3,4\}\},\quad \{\{2\}\{1,3,4\}\},\quad \{\{3\}\{1,2,4\}\},\quad \{\{4\}\{1,2,3\},$$
$$\{1,2\}\{3,4\}\},\quad \{\{1,3\}\{2,4\}\},\quad \{\{1,4\}\{2,3\}\}.$$

As a result, $\left\{{4 \atop 2}\right\} = 7$.

Theorem 1.5. *The ordinary generating function of Stirling numbers of the second kind is given by:*

$$\sum_{n=k}^{\infty} \left\{{n \atop k}\right\} t^n = \frac{1}{(1-t)(1-2t)\cdots(1-kt)}$$
$$= \prod_{j=1}^{k} \frac{1}{(1-jt)}.$$

The exponential generating function of Stirling numbers of the second kind is given by:

$$\sum_{n=k}^{\infty} \left\{{n \atop k}\right\} \frac{t^n}{n!} = \frac{1}{k!}(e^t-1)^k. \tag{1.14}$$

Theorem 1.6. *The Stirling numbers of the second kind are given by the following explicit formula:*

$$\left\{{n \atop k}\right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n.$$

Theorem 1.7. *The Stirling numbers of the second kind satisfy the following recurrence relation:*

$$\begin{cases} \left\{{n \atop 0}\right\} = \left\{{0 \atop k}\right\} = 0 \quad \textit{except} \quad \left\{{0 \atop 0}\right\} = 1 \\ \\ \left\{{n \atop k}\right\} = \left\{{n-1 \atop k-1}\right\} + k\left\{{n-1 \atop k}\right\} \; n \geq k \geq 0. \end{cases} . \tag{1.15}$$

Table of Stirling Numbers of Second Kind

Using the triangular recurrence relation (1.15), we can calculate the Stirling numbers of the second kind $\left\{{n \atop k}\right\}$:

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	1					
3	0	1	3	1				
4	0	1	7	6	1			
5	0	1	15	25	10	1		
6	0	1	31	90	65	15	1	
7	0	1	63	301	350	140	20	1

Table 1.3: Stirling Numbers of the Second Kind

1.10 The Generalized Stirling Numbers

The extension of Stirling numbers includes the classical Stirling numbers of the first and second kinds within larger families, such as r-Stirling numbers, q-Stirling numbers, and other related generalizations. These extended versions preserve the essential combinatorial structure of the original numbers while introducing extra parameters, which provide more flexible interpretations and broaden their applications in combinatorics and number theory.

1.10.1 r-Stirling Numbers of The First Kind

Broder, in [8], presented combinatorial interpretations of the r-Stirling numbers of the first kind and established several of their algebraic properties, along with various identities, recurrence relations, and corresponding generating functions [18].

Definition 1.9. *The r-Stirling numbers of the first kind, denoted $\begin{bmatrix} n \\ k \end{bmatrix}_r$, count the number of permutations of a set of n elements into k cycles such that the first r elements belong to distinct cycles,their exponential generating function is given by:*

$$\sum_{n=k}^{\infty} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \frac{t^n}{n!} = \frac{1}{k!} \frac{1}{(1-t)^r} \left(\ln\left(\frac{1}{1-t} \right) \right)^k. \tag{1.16}$$

In particular, for $r = 0$, we obtain the classical Stirling numbers of the first kind (1.11).

Example 1.6. *The set $[3] = \{1, 2, 3\}$ has 2 permutations into 2 cycles such that the first two elements 1 and 2 belong to distinct cycles. The permutations are:*

$$(1)(2\,3), \quad (2)(1\,3).$$

As a result:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}_2 = 2.$$

Properties

1. **Generating function**: The ordinary generating functions for the sequence of r-Stirling numbers of the first kind are given by:

$$\sum_{n=k}^{\infty} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r t^n = t^r (1-rt)(t+r+1)\dots(t+n-1), \quad n \geq r \geq 0.$$

2. **Reccurrence relations**: The r-Stirling numbers of the first kind satisfy the following recurrence relation:

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_r &= 0, \quad n < r \\ \begin{bmatrix} n \\ k \end{bmatrix}_r &= \delta_{k,r}, \quad n = r \\ \begin{bmatrix} n \\ k \end{bmatrix}_r &= (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix}_r + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_r, \quad n > r. \end{aligned} \tag{1.17}$$

Table of r-Stirling Numbers of First Kind

Using the triangular recurrence relation (1.17), we can calculate the 2-Stirling numbers of the first kind:

$n \backslash k$	2	3	4	5	6	7
2	1					
3	2	1				
4	6	5	1			
5	24	26	9	1		
6	120	154	71	14	1	
7	720	1044	580	155	20	1

Table 1.4: The First Values of The 2-Stirling Numbers of The First Kind

1.10.2 r-Stirling Numbers of The Second Kind

Broder, in [8], provided combinatorial interpretations and several algebraic properties of the r-Stirling numbers of the second kind, as well as various identities, recurrence relations, and their associated generating functions.

Definition 1.10. *The r-Stirling numbers of the second kind, denoted $\left\{ {n \atop k} \right\}_r$, count the number of ways to partition a set of n elements into k nonempty subsets such that the first r elements belong to distinct subset (blocks) [18]. Their exponential generating function is given by:*

$$\sum_{n=k}^{\infty} \left\{ {n+r \atop k+r} \right\}_r \frac{t^n}{n!} = \frac{1}{k!} e^{rt} \left(e^t - 1 \right)^k . \tag{1.18}$$

Special Case

When we substitute $r = 0$ or $r = 1$, we obtain the classical Stirling numbers of the second kind (1.14). They also arise analytically as the coefficients of the following polynomials:

$$(t+r)^n = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r t^{\underline{k}}. \tag{1.19}$$

Example 1.7. *The set $[3] = \{1, 2, 3\}$ has 2 partitions into two blocks, such that the first two elements 1 and 2 are in distinct blocks. The partitions are:*

$$\{1\}\{2, 3\}, \quad \{2\}\{1, 3\}.$$

As a result:

$$\left\{ {3 \atop 2} \right\}_2 = 2.$$

Properties

1. **Generating function**: The ordinary generating functions for the sequence of r-Stirling numbers of the second kind are given by:

$$\sum_{n=k}^{\infty} \left\{ {n+r \atop k+r} \right\}_r t^n = \frac{t^k}{(1-rt)(1-(r+1)t)\dots(1-(r+k)t)}, \quad k \geq r \geq 0.$$

2. **Explicit formula**: For $\left\{ {n \atop k} \right\}_r$, the explicit formula is given by :

$$\left\{ {n+r \atop k+r} \right\}_r = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (r+j)^n .$$

3. **Reccurrence relations**: The r-Stirling numbers of the second kind satisfy the following recurrence relation:

$$\begin{aligned} \left\{ {n \atop k} \right\}_r &= 0, \quad n < r, \\ \left\{ {n \atop k} \right\}_r &= \delta_{k,r}, \quad n = r, \\ \left\{ {n \atop k} \right\}_r &= k \left\{ {n-1 \atop k} \right\}_r + \left\{ {n-1 \atop k-1} \right\}_r, \quad n > r. \end{aligned} \tag{1.20}$$

Table of r-Stirling Numbers of Second Kind

Using the triangular recurrence relation (1.20), we can calculate the 2-Stirling numbers of the first kind:

$n \backslash k$	2	3	4	5	6	7	8	9
2	1							
3	2	1						
4	4	5	1					
5	8	19	9	1				
6	16	65	55	14	1			
7	32	211	285	125	20	1		
8	64	655	1351	910	245	27	1	
9	128	2059	6069	5901	2380	434	35	1

Table 1.5: The First Values of The 2-Stirling Numbers of The Second Kind

1.10.3 The Weighted Stirling Numbers of the Second Kind

Definition 1.11. *The weighted Stirling numbers of the second kind [10, 18], usually denoted $S_n^k(x)$, their exponential generating function is:*

$$\sum_{n=0}^{\infty} S_n^k(x) \frac{z^n}{n!} = e^{xz} \frac{(e^z - 1)^k}{k!}.$$

Theorem 1.8. *Their explicit formula is given by:*

$$S_n^k(x) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (x+j)^n,$$

where

$$S_n^k(x) = \frac{1}{k!} \triangle^k x^n,$$

with

$$\frac{1}{k!} \triangle^k x^n|_{x=0} = \begin{Bmatrix} n \\ k \end{Bmatrix} \quad \text{and} \quad S_n^k(r) = \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r.$$

Remark 2. *Reduce to the classical Stirling numbers of the second kind when $r = 0$, that is:*

$$\sum_{n=0}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix} \frac{z^n}{n!} = \frac{(e^z - 1)^k}{k!}.$$

1.10.4 Whitney Numbers of the Second Kind

Definition 1.12. *The Whitney numbers, denoted by $W_m(n,k)$, are introduced and defined by the exponential generating function [18, 29]:*

$$\sum_{n=k}^{\infty} W_m(n,k) \frac{z^n}{n!} = \frac{e^z}{m^k} \cdot \frac{(e^{mz} - 1)^k}{k!}.$$

Properties

1. **Explicit formula**: Their explicit formula is given by:

$$W_m(n,k) = \frac{1}{m^k k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (mj+1)^n.$$

2. **Recurrence relation**: The Whitney numbers are defined by the recurrence relation:

$$W_m(n,k) = (1+mk) W_m(n-1,k) + W_m(n-1,k-1), \quad n \geq k \geq 1.$$

1.10.5 The r-Whitney Numbers of the First and the Second Kinds

Mező and Ramírez, in [22, 23] provided the r-Whitney numbers of the first kind $w_{m,r}(n,k)$ and of the second kind $W_{m,r}(n,k)$, as:

Definition 1.13. *The r-Whitney numbers of the first and the second kinds are defined by the following generating functions:*

$$\sum_{n=k}^{\infty} w_{m,r}(n,k)\frac{z^n}{n!} = (1+mz)^{-r/m}\,\frac{\left(\ln(1+mz)\right)^k}{m^k\,k!},$$

$$\sum_{n=k}^{\infty} W_{m,r}(n,k)\frac{z^n}{n!} = \frac{e^{rz}}{m^k}\frac{(e^{mz}-1)^k}{k!},$$

respectively, they can be expressed via the following change of basis:

$$m^n x^{\underline{k}} = \sum_{k=0}^{n} w_{m,r}(n,k)\,(mx+r)^k,$$

and

$$(mx+r)^n = \sum_{k=0}^{n} W_{m,r}(n,k) m^k x^{\underline{k}}.$$

Properties

1. **Explicit formula:** For the r-Whitney numbers of the second kind, we have the following formula:

$$W_{m,r}(n,k) = \frac{1}{m^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(mj+r)^n.$$

   In particular, we have:

$$W_{1,0}(n,k) = \left\{ {n \atop k} \right\}, \quad W_{1,r}(n,k) = \left\{ {n+r \atop k+r} \right\}_r, \quad \text{and} \quad W_{m,1}(n,k) = W_m(n,k).$$

2. **Recurrence relation:** The r-Whitney numbers of the first kind and the second kind satisfy the following recurrence relations respectively:

$$w_{m,r}(n,k) = w_{m,r}(n-1,k-1) + (m-nm-r)w_{m,r}(n-1,k)$$

   and

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (km+r)W_{m,r}(n-1,k).$$

   Remark 3. *The relationship between the r-Whitney numbers and the Stirling weighted numbers of the second kind is given by:*

$$S_n^k\left(\frac{r}{m}\right) = \frac{1}{m^{\,n-k}}\,W_{m,r}(n,k).$$

1.11 The Bernoulli Polynomials

1.11.1 Bernoulli Numbers

In 1755, Euler posed the following definition for the Bernoulli numbers, which remains the most common modern definition for the sequence [7, 17, 18, 19] .

Definition 1.14. *The Bernoulli numbers are the coefficients of the exponential generating function:*

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}. \tag{1.21}$$

Properties

1. **Explicit formula:** The Bernoulli numbers B_n, we have the following formula:

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} \left\{ n \atop k \right\}. \tag{1.22}$$

2. **Recurrence relation:**The Bernoulli numbers B_n, satisfy the following recurrence relations respectively:

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0, \quad \forall n \geq 2.$$

Remark 4. *The Bernoulli numbers B_n with odd indices are zero, that is:*

$$\forall n \geq 1, \;\; B_{2n+1} = 0.$$

1.11.2 Bernoulli Polynomials

In this part, we study the Bernoulli polynomials [3, 17]. We define them and give some of their main properties. These polynomials are closely related to Bernoulli numbers and appear in many areas of mathematics, especially in calculus.

Definition 1.15. *The Bernoulli polynomials are a sequence of polynomials, $B_n(x)$, defined by the following power series expansion:*

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1}. \tag{1.23}$$

The generating function for the Bernoulli polynomials is the generating function for the Bernoulli numbers multiplied by a term of e^{tx}. Our first observation of the Bernoulli polynomials is that the constant term of $B_n(x)$ is in fact B_n. If we set $x = 0$, then:

$$\frac{te^{t(0)}}{e^t - 1} = \frac{t}{e^t - 1}$$

and so $B_n(0) = B_n$.

There are further connections: we can use the generating function for the Bernoulli numbers to develop a recurrence relation for the Bernoulli polynomials.

Theorem 1.9. *The Bernoulli polynomials, $B_n(x)$ satisfy the recurrence relation:*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Table of Bernoulli polynomials

In according with the recurrence relation established above, we obtain the following Table 1.6:

n	$B_n(x)$
0	1
1	$x - \frac{1}{2}$
2	$x^2 - x + \frac{1}{6}$
3	$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$
4	$x^4 - 2x^3 + x^2 - \frac{1}{30}$
5	$x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$
6	$x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$

Table 1.6: Table of Bernoulli Polynomials

Theorem 1.10. *For $n \geq 1$, the Bernoulli polynomial $B_n(x)$ satisfies the following identity:*

$$B_n'(x) = nB_{n-1}(x).$$

1.12 The Lah Polynomials

1.12.1 Lah Numbers

Definition 1.16. *The (unsigned) Lah numbers $L(n,k)$, counts the number of partitions of the set $[n] = \{1, 2, \ldots, n\}$ into k nonempty ordered lists.*

Definition 1.17. *The (unsigned) Lah numbers, denoted by $L(n,k)$, provide the connection between rising factorials and falling factorials [31]. They are defined by the relations:*

$$t^{\overline{n}} = \sum_{k=0}^{n} L(n,k)\, t^{\underline{k}}$$

and

$$t^{\underline{k}} = \sum_{k=0}^{n} (-1)^{n-k} L(n,k)\, t^{\overline{n}},$$

where $t^{\underline{k}}$ and $t^{\overline{n}}$ denote the rising and falling factorials, respectively.

Properties

1. **Generating function**: The exponential generating functions of the Lah numbers $L(n,k)$ are given by:
$$\sum_{n \geq k} L(n,k) \frac{t^n}{n!} = \frac{1}{k!} \left(\frac{t}{1-t} \right)^k. \tag{1.24}$$

2. **Explicit formula**: Their explicit formula is given by:
$$L(n,k) = \frac{n!}{k!} \binom{n-1}{k-1}, \qquad n \geq k \geq 1, \tag{1.25}$$
with $L(0,0) = 1$.

3. **Recurrence relation:** For the Lah numbers $L(n,k)$, we have the following formula:
$$L(n,k) = L(n-1,k-1) + (n+k-1)L(n-1,k), \qquad n \geq k \geq 1, \tag{1.26}$$
In particular, we have: $L(n,0) = \delta_{n,0}$ for $n \geq 0$, and $L(0,k) = \delta_{k,0}$ for $k \geq 0$.

Table of the Lah numbers

Using the recurrence relation (1.26), we can calculate the first values of the Lah numbers:

$n \diagdown k$	1	2	3	4	5	6	7	8
1	1							
2	2	1						
3	6	6	1					
4	24	36	12	1				
5	120	240	120	20	1			
6	720	1800	1200	300	30	1		
7	5040	15120	12600	4200	630	42	1	
8	40320	141120	141120	58800	11760	1176	56	1

Table 1.7: The First Values of The Lah Numbers

1.12.2 Lah Polynomials

Definition 1.18. *The Lah Polynomials, denoted by $L_n(x)$, are defined by:*

$$L_n(x) := \sum_{k=0}^{n} L(n,k)\, x^k, \tag{1.27}$$

The first values of these polynomials are:

$$\begin{aligned}
L_0(x) &= 1,\\
L_1(x) &= x,\\
L_2(x) &= x^2 + 2x,\\
L_3(x) &= x^3 + 6x^2 + 6x,\\
L_4(x) &= x^4 + 12x^3 + 36x^2 + 24x.
\end{aligned}$$

2

The Bell Numbers and Polynomials

In this chapter, we study Bell numbers and Bell polynomials, which play a central role in combinatorics. We begin by introducing Bell numbers and their combinatorial interpretation in counting set partitions. We then present Bell polynomials, highlighting their main properties, generating functions, and their connections with Stirling numbers. For more details, refer to the following references [18, 31, 32].

2.1 Bell Numbers

The Bell numbers denoted by ϕ_n, represent the number of possible partitions of a finite set with n elements. They can equivalently be expressed as the sum of the Stirling numbers of the second kind:

$$\phi_n = \sum_{k=0}^{n} \left\{ n \atop k \right\}. \tag{2.1}$$

Theorem 2.1. *The Bell numbers satisfy the following recurrence relation:*

$$\phi_0 = 1 \quad \textit{and} \quad \phi_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \phi_k. \tag{2.2}$$

The first values of the Bell numbers are listed in the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12
ϕ_n	1	2	5	15	52	203	877	4140	21147	115975	678570	4213597

Table 5 *– Initial values of the Bell numbers.*

Theorem 2.2. *The exponential generating functions of the Bell numbers are given by:*

$$\sum_{n=0}^{\infty} \phi_n \frac{z^n}{n!} = \exp(e^z - 1). \tag{2.3}$$

Proof. By using the exponential generating function of the Stirling numbers of the second kind:

$$\frac{(e^z - 1)^k}{k!} = \sum_{n=k}^{\infty} \left\{ n \atop k \right\} \frac{z^n}{n!},$$

we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty}\phi_n\frac{z^n}{n!} &= \exp(e^z-1)\\ &= \sum_{k=0}^{\infty}\frac{1}{k!}(e^z-1)^k\\ &= \sum_{n=0}^{\infty}\left(\sum_{n=k}^{\infty}\left\{{n\atop k}\right\}\frac{z^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{n=0}^{n}\left\{{n\atop k}\right\}\right)\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\{{n\atop k}\right\}\right)\frac{z^n}{n!}.\end{aligned}$$

By matching the coefficients of $\frac{z^n}{n!}$ on both sides, we obtain the desired relation.

□

Theorem 2.3. *Dobinski's formula provides an explicit representation of the Bell numbers:*

$$\phi_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}.$$

Proof. According to formula (2.3),

$$\sum_{n=0}^{\infty}\phi_n\frac{z^n}{n!} = \exp(e^z-1)$$

and

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!},$$

we get

$$\begin{aligned}e^{e^z-1} &= e^{e^z}e^{-1}\\ &= \frac{1}{e}\sum_{k=0}^{\infty}\frac{(e^z)^k}{k!}\\ &= \frac{1}{e}\sum_{k=0}^{\infty}\frac{e^{kz}}{k!}\\ &= \frac{1}{e}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(kz)^n}{k!\,n!}\\ &= \frac{1}{e}\sum_{n=0}^{\infty}\frac{z^n}{n!}\left(\sum_{k=0}^{\infty}\frac{k^n}{k!}\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}\right)\frac{z^n}{n!}.\end{aligned}$$

Consequently

$$\sum_{n=0}^{\infty}\phi_n\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}\right)\frac{z^n}{n!}.$$

By identifying the coefficients of $\frac{z^n}{n!}$, we obtain:

$$\phi_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}.$$

□

An alternative explicit representation, introduced by B.N. Gao and F. Qi in [20] the confluent hypergeometric function of Kummer and the Stirling numbers of the second kind:

$$\phi_n = \frac{1}{e}\sum_{k=1}^{n}(-1)^{n-k}k!\left\{ n \atop k \right\} {}_1F_1\left(\begin{matrix} k+1 \\ 2 \end{matrix} ;1 \right)$$

Theorem 2.4. *The integral representation of the Bell numbers [4, 9, 12].*

$$\phi_n = \frac{2n!}{\pi e}\,\mathrm{Im}\int_0^{\pi}\exp\Big(\exp\Big(e^{i\theta}\Big)\Big)\sin(n\theta)\,d\theta.$$

2.2 Bell Polynomials

For all $n \in \mathbb{N}$, Bell polynomials $\phi_n(x)$ are defined by:

$$\phi_n(x) = \sum_{k=0}^{n}\left\{ n \atop k \right\} x^k. \tag{2.4}$$

These polynomials are also known as single-variable Bell polynomials or exponential polynomials. In particular, for $x = 1$ in (2.4) we get the classical Bell numbers.

Theorem 2.5. *Their exponential generating function is given by:*

$$\sum_{n=0}^{\infty}\phi_n(x)\frac{z^n}{n!} = \exp\left(x(e^z-1)\right). \tag{2.5}$$

Proof. By using the exponential generating function (2.5)

$$\sum_{n=0}^{\infty}\phi_n(x)\frac{z^n}{n!} = \exp\left(x(e^z-1)\right).$$

and the exponential generating function of the Stirling numbers of the second kind:

$$\frac{(e^z-1)^k}{k!} = \sum_{n=k}^{\infty}\left\{ n \atop k \right\}\frac{z^n}{n!},$$

we obtain:

$$\begin{aligned} e^{x(e^z-1)} &= \sum_{k=0}^{\infty}(e^z-1)^k\frac{x^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(k!\sum_{n=k}^{\infty}\left\{ n \atop k \right\}\frac{z^n}{n!}\right)\frac{x^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(k!\sum_{n=0}^{n}\left\{ n \atop k \right\}\frac{x^k}{k!}\right)\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\{ n \atop k \right\}x^k\right)\frac{z^n}{n!}. \end{aligned}$$

By matching the coefficients of $\frac{z^n}{n!}$ on both sides, we obtain the desired relation.

□

Theorem 2.6. ***(Dobinski's formula)***
The Bell polynomials $\phi_n(x)$ admit the following representation:

$$\phi_n(x) = e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k. \tag{2.6}$$

Proof. *The proof is similar to the proof of Theorem 2.3.* □

Theorem 2.7. *The sequence of polynomials $\phi_n(x)$ satisfies the recurrence relation*

$$\phi_0(x) = 1,$$

$$\phi_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \phi_k(x).$$

Proof. *The exponential generating function of the Bell polynomials (2.5)*

$$\sum_{n=0}^{\infty} \phi_n(x) \frac{z^n}{n!} = \exp\left(x(e^z - 1)\right)$$

and differentiating both sides with respect to z, we get:

$$\sum_{n=1}^{\infty} \phi_n(x) \frac{z^{n-1}}{(n-1)!} = \exp\left(x(e^z - 1)\right) x e^z.$$

We set $n - 1 = n$, we find:

$$\begin{aligned}
\sum_{n=0}^{\infty} \phi_{n+1}(x) \frac{z^n}{n!} &= \exp\left(x(e^z - 1)\right) x e^z \\
&= x \left(\sum_{n=0}^{\infty} \phi_n(x) \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(x \sum_{k=0}^{n} \binom{n}{k} \phi_k(x) \right) \frac{z^n}{n!}.
\end{aligned}$$

By comparing the coefficients of $\frac{z^n}{n!}$, we obtain:

$$\phi_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \phi_k(x).$$

□

First Bell Polynomials

The first few Bell polynomials are listed below:

n	$\phi_n(x)$
0	1
1	x
2	$x + x^2$
3	$x + 3x^2 + x^3$
4	$x + 7x^2 + 6x^3 + x^4$

Table 2.9 – Bell Polynomials

Special Case

When we substitute $x = 1$ in the Bell polynomials (2.5), we obtain the Bell numbers.

$$\sum_{n=0}^{\infty} \phi_n \frac{z^n}{n!} = \exp(e^z - 1).$$

3

The Generalized Bell Polynomials

The generalized Bell polynomials extend the classical Bell polynomials, which are closely related to set partitions in combinatorics. They retain many of the original properties while introducing extra parameters that lead to more flexible and broader applications.
In this chapter, we study different generalizations, including the p-Bell numbers, p-Bell polynomials, poly-Bell polynomials, as well as the r-Bell numbers and r-Bell polynomials. We present their definitions, main properties, and key identities. For more details, the reader may refer to [18, 19, 31].

3.1 The p-Bell Numbers

In this section, we study a new class of numbers that extends the classical Bell numbers, which we refer to as the p-Bell numbers. According to [18, 19, 31].

Definition 3.1. *For every integer $p \geq 0$, we define a sequence of rational numbers, denoted by $B_{n,p}$ for $n \geq 0$, through the following generating function:*

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,p} \frac{z^n}{n!} = {}_1F_1\left(\begin{matrix} 1 \\ p+1 \end{matrix} ; e^z - 1 \right) = \sum_{n=0}^{\infty} \binom{n+p}{p}^{-1} \frac{(e^z-1)^n}{n!}, \tag{3.1}$$

where $B_{n,0} := \phi_n$ are the classical Bell numbers, and ${}_1F_1\left(\begin{matrix} a \\ c \end{matrix} ; z \right)$ denotes the confluent hypergeometric function of Kummer.

The first three exponential generating functions are:

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{B}_{n,1} \frac{z^n}{n!} &= \frac{\exp(e^z-1)-1}{e^z-1}, \\ \sum_{n=0}^{\infty} \mathcal{B}_{n,2} \frac{z^n}{n!} &= \frac{2\left(\exp(e^z-1)-e^z\right)}{(e^z-1)^2}, \\ \sum_{n=0}^{\infty} \mathcal{B}_{n,3} \frac{z^n}{n!} &= \frac{3\left(2\exp(e^z-1)-e^{2z}-1\right)}{(e^z-1)^3}. \end{aligned} \tag{3.2}$$

Theorem 3.1. *The explicit formula for p-Bell numbers $\mathcal{B}_{n,p}$ involving Stirling numbers of the second kind.*

For $p \geq 0$, we have:

$$\mathcal{B}_{n,p} = \sum_{k=0}^{n} \binom{k+p}{k}^{-1} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}. \tag{3.3}$$

Proof. *By using* (1.14), *we obtain:*

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{k+p}{k}^{-1}\left\{ n \atop k \right\}\frac{z^n}{n!} &= \sum_{k=0}^{\infty}\binom{k+p}{k}^{-1}\sum_{n=k}^{\infty}\left\{ n \atop k \right\}\frac{z^n}{n!}\\ &= \sum_{k=0}^{\infty}\frac{p!\,k!}{(k+p)!}\frac{(e^z-1)^k}{k!}\\ &= \sum_{k=0}^{\infty}\frac{1^{\overline{k}}}{(p+1)^{\overline{k}}}\frac{(e^z-1)^k}{k!}\\ &= {}_1F_1\left(\begin{matrix}1\\ p+1\end{matrix};e^z-1\right)\\ &= \sum_{n=0}^{\infty}\mathcal{B}_{n,p}\frac{z^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$ *on both sides, we obtain the result* (3.3). *In particular, for* $p=1$, *we have :*

$$\begin{aligned}\mathcal{B}_{n,1} &= \sum_{k=0}^{n}\frac{1}{k+1}\left\{ n \atop k \right\}\\ &= \int_0^1 \phi_n(t)\,dt.\end{aligned}$$

Also, we can write $B_{n,1}$ *in the following form.*

□

Theorem 3.2. *For* $n\geq 0$, *we have:*

$$\mathcal{B}_{n,1} = \sum_{k=0}^{n}\binom{n}{k}\frac{\phi_{k+1}B_{n-k}}{k+1}, \tag{3.4}$$

where B_n *denotes the* n*-th Bernoulli number [21], which is defined by the following generating function* (1.21)*:*

$$\sum_{n\geq 0} B_n\frac{z^n}{n!} = \frac{z}{e^z-1}.$$

Proof. *We can rewrite* (3.2) *as follows:*

$$\begin{aligned}\sum_{n\geq 0}\mathcal{B}_{n,1}\frac{z^n}{n!} &= \frac{\exp(e^z-1)-1}{e^z-1}\\ &= \frac{z}{e^z-1}\left(\frac{\exp(e^z-1)-1}{z}\right)\\ &= \sum_{n=0}^{\infty}B_n\frac{z^n}{n!}\left(\frac{\exp(e^z-1)-1}{z}\right).\end{aligned}$$

We have

$$\sum_{n=0}^{\infty}\phi_n\frac{z^n}{n!} = \exp(e^z-1)$$

and

$$\exp(e^z-1)-1 = \sum_{n=1}^{\infty}\phi_n\frac{z^n}{n!},$$

which gives

$$\frac{\exp(e^z-1)-1}{z} = \frac{1}{z}\sum_{n=1}^{\infty}\phi_n\frac{z^n}{n!} = \sum_{n=1}^{\infty}\phi_n\frac{z^{n-1}}{n!}.$$

We put $n := n-1$, we obtain

$$\frac{\exp(e^z-1)-1}{z} = \sum_{n=0}^{\infty}\phi_{n+1}\frac{z^n}{(n+1)!} = \sum_{n=0}^{\infty}\frac{\phi_{n+1}}{n+1}\frac{z^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,1}\frac{z^n}{n!} = \left(\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{\phi_{n+1}}{n+1}\frac{z^n}{n!}\right).$$

Using the Cauchy product formula for exponential generating functions, we get

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,1}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}B_{n-k}\frac{\phi_{k+1}}{k+1}\right)\frac{z^n}{n!}.$$

By identifying the coefficients of $\frac{z^n}{n!}$, we obtain the following desired result:

$$\mathcal{B}_{n,1} = \sum_{k=0}^{n}\binom{n}{k}\frac{\phi_{k+1}B_{n-k}}{k+1}.$$

□

Remark 5. *We note that formula (3.4) is a particular case of Ramanujan's identity [21, Example 4, p. 51].*

Theorem 3.3. *The double exponential generating function of $B_{n,p}$ is given by:*

$$\sum_{p=0}^{\infty}\sum_{n=p}^{\infty}\mathcal{B}_{n,p}\frac{z^n}{n!}\frac{y^p}{p!} = \frac{(e^z-1)\exp(e^z-1)}{e^z-1-y}.$$

Proof. By using (3.1) and putting

$$y^p = (e^z-1)^p\left(\frac{y}{e^z-1}\right)^p,$$

we get

$$\begin{aligned}
\sum_{p=0}^{\infty}\sum_{n=p}^{\infty}\mathcal{B}_{n,p}\frac{z^n}{n!}\frac{y^p}{p!} &= \sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\frac{(e^z-1)^n(e^z-1)^p\left(\frac{y}{e^z-1}\right)^p}{(n+p)!}\\
&= \sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\frac{(e^z-1)^n(e^z-1)^p}{(n+p)!}\left(\frac{y}{e^z-1}\right)^p\\
&= \sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\frac{(e^z-1)^{n+p}}{(n+p)!}\left(\frac{y}{e^z-1}\right)^p\\
&= \sum_{n=0}^{\infty}\frac{(e^z-1)^n}{n!}\sum_{p=0}^{\infty}\left(\frac{y}{e^z-1}\right)^p\\
&= \frac{(e^z-1)\exp(e^z-1)}{e^z-1-y}.
\end{aligned}$$

Which completes the proof.

□

Now, by means of the generalized Stirling transform [29], we obtain the following result.

Theorem 3.4. *For $n, m \geq 0$, we have:*

$$\sum_{k=0}^{m} s(m,k)\,\mathcal{B}_{n+k,p} = \sum_{k=0}^{n} \left\{ {n+m \atop k+m} \right\}_m \binom{m+k+p}{p}^{-1}. \tag{3.5}$$

Proof. From (3.3), we have

$$\mathcal{B}_{n,p} = \sum_{k=0}^{n} \binom{k+p}{k}^{-1} \left\{ {n \atop k} \right\}.$$

By taking

$$a_k = \binom{k+p}{p}^{-1}.$$

Now, by the generalized Stirling transform (see [29]), we have

$$\sum_{k=0}^{m} s(m,k)\,b_{n+k} = \sum_{k=0}^{n} \left\{ {n+m \atop k+m} \right\}_m a_{m+k},$$

where

$$b_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_m a_k.$$

Taking $b_n = \mathcal{B}_{n,p}$ and $a_k = \binom{k+p}{p}^{-1}$, we obtain

$$\sum_{k=0}^{m} s(m,k)\,\mathcal{B}_{n+k,p} = \sum_{k=0}^{n} \left\{ {n+m \atop k+m} \right\}_m \binom{m+k+p}{m+k}^{-1}.$$

This completes the proof. □

Theorem 3.5. *For $p \geq 1$, we have:*

$$\sum_{n\geq 0} \mathcal{B}_{n,p} \frac{z^n}{n!} = p \int_0^1 \exp((e^z-1)t)\,(1-t)^{p-1}\,dt. \tag{3.6}$$

Proof. We know that (1.10), by setting $a = 1$, $b = 1+p$, we obtain:

$$\begin{aligned} {}_1F_1\left(\begin{matrix} 1 \\ 1+p \end{matrix} ; z \right) &= \frac{\Gamma(1+p)}{\Gamma((1+p)-1)\Gamma(1)} \int_0^1 e^{zt} t^0 (1-t)^{1+p-1-1}\,dt \\ &= \frac{\Gamma(1+p)}{\Gamma((p)\Gamma(1)} \int_0^1 e^{zt}(1-t)^{p-1}\,dt. \end{aligned}$$

Using $\frac{\Gamma(p+1)}{\Gamma(p)} = p$ and $\Gamma(1) = 1$ we obtain:

$${}_1F_1\left(\begin{matrix} 1 \\ 1+p \end{matrix} ; z \right) = p \int_0^1 e^{zt}(1-t)^{p-1}\,dt.$$

Substitute $z := e^z - 1$, we obtain:

$${}_1F_1\left(\begin{matrix} 1 \\ 1+p \end{matrix} ; e^z-1 \right) = p \int_0^1 \exp((e^z-1)t)(1-t)^{p-1}\,dt.$$

□

Corollary 1. *For $p \geq 1$, we have :*

$$\mathcal{B}_{n,p} = p \int_0^1 (1-t)^{p-1}\ \phi_n(t) dt. \tag{3.7}$$

Proof. We have:

$$\begin{aligned}\sum_{n\geq 0} \mathcal{B}_{n,p} \frac{z^n}{n!} &= p \int_0^1 \sum_{n\geq 0} \phi_n(t) \frac{z^n}{n!} (1-t)^{p-1}\, dt \\ &= \sum_{n\geq 0} \left(p \int_0^1 (1-t)^{p-1}\ \phi_n(t)\, dt \right) \frac{z^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$, on both sides, we arrive at the result (3.7). □

The generating functions

According to [19], the generating function of $\mathcal{B}_{n,p}$ admits several representations. In what follows, we present some of these expressions.

Theorem 3.6. *For $p \geq 1$, according to incomplete Gamma function $\gamma(s,z)$ (1.7), the exponential generating function for p-Bell numbers is given by:*

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,p} \frac{z^n}{n!} = \frac{p\,(\exp(e^z-1))}{(e^z-1)^p}\, \gamma(p, e^z-1). \tag{3.8}$$

Proof. This can be achieved by the change of variable $t = 1 - \frac{y}{e^z-1}$, which transforms this integral .

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,p} \frac{z^n}{n!} = p \int_0^1 \exp((e^z-1)t)\, (1-t)^{p-1}\, dt.$$

Then,

$$dt = -\frac{1}{e^z-1}\, dy, \qquad (1-t)^{p-1} = \left(\frac{y}{e^z-1}\right)^{p-1},$$

and

$$(e^z-1)t = (e^z-1) - y.$$

moreover, the limits become : $t = 0 \Rightarrow y = e^z - 1, \quad t = 1 \Rightarrow y = 0$
substituting into the integral, we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{B}_{n,p} \frac{z^n}{n!} &= p \int_{e^z-1}^0 \exp(e^z-1)\ e^{-y} \left(\frac{y}{e^z-1}\right)^{p-1} \left(-\frac{dy}{e^z-1}\right) \\ &= \frac{p \exp(e^z-1)}{(e^z-1)^p} \int_0^{e^z-1} y^{p-1} e^{-y}\, dy\end{aligned}$$

Using the definition of the lower incomplete Gamma function (1.7), we deduce:

$$\int_0^{e^z-1} e^{-y} y^{p-1}\, dy = \gamma(p, e^z-1),$$

we arrive at the result (3.1).

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,p} \frac{z^n}{n!} = \frac{p\,(\exp(e^z-1))}{(e^z-1)^p}\, \gamma(p, e^z-1).$$

□

Theorem 3.7. *The exponential generating function for p-Bell numbers is given by:*

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,p} \frac{z^n}{n!} = \frac{p!\,(\exp(e^z-1))}{(e^z-1)^p} - \sum_{k=1}^{p} \frac{p^{\underline{k}}}{(e^z-1)^k} \ . \tag{3.9}$$

Theorem 3.8. *It is well known that the p-Bell numbers are closely related to both the classical Bell numbers and the generalized Bernoulli polynomials. The latter, denoted by $B_n^{(\alpha)}(x)$ and of degree n in x, are defined by the exponential generating function [7, 33]:*

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\,\frac{t^n}{n!} \qquad (|t|<2\pi,\ 1^{\alpha}:=1)$$

Remark 6. *For an arbitrary parameter α, the quantities $B_n^{(\alpha)} := B_n^{(\alpha)}(0)$ are called the generalized Bernoulli numbers of order α. In particular, when $\alpha = 1$, the polynomials $B_n^{(1)}(x)$ reduce to the classical Bernoulli polynomials $B_n(x)$, whose exponential generating function is given by:*

$$\sum_{n=0}^{\infty} B_n(x)\,\frac{z^n}{n!} = \frac{ze^{xz}}{e^z-1}.$$

Corollary 2. *For $n, p \geq 0$, we have:*

$$\mathcal{B}_{n,p} = \sum_{k=0}^{n+p} \binom{n+p}{k} \binom{n+p}{p}^{-1} \phi_{\,n+p-k}\, B_k^{(p)} - \sum_{k=1}^{p} \binom{n+k}{k}^{-1} \binom{k}{p} B_{n+k}^{(k)}. \tag{3.10}$$

Proof. From (3.9), we have

$$\sum_{n=0}^{\infty} B_{n,p} \frac{z^n}{n!} = \frac{p!\,\exp(e^z-1)}{(e^z-1)^p} - \sum_{k=1}^{p} \frac{p^{\underline{k}}}{(e^z-1)^k}.$$

We expand the first term, and by the Cauchy product, we obtain:

$$\begin{aligned} \frac{p!\,\exp(e^z-1)}{(e^z-1)^p} &= p! \left(\sum_{n=0}^{\infty} \phi_n \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} B_n^{(p)} \frac{z^{n-p}}{n!}\right) \\ &= p! \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \phi_{n-k} B_k^{(p)}\right) \frac{z^{n-p}}{n!}. \end{aligned}$$

Setting $n := n+p$, we get:

$$\frac{p!\,\exp(e^z-1)}{(e^z-1)^p} = p! \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n+p} \binom{n+p}{k} \phi_{n+p-k} B_k^{(p)}\right) \frac{z^n}{(n+p)!}.$$

Since $(n+p)! = n!\,p!\,\binom{n+p}{p}$, it follows that

$$\frac{p!}{(n+p)!} = \frac{1}{n!} \binom{n+p}{p}^{-1}.$$

Hence,

$$\frac{p!\,\exp(e^z-1)}{(e^z-1)^p} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n+p} \binom{n+p}{k} \binom{n+p}{p}^{-1} \phi_{n+p-k} B_k^{(p)}\right) \frac{z^n}{n!}.$$

We expand the second term:

$$\sum_{k=1}^{p} \frac{p^{\underline{k}}}{(e^z-1)^k}.$$

Using the identity

$$\frac{1}{(e^z-1)^k}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{z^{n-k}}{n!},$$

we obtain

$$\sum_{k=1}^{p}\frac{p^{\underline{k}}}{(e^z-1)^k}=\sum_{k=1}^{p}p^{\underline{k}}\sum_{n=0}^{\infty}B_n^{(k)}\frac{z^{n-k}}{n!}.$$

Setting $n:=n+k$, we get

$$\sum_{k=1}^{p}\frac{p^{\underline{k}}}{(e^z-1)^k}=\sum_{n=0}^{\infty}\sum_{k=1}^{p}p^{\underline{k}}B_{n+k}^{(k)}\frac{z^n}{(n+k)!},$$

now,

$$\frac{p^{\underline{k}}}{(n+k)!}=\frac{1}{n!}\binom{n+k}{k}^{-1}\binom{p}{k},$$

thus,

$$\sum_{k=1}^{p}\frac{p^{\underline{k}}}{(e^z-1)^k}=\sum_{n=0}^{\infty}\left(\sum_{k=1}^{p}\binom{n+k}{k}^{-1}\binom{p}{k}B_{n+k}^{(k)}\right)\frac{z^n}{n!}.$$

Combining both expansions, we obtain:

$$\sum_{n=0}^{\infty}B_{n,p}\frac{z^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n+p}\binom{n+p}{k}\binom{n+p}{p}^{-1}\phi_{n+p-k}B_k^{(p)}\right)\frac{z^n}{n!}$$

$$-\sum_{n=0}^{\infty}\left(\sum_{k=1}^{p}\binom{n+k}{k}^{-1}\binom{p}{k}B_{n+k}^{(k)}\right)\frac{z^n}{n!}.$$

By identifying the coefficients of $\frac{z^n}{n!}$, we get the result (3.10):

$$B_{n,p}=\sum_{k=0}^{n+p}\binom{n+p}{k}\binom{n+p}{p}^{-1}\phi_{n+p-k}B_k^{(p)}-\sum_{k=1}^{p}\binom{n+k}{k}^{-1}\binom{p}{k}B_{n+k}^{(k)}.$$

□

Lemma 3.1. *For $p\geq 0$, we have*

$${}_1F_1\left(\begin{matrix}p+1\\ p+2\end{matrix};1-e^z\right)=(-1)^p(p+1)\left(e^{-z}\frac{d}{dz}\right)^p{}_1F_1\left(\begin{matrix}1\\ 2\end{matrix};1-e^z\right)\tag{3.11}$$

and

$${}_1F_1\left(\begin{matrix}1\\ 2\end{matrix};1-e^z\right)=\left(\frac{1-\exp(1-e^z)}{e^z-1}\right).$$

Theorem 3.9. *For $p\geq 1$, we have:*

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,p}\frac{z^n}{n!}=\exp(e^z-1)\,(-1)^{p-1}\,p\left(e^{-z}\frac{d}{dz}\right)^{p-1}\left(\frac{1-\exp(1-e^z)}{e^z-1}\right).$$

Theorem 3.10. *We have the integral representation of Cesàro type [11]. For $n\geq 1$, we have:*

$$\mathcal{B}_{n,p}=\frac{2\,n!\,p!}{\pi e}\,\mathrm{Im}\int_0^{\pi}\left(\frac{\exp(\left(\exp(e^{i\theta})\right))}{\left(\exp(e^{i\theta})-1\right)^p}-e\sum_{\ell=0}^{p-1}\frac{\left(\exp(e^{i\theta})-1\right)^{\ell-p}}{\ell!}\right)\sin(n\theta)\,d\theta,$$

where $\mathrm{Im}(z)$ denotes the imaginary part of z.

Proof. Using the explicit formula (3.3), and we use the integral representation of the Stirling numbers of the second kind:

$$\left\{ {n \atop k} \right\} = \frac{2\,n!}{\pi}\ \mathrm{Im}\int_0^{\pi} \frac{(\exp(e^{i\theta})-1)^k}{k!}\sin(n\theta)\,d\theta,$$

we obtain :

$$\mathcal{B}_{n,p} = \frac{2\,n!}{\pi}\ \mathrm{Im}\left(\int_0^{\pi}\sum_{k=0}^{n}\binom{k+p}{k}^{-1}\frac{(\exp(e^{i\theta})-1)^k}{k!}\right)\sin(n\theta)\,d\theta.$$

Since $\binom{k+p}{k}^{-1} = \dfrac{k!\,p!}{(k+p)!}$, *this becomes*

$$\mathcal{B}_{n,p} = \frac{2\,n!\,p!}{\pi}\ \mathrm{Im}\left(\int_0^{\pi}\sum_{k=0}^{n}\frac{(\exp(e^{i\theta})-1)^k}{(k+p)!}\right)\sin(n\theta)\,d\theta.$$

Setting $x = \exp(e^{i\theta})-1$ *and using the identity*

$$\sum_{k=0}^{\infty}\frac{x^k}{(k+p)!} = \frac{e^x}{x^p} - \sum_{\ell=0}^{p-1}\frac{x^{\ell-p}}{\ell!},$$

we deduce

$$\mathcal{B}_{n,p} = \frac{2\,n!\,p!}{\pi}\ \mathrm{Im}\int_0^{\pi}\left(\frac{\exp(\exp(e^{i\theta})-1)}{(\exp(e^{i\theta})-1)^p} - \sum_{\ell=0}^{p-1}\frac{(\exp(e^{i\theta})-1)^{\ell-p}}{\ell!}\right)\sin(n\theta)\,d\theta.$$

Finally, using $\exp(\exp(e^{i\theta})-1) = \exp(\exp(e^{i\theta}))/e$, *we obtain:*

$$\mathcal{B}_{n,p} = \frac{2\,n!\,p!}{\pi e}\ \mathrm{Im}\int_0^{\pi}\left(\frac{\exp(\exp(e^{i\theta}))}{(\exp(e^{i\theta})-1)^p} - e\sum_{\ell=0}^{p-1}\frac{(\exp(e^{i\theta})-1)^{\ell-p}}{\ell!}\right)\sin(n\theta)\,d\theta.$$

This completes the proof. □

The Recurrence relations

The p-Bell numbers admits outher recurrence relations we present some of these expressions.

Theorem 3.11. *The p-Bell numbers satisfies the following recurrence relation*

$$\mathcal{B}_{n+1,p} = (n+1)\,\mathcal{B}_{n,p} - \sum_{k=0}^{n-2}\binom{n}{k}(-1)^{n-k}\mathcal{B}_{k+1,p} - \frac{p}{p+1}\,\mathcal{B}_{n,p+1}, \tag{3.12}$$

with the initial sequence $\mathcal{B}_{0,p} = 1$.

By using (3.12), we list the p-Bell numbers $\mathcal{B}_{n,p}$ for $0 \le n \le 6$ and $0 \le p \le 3$.

$$\mathcal{S}_p = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \mathcal{B}_{0,p} \\ 1 & 1/2 & 1/3 & 1/4 & \cdots \mathcal{B}_{1,p} \\ 2 & 5/6 & 1/2 & 7/20 & \cdots \mathcal{B}_{2,p} \\ 5 & 7/4 & 14/15 & 3/5 & \cdots \mathcal{B}_{3,p} \\ 15 & 68/15 & 13/6 & 179/140 & \cdots \mathcal{B}_{4,p} \\ 52 & 167/12 & 127/21 & 185/56 & \cdots \mathcal{B}_{5,p} \\ 203 & 2057/42 & 235/12 & 8389/840 & \cdots \mathcal{B}_{6,p} \\ \vdots & \vdots & \vdots & \vdots & \\ \mathcal{B}_{n,0} & \mathcal{B}_{n,1} & \mathcal{B}_{n,2} & \mathcal{B}_{n,3} & \end{pmatrix}$$

Theorem 3.12. *Let $p \geq 0$ be a fixed parameter. We define the double-index sequence $Z_{n;m}^{(p)}$ by:*

$$\mathcal{Z}_{n,m}(p) = \binom{m+p}{p} \sum_{k=0}^{m} s(m,k)\, \mathcal{B}_{n+k,p}, \tag{3.13}$$

where $s(m,k)$ denotes the Stirling numbers of the first kind and $\mathcal{B}_{n,p}$ are the p-Bell numbers.

Then, using (3.5), we obtain the equivalent form

$$\mathcal{Z}_{n,m}(p) = \binom{m+p}{p} \sum_{k=0}^{n} \begin{Bmatrix} n+m \\ k+m \end{Bmatrix}_m \binom{m+k+p}{p}^{-1}.$$

Moreover, the sequence satisfies the initial conditions:

$$\mathcal{Z}_{0,m}(p) = 1, \qquad \mathcal{Z}_{n,0}(p) = \mathcal{B}_{n,p}.$$

Theorem 3.13. *The $\mathcal{Z}_{n,m}(p)$ satisfy the following three-term recurrence relation.*

$$\mathcal{Z}_{n+1,m}(p) = \frac{m+1}{m+p+1}\, \mathcal{Z}_{n,m+1}(p) + m\, \mathcal{Z}_{n,m}(p), \tag{3.14}$$

with the initial sequence given by $\mathcal{Z}_{0,m}(p) = 1$.

Proof. Starting from (3.13), we write

$$\mathcal{Z}_{n,m+1}(p) = \binom{m+p+1}{p} \sum_{k=0}^{m+1} s(m+1,k)\, \mathcal{B}_{n+k,p}.$$

Using the recurrence relation of the Stirling numbers of the first kind

$$s(m+1,k) = s(m,k-1) - m\, s(m,k),$$

we obtain

$$\begin{aligned} \mathcal{Z}_{n,m+1}(p) &= \binom{m+p+1}{p} \sum_{k=0}^{m+1} \Big(s(m,k-1) - m\, s(m,k)\Big) \mathcal{B}_{n+k,p} \\ &= \binom{m+p+1}{p} \sum_{k=0}^{m+1} s(m,k-1) \mathcal{B}_{n+k,p} \\ &\quad - m \binom{m+p+1}{p} \sum_{k=0}^{m+1} s(m,k) \mathcal{B}_{n+k,p}. \end{aligned}$$

We put $j = k-1$, we get:

$$\sum_{k=0}^{m+1} s(m,k-1) \mathcal{B}_{n+k,p} = \sum_{j=-1}^{m} s(m,j) \mathcal{B}_{n+j+1,p}.$$

Since $s(m,-1) = 0$, so

$$\sum_{k=0}^{m+1} s(m,k-1) \mathcal{B}_{n+k,p} = \sum_{k=0}^{m} s(m,k) \mathcal{B}_{n+k+1,p}.$$

For the second sum, usings$(m,m+1) = 0$, we have:

$$\sum_{k=0}^{m+1} s(m,k) \mathcal{B}_{n+k,p} = \sum_{k=0}^{m} s(m,k) \mathcal{B}_{n+k,p}.$$

Therefore,

$$\mathcal{Z}_{n,m+1}(p) = \binom{m+p+1}{p} \sum_{k=0}^{m} s(m,k)\mathcal{B}_{n+k+1,p} - m\binom{m+p+1}{p} \sum_{k=0}^{m} s(m,k)\mathcal{B}_{n+k,p}.$$

By using the definition

$$\mathcal{Z}_{n,m}(p) = \binom{m+p}{p} \sum_{k=0}^{m} s(m,k)\mathcal{B}_{n+k,p},$$

we obtain

$$\sum_{k=0}^{m} s(m,k)\mathcal{B}_{n+k+1,p} = \binom{m+p}{p}^{-1} \mathcal{Z}_{n+1,m}(p)$$

and

$$\sum_{k=0}^{m} s(m,k)\mathcal{B}_{n+k,p} = \binom{m+p}{p}^{-1} \mathcal{Z}_{n,m}(p).$$

Substituting these identities into the previous relation yields

$$\mathcal{Z}_{n,m+1}(p) = \binom{m+p+1}{p}\binom{m+p}{p}^{-1} \mathcal{Z}_{n+1,m}(p) - m\binom{m+p+1}{p}\binom{m+p}{p}^{-1} \mathcal{Z}_{n,m}(p).$$

Since

$$\frac{\binom{m+p+1}{p}}{\binom{m+p}{p}} = \frac{m+p+1}{m+1},$$

we deduce

$$\mathcal{Z}_{n,m+1}(p) = \frac{m+p+1}{m+1}\mathcal{Z}_{n+1,m}(p) - m\frac{m+p+1}{m+1}\mathcal{Z}_{n,m}(p).$$

Hence, we deduce that:

$$\mathcal{Z}_{n+1,m}(p) = \frac{m+1}{m+p+1}\mathcal{Z}_{n,m+1}(p) + m\,\mathcal{Z}_{n,m}(p).$$

Which is obviously equivalent to (3.14).

□

3.2 The Poly-Bell Numbers

It is well known that the Cauchy formula for repeated integration [27] is given by:

$$\mathcal{D}^{-n} f(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\,dt, \tag{3.15}$$

where $\mathcal{D}^{-n}$ denotes the n-fold integral and f be a continuous function on the real line. Comparing the formula (3.15) with the formula (3.7) we obtain:

Theorem 3.14. *For $p \geq 0$, we have:*

$$\begin{aligned}\mathcal{B}_{n,p} &= p!\,\mathcal{D}^{-p}\phi_n(x)\big|_{x=1} \\ &= p!\int_0^1 dx_p \int_0^{x_p} dx_{p-1}\cdots\int_0^{x_2} \phi_n(x_1)\,dx_1.\end{aligned} \tag{3.16}$$

The iterated integral expression (3.16) *motivates a definition of poly-Bell numbers. Define the poly-Bell numbers $\mathbb{B}_n^{(p)}$ as follows:*

$$\mathbb{B}_n^{(p)} = \frac{\mathcal{B}_{n,p}}{p!}.$$

Theorem 3.15. *For $\mathbb{B}_n^{(p)}$ the generating function is given by:*

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(p)} \frac{z^n}{n!} = \sum_{n\geq 0} \frac{(e^z-1)^n}{\Gamma(n+p+1)} \tag{3.17}$$

and can be extended for p negative as:

$$\sum_{n=0}^{\infty} \mathbb{B}_n^{(-p)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(e^z-1)^n}{\Gamma(n-p+1)}. \tag{3.18}$$

An explicit formula of the poly-Bell numbers at a negative upper index p can be obtained by a direct use of (3.18).

Theorem 3.16. *The poly-Bell numbers $\mathbb{B}_n^{(-p)}$ are positive and we have:*

$$\mathbb{B}_n^{(-p)} = \sum_{k=p}^{n} \frac{k!}{(k-p)!} \left\{ n \atop k \right\}. \tag{3.19}$$

Corollary 3. *For $p \geq 1$, we have:*

$$\begin{aligned} \mathbb{B}_n^{(-p)} &= \frac{d^p}{dx^p} \phi_n(x)\Big|_{x=1} \\ &= p! \sum_{j=p}^{n} \binom{n}{j} \left\{ j \atop p \right\} \phi_{n-j}. \end{aligned}$$

Proof. From (2.4), we have:

$$\begin{aligned} \frac{d^p}{dx^p} \phi_n(x) &= \sum_{k=0}^{n} \left\{ n \atop k \right\} \frac{d^p}{dx^p} x^k \\ &= p! \sum_{k=p}^{n} \binom{k}{p} \left\{ n \atop k \right\} x^{k-p}. \end{aligned}$$

which reduces to $\mathbb{B}_n^{(-p)}$ by setting $x=1$. On the other hand, from (2.5), we have:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{d^p}{dx^p} \phi_n(x) \frac{z^n}{n!} &= \frac{d^p}{dx^p} \exp\left(x(e^z-1)\right) \\ &= (e^z-1)^p \exp\left(x(e^z-1)\right) \\ &= p! \sum_{n=0}^{\infty} \left\{ n \atop p \right\} \frac{z^n}{n!} \sum_{n=0}^{\infty} \phi_n(x) \frac{z^n}{n!} \\ &= p! \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{n}{j} \left\{ j \atop p \right\} \phi_{n-j}(x) \right) \frac{z^n}{n!}. \end{aligned}$$

Equating coefficients of $\frac{z^n}{n!}$ and setting $x=1$, we get the result. □

Theorem 3.17. *The double generating function of $\mathbb{B}_n^{(-p)}$ is given by:*

$$\sum_{p=0}^{\infty} \sum_{n=p}^{\infty} \mathbb{B}_n^{(-p)} \frac{z^n}{n!} \frac{y^p}{p!} = \exp\left((y+1)(e^z-1)\right).$$

Proof. The double generating function of $\mathbb{B}_n^{(-p)}$ can be obtained by using (3.18), since $\Gamma(n-p+1) = (n-p)!$ for $n \geq p$, we rewrite:

$$\sum_{p=0}^{\infty} \sum_{n=p}^{\infty} \mathbb{B}_n^{(-p)} \frac{z^n}{n!} \frac{y^p}{p!} = \sum_{p=0}^{\infty} \sum_{n=p}^{\infty} \frac{(e^z-1)^n}{(n-p)!} \frac{y^p}{p!},$$

we set $n = p + i$ with $i \geq 0$, hence $(n-p)! = i!$, and $(e^z-1)^n = (e^z-1)^{p+i}$. Then:

$$\begin{aligned}\sum_{p=0}^{\infty}\sum_{n=p}^{\infty}\mathbb{B}_n^{(-p)}\frac{z^n}{n!}\frac{y^p}{p!} &= \sum_{p=0}^{\infty}\frac{(y(e^z-1))^p}{p!}\sum_{i\geq 0}\frac{(e^z-1)^i}{i!}\\ &= \sum_{p\geq 0}\frac{(y(e^z-1))^p}{p!}\sum_{i\geq 0}\frac{(e^z-1)^i}{i!}\\ &= \exp\left(y(e^z-1)\right)\exp(e^z-1)\\ &= \exp\left((y+1)(e^z-1)\right).\end{aligned}$$

□

Corollary 4. *We have :*

$$\sum_{p=0}^{\infty}\mathbb{B}_n^{(-p)}\frac{y^p}{p!} = \phi_n(1+y).$$

Proof. It follows immediately from:

$$\exp\left((y+1)(e^z-1)\right) = \sum_{n=0}^{\infty}\phi_n(1+y)\frac{z^n}{n!}.$$

□

Here is a small table of the numbers $\mathbb{B}_n^{(-p)}$ $(-4 \leq p \leq -1,\ 0 \leq n \leq 9)$.

$p\backslash n$	0	1	2	3	4	5	6	7	8	9
-1	0	1	3	10	37	151	674	3263	17007	94828
-2	0	0	2	12	62	320	1712	9604	56674	351792
-3	0	0	0	6	60	450	3120	21336	147756	1048830
-4	0	0	0	0	24	360	3720	33600	287784	2424744

Table 3.1: First values of poly-Bell numbers with negative upper index

3.3 The p-Bell Polynomials

The p-Bell polynomials are a generalization of the classical Bell polynomials. They appear in the study of ordered partitions and related combinatorial structures and can be described through their generating function, as discussed in [18, 19, 31].

Definition 3.2. *The p-Bell polynomials $\mathcal{B}_{n,p}(x)$ are defined by the following generating function:*

$$\sum_{n=0}^{\infty}\mathcal{B}_{n,p}(x)\frac{z^n}{n!} = {}_1F_1\left(\begin{matrix}1\\ p+1\end{matrix};e^z-1\right)e^{xz}. \tag{3.20}$$

Theorem 3.18. *The explicit formula for the p-Bell polynomials, for $n \geq 0$, we have:*

$$\mathcal{B}_{n,p}(x) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{k,p}x^{n-k}, \tag{3.21}$$

Proof. By using the generating function (3.20), we have:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{B}_{n,p}(x)\frac{z^n}{n!} &= {}_1F_1\left(\begin{matrix}1\\ p+1\end{matrix};e^z-1\right)e^{xz}\\ &= \left(\sum_{n=0}^{\infty}\mathcal{B}_{n,p}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}x^n\frac{z^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{k,p}x^{n-k}\right),\end{aligned}$$

By identification, we have:

$$\mathcal{B}_{n,p}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_{k,p} x^{n-k}.$$

□

The first few p-Bell polynomials are:

$$\mathcal{B}_{0,p}(x) = 1,$$

$$\mathcal{B}_{1,p}(x) = x + \frac{1}{p+1},$$

$$\mathcal{B}_{2,p}(x) = x^2 + \frac{2x}{p+1} + \frac{p+4}{(p+1)(p+2)},$$

$$\mathcal{B}_{3,p}(x) = x^3 + \frac{3x^2}{p+1} + \frac{3(p+4)x}{(p+1)(p+2)} + \frac{p^2+11p+30}{(p+1)(p+2)(p+3)}.$$

and

$$\mathcal{B}_{4,p}(x) = x^4 + \frac{4x^3}{p+1} + \frac{6(p+4)x^2}{(p+1)(p+2)} + \frac{4(p^2+11p+30)x}{(p+1)(p+2)(p+3)} + \frac{p^3+23p^2+160p+360}{(p+1)(p+2)(p+3)(p+4)}.$$

Theorem 3.19. *For $p \geq 0$, we have:*

$$\mathcal{B}_{n,p}(x) = \sum_{k=0}^{n} \binom{k+p}{k}^{-1} S_n^k(x), \tag{3.22}$$

Proof. From the right-hand side of (3.22) and the generating function of $S_n^k(x)$.

$$\frac{1}{k!} e^{xz} (e^z - 1)^k = \sum_{n=k}^{\infty} S_k^n(x) \frac{z^n}{n!},$$

we have:

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{k+p}{k}^{-1} S_n^k(x) \right) \frac{z^n}{n!} &= \sum_{k=0}^{\infty} \binom{k+p}{k}^{-1} \sum_{n \geq 0} S_n^k(x) \frac{z^n}{n!} \\
&= \sum_{k=0}^{\infty} \binom{k+p}{k}^{-1} \frac{1}{k!} e^{xz} (e^z - 1)^k \\
&= e^{xz} \sum_{k=0}^{\infty} \frac{p!}{(k+p)!} (e^z - 1)^k \\
&= e^{xz} \sum_{n=0}^{\infty} B_{n,p} \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} B_{n,p}(x) \frac{z^n}{n!}.
\end{aligned}$$

Since the exponential generating functions are identical, equating coefficients yields (3.22):

$$B_{n,p}(x) = \sum_{k=0}^{n} \binom{k+p}{k}^{-1} S_n^k(x).$$

□

Theorem 3.20. *The polynomials $\mathcal{Z}_{n,m}(x;p)$ satisfy the following three-term recurrence relation.*

$$\mathcal{Z}_{n+1,m}(x;p) = \frac{m+1}{m+p+1}\,\mathcal{Z}_{n,m+1}(x;p) + (m+x)\,\mathcal{Z}_{n,m}(x;p), \tag{3.23}$$

with the initial sequence given by:

$$\mathcal{Z}_{0,m}(x;p) = 1.$$

The polynomials $G_{n,x}(t)$ defined a particular case of the more general polynomials considered by Corcino et al. in [16] defined by:

Definition 3.3. *The polynomials $G_{n,x}(t)$ are defined by:*

$$G_{n,x}(t) = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}(x+k)^n,$$

These polynomials will be used in the following integral representation of the p-Bell polynomials.

Theorem 3.21. *The p-Bell polynomials $\mathcal{B}_{n,p}(x)$ satisfy the following integral representation.*

$$\mathcal{B}_{n,p}(x) = p\int_0^1 (1-t)^{p-1}G_{n,x}(t)\,dt. \tag{3.24}$$

Now using their properties, we obtain the following corollaries.

Corollary 5. ***(Dobinski's formula)*** *For $p \geq 1$, we have*

$$\mathcal{B}_{n,p}(x) = p\sum_{k\geq 0}\frac{(x+k)^n}{k!}\,{}_1F_1\left(\begin{matrix}k+1\\ p+k+1\end{matrix};-1\right).$$

Proof. From the Dobinski formula for the $(r,1)$-Bell numbers [16, Example(1.18)]. we have:

$$G_{n,x}(t) = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}(x+k)^n,$$

we obtain:

$$\begin{aligned}\mathcal{B}_{n,p}(x) &= p\int_0^1 (1-t)^{p-1}\left(e^{-t}\sum_{k\geq 0}\frac{t^k}{k!}(x+k)^n\right)dt\\ &= p\sum_{k=0}^{\infty}\frac{(x+k)^n}{k!}\int_0^1 e^{-t}t^k(1-t)^{p-1}dt\\ &= \sum_{k=0}^{\infty}\frac{(x+k)^n}{k!}\,{}_1F_1\left(\begin{matrix}k+1\\ p+k+1\end{matrix};-1\right).\end{aligned}$$

□

Corollary 6. ***(Recurrence relations)*** *For $p \geq 1$, we have:*

$$\mathcal{B}_{n+1,p}(x) = x\mathcal{B}_{n,p}(x) - \sum_{k=0}^{n}\binom{n}{k}\left(\frac{p}{p+1}\mathcal{B}_{k,p+1}(x) - \mathcal{B}_{k,p}(x)\right). \tag{3.25}$$

Proof. From the recurrence relations for the $(r,1)$-Bell numbers [16, Equation (2.9)], we have:

$$G_{n+1,x}(t) = x\,G_{n,x}(t) + \sum_{k=0}^{n}t\binom{n}{k}G_{n-k,x}(t),$$

we obtain

$$\mathcal{B}_{n+1,p}(x) = p\int_0^1 (1-t)^{p-1}\left(x\,G_{n,x}(t) + \sum_{k=0}^{n} t\binom{n}{k} G_{n-k,x}(t)\right)dt$$
$$= x\,p\int_0^1 (1-t)^{p-1}G_{n,x}(t)\,dt - p\sum_{k=0}^{n}\binom{n}{k}\int_0^1((1-t)-1)(1-t)^{p-1}G_{n-k,x}(t)\,dt.$$

After some manipulation we get (3.25). □

Note that when $p = 0$, (3.25) gives:

$$\phi_{n+1}(x) = x\,\phi_n(x) + \sum_{k=0}^{n}\binom{n}{k}\phi_k(x),$$

where $\phi_n(x) := B_{n;0}(x)$.

3.4 The r-Bell Numbers

The r-Bell number $\phi_{n,r}$ is the number of partitions of a set with $n+r$ elements such that, in each partition, the first r elements belong to distinct subsets. As discussed in [4, 9, 10, 12, 24, 25].

Definition 3.4. *The r-Bell numbers $\phi_{n,r}$ are defined by:*

$$\phi_{n,r} = \sum_{k=0}^{n}\left\{ {n+r \atop k+r} \right\}_r. \tag{3.26}$$

In particular, for $r = 0$, we obtain the Bell numbers (2.1)

Example and Tables

Example 3.1. *The following example illustrates again the meaning of the r-Bell numbers. By definition,*

$$\phi_{2,2} = \left\{ {4 \atop 2} \right\}_2 + \left\{ {4 \atop 3} \right\}_2 + \left\{ {4 \atop 4} \right\}_2.$$

$\left\{ {4 \atop 2} \right\}_2$ *counts the partitions of 4 element into 2 subsets such that the first 2 element are in distinct subsets:*

$$\{1,3,4\},\{2\};\quad \{1\},\{2,3,4\};\quad \{1,3\},\{2,4\};\quad \{1,4\},\{2,3\}.$$

$\left\{ {4 \atop 3} \right\}_2$ *belongs to the partitions*

$$\{1\},\{2\},\{3,4\};\quad \{1,3\},\{2\},\{4\};\quad \{1,4\},\{2\},\{3\};\quad \{1\},\{2,3\},\{4\};\quad \{1\},\{2,4\},\{3\}.$$

Finally, $\left\{ {4 \atop 4} \right\}_2$ *equals to the number of partitions of 4 elements into 4 subsets (and necessarily,the first two elements are in distinct subsets):*

$$\{1\},\{2\},\{3\},\{4\}.$$

Hence,

$$\phi_{2,2} = \left\{ {4 \atop 2} \right\}_2 + \left\{ {4 \atop 3} \right\}_2 + \left\{ {4 \atop 4} \right\}_2 = 4 + 5 + 1 = 10.$$

This is the number of partitions of the set $\{1,2,3,4\}$ such that the first two elements belong to distinct subsets.

$r\backslash n$	0	1	2	3	4	5	6
0	1	1	2	5	15	52	203
1	1	2	5	15	52	203	877
2	1	3	10	37	151	674	3263
3	1	4	17	77	372	1915	10481
4	1	5	26	141	799	4736	29371
5	1	6	37	235	1540	10427	73013
6	1	7	50	365	2727	20878	163967

Table 3.2: The First r-Bell Numbers

Using the recurrence relation (3.26), we can compute the first values of the r-Bell numbers presented in Table 3.2.

Theorem 3.22. *Their exponential generating function is given by:*

$$\sum_{n=0}^{\infty} \phi_{n,r} \frac{z^n}{n!} = \exp(e^z - 1 + rz). \tag{3.27}$$

Proof. Starting from the definition (3.26), and substituting it into the exponential generating function, we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_{n,r} \frac{z^n}{n!} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r \right) \frac{z^n}{n!} \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \left\{ {n+r \atop k+r} \right\}_r \frac{z^n}{n!}.\end{aligned}$$

Using the exponential generating function of the r-Stirling numbers, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_{n,r} \frac{z^n}{n!} &= \frac{(e^z - 1)^k}{k!} e^{rz} \\ &= e^{rz} \sum_{k=0}^{\infty} \frac{(e^z - 1)^k}{k!} \\ &= e^{rz} \exp(e^z - 1) \\ &= \exp(e^z - 1 + rz),\end{aligned}$$

which completes the proof. □

Theorem 3.23. *The generalized Dobinski's Formula for r-Bell numbers, is given by :*

$$\phi_{n,r} = \frac{1}{e} \sum_{i=0}^{\infty} \frac{(i+r)^n}{i!}. \tag{3.28}$$

Proof. Using the generating function of the r-Stirling numbers (1.18), we obtain:

$$\frac{(i+r)^n}{i!} = \sum_{k=0}^{i} \left\{ {n+r \atop k+r} \right\}_r \frac{1}{(i-k)!}.$$

$$\begin{aligned}\sum_{i=0}^{\infty} \frac{(i+r)^n}{i!} &= \sum_{i=0}^{\infty} \sum_{k=0}^{i} \left\{ {n+r \atop k+r} \right\}_r \frac{1}{(i-k)!} \\ &= \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r \sum_{i=k}^{\infty} \frac{1}{(i-k)!}.\end{aligned}$$

Setting $j = i - k$, we obtain:

$$\sum_{i=0}^{\infty} \frac{(i+r)^n}{i!} = \sum_{k=0}^{n} \left\{ n+r \atop k+r \right\}_r \sum_{j=0}^{\infty} \frac{1}{j!}$$
$$= e \sum_{k=0}^{n} \left\{ n+r \atop k+r \right\}_r.$$

Finally, we get the result.

$$\phi_{n,r} = \frac{1}{e} \sum_{i=0}^{\infty} \frac{(i+r)^n}{i!}.$$

□

Theorem 3.24. *the r-Bell numbers satisfy the following recurrence relation given by:*

$$\phi_{n,r+1} = \sum_{i=0}^{n} \binom{n}{i} \phi_{i,r}.$$

Theorem 3.25. *The r-Bell numbers have the integral representation given by:*

$$\phi_{n,r} = \frac{2n!}{\pi e} \operatorname{Im} \int_0^{\pi} \exp(\exp(e^{i\theta})) \exp(re^{i\theta}) \sin(n\theta)\, d\theta.$$

Proof. We start from the r-Stirling numbers identity:

$$k! \left\{ n+r \atop k+r \right\}_r = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j+r)^n.$$

Using the integral representation:

$$(j+r)^n = \frac{2n!}{\pi} \operatorname{Im} \int_0^{\pi} \exp((j+r)e^{i\theta}) \sin(n\theta)\, d\theta,$$

we get

$$k! \left\{ n+r \atop k+r \right\}_r = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{2n!}{\pi} \operatorname{Im} \int_0^{\pi} \exp((j+r)e^{i\theta}) \sin(n\theta)\, d\theta,$$
$$= \frac{2n!}{\pi} \operatorname{Im} \int_0^{\pi} \left(\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \exp((j+r)e^{i\theta}) \right) \sin(n\theta)\, d\theta.$$
$$= \frac{2n!}{\pi} \operatorname{Im} \int_0^{\pi} \left(\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \exp(je^{i\theta}) \right) \exp(re^{i\theta}) \sin(n\theta)\, d\theta.$$

Using the binomial identity:

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} x^j = (x-1)^k,$$

with $x = \exp(e^{i\theta})$, we get:

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \exp(je^{i\theta}) = (\exp(e^{i\theta}) - 1)^k.$$

Thus:

$$k! \left\{ n+r \atop k+r \right\}_r = \frac{2n!}{\pi} \operatorname{Im} \int_0^{\pi} (\exp(e^{i\theta}) - 1)^k \exp(re^{i\theta}) \sin(n\theta)\, d\theta.$$

Dividing by $k!$ and summing over $k \geq 0$:

$$\begin{aligned}
\sum_{k=0}^{\infty} \left\{ n+r \atop k+r \right\}_r &= \frac{2n!}{\pi} \operatorname{Im} \int_0^{\pi} \sum_{k=0}^{\infty} \frac{(\exp(e^{i\theta})-1)^k}{k!} \exp(re^{i\theta}) \sin(n\theta)\, d\theta \\
&= \frac{2n!}{\pi} \operatorname{Im} \int_0^{\pi} \exp(\exp(e^{i\theta})-1) \exp(re^{i\theta}) \sin(n\theta)\, d\theta \\
&= \frac{2n!}{\pi} \operatorname{Im} \int_0^{\pi} \exp(-1) \exp(\exp(e^{i\theta})) \exp(re^{i\theta}) \sin(n\theta)\, d\theta \\
&= \frac{2n!}{\pi e} \operatorname{Im} \int_0^{\pi} \exp(\exp(e^{i\theta})) \exp(re^{i\theta}) \sin(n\theta)\, d\theta.
\end{aligned}$$

This completes the proof. □

3.5 The r-Bell Polynomials

Definition 3.5. *The r-Bell polynomials $\phi_{n,r}(x)$ are defined by:*

$$\phi_{n,r}(x) = \sum_{k=0}^{n} \left\{ n+r \atop k+r \right\}_r x^k. \tag{3.29}$$

In particular, for $r = 0$, we obtain the Bell polynomials (2.4).

Theorem 3.26. *Their exponential generating function is given by:*

$$\sum_{n=0}^{\infty} \phi_{n,r}(x) \frac{z^n}{n!} = \exp(x(e^z - 1) + rz). \tag{3.30}$$

Proof. Starting from the definition (3.29), we get:

$$\begin{aligned}
\sum_{n=0}^{\infty} \phi_{n,r}(x) \frac{z^n}{n!} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left\{ n+r \atop k+r \right\}_r x^k \right) \frac{z^n}{n!} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \left\{ n+r \atop k+r \right\}_r x^k \frac{z^n}{n!}.
\end{aligned}$$

Using the exponential generating function of the r-Stirling numbers, we obtain:

$$\sum_{n=k}^{\infty} \left\{ n+r \atop k+r \right\}_r \frac{z^n}{n!} = \frac{(e^z-1)^k}{k!} e^{rz}.$$

Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty} \phi_{n,r}(x) \frac{z^n}{n!} &= \sum_{k=0}^{\infty} x^k \frac{(e^z-1)^k}{k!} e^{rz} \\
&= e^{rz} \sum_{k=0}^{\infty} \frac{\left(x(e^z-1)\right)^k}{k!} \\
&= e^{rz} \exp(x(e^z-1)) \\
&= \exp(x(e^z-1) + rz),
\end{aligned}$$

which completes the proof.

□

Theorem 3.27. *the r-Bell polynomials satisfy the following recurrence relation given by:*

$$\phi_{n,r+1}(x) = \sum_{i=0}^{n} \binom{n}{i} \phi_{i,r}(x).$$

Proof. Using the exponential generating function of the r-Bell polynomials:

$$\sum_{n=0}^{\infty} \phi_{n,r}(x)\frac{z^n}{n!} = e^{x(e^z-1)+rz}.$$

Replacing r by r + 1, we obtain:

$$\sum_{n=0}^{\infty} \phi_{n,r+1}(x)\frac{z^n}{n!} = e^{x(e^z-1)+(r+1)z} = e^z\, e^{x(e^z-1)+rz}.$$

Using the generating function of $\phi_{n,r}(x)$:

$$\sum_{n=0}^{\infty} \phi_{n,r+1}(x)\frac{z^n}{n!} = e^z \sum_{n=0}^{\infty} \phi_{n,r}(x)\frac{z^n}{n!},$$

since

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!},$$

we get:

$$\sum_{n=0}^{\infty} \phi_{n,r+1}(x)\frac{z^n}{n!} = \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)\left(\sum_{n=0}^{\infty} \phi_{n,r}(x)\frac{z^n}{n!}\right).$$

By the Cauchy product of series, we obtain:

$$\phi_{n,r+1}(x) = \sum_{i=0}^{n} \binom{n}{i} \phi_{i,r}(x).$$

Hence the result is proved. □

The first few r-Bell polynomials are:

$$\begin{aligned}
\phi_{0,r}(x) &= 1\\
\phi_{1,r}(x) &= x + r\\
\phi_{2,r}(x) &= x^2 + (2r+1)x + r^2\\
\phi_{3,r}(x) &= x^3 + (3r+3)x^2 + (3r^2+3r+1)x + r^3\\
B_{4,r}(x) &= x^4 + (4r+6)x^3 + (6r^2+12r+7)x^2\\
&\quad + (4r^3+6r^2+4r+1)x + r^4.
\end{aligned}$$

Theorem 3.28. ***(Dobinski's formula)*** *The r-Bell polynomials satisfy the identity:*

$$\phi_{n,r}(x) = \frac{1}{e^x}\sum_{k=0}^{\infty} \frac{(k+r)^n}{k!}\, x^k. \tag{3.31}$$

Proof. Using the generating function of the r-Stirling numbers for fixed n and r (1.19), we get:

$$\begin{aligned}
\frac{(i+r)^n}{i!} &= \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r \frac{1}{(i-k)!},\\
\sum_{i=0}^{\infty} \frac{(i+r)^n}{i!}x^i &= \sum_{i=0}^{\infty}\sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r \frac{x^i}{(i-k)!}\\
&= \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r \sum_{i=k}^{\infty} \frac{x^i}{(i-k)!}\\
&= \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r \sum_{i=k}^{\infty} \frac{x^i}{(i-k)!}.
\end{aligned}$$

Setting $j = i - k$, we obtain:

$$\begin{aligned}
\sum_{i=0}^{\infty} \frac{(i+r)^n}{i!} x^i &= \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r \sum_{j=0}^{\infty} \frac{x^{j+k}}{j!} \\
&= \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r x^k \sum_{j=0}^{\infty} \frac{x^j}{j!} \\
&= e^x \left(\sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r x^k \right) \\
&= e^x \phi_{n,r}(x).
\end{aligned}$$

□

4

The Exponential Partial Bell Polynomials

In this chapter, we present the **(Exponential) partial Bell polynomials** $B_{n,k}(x_1, x_2, \ldots, x_n)$, introduced by Bell as an effective tool for representing higher-order derivatives of composite functions. We study their main properties, explicit formulas, and recurrence relations.

We then introduce the **(Exponential) complete Bell polynomials** $B_n(x_1, x_2, \ldots, x_n)$, defined as the sum of all partial Bell polynomials over all possible degrees. We present their exponential generating function, their explicit formula, and we highlight the fundamental relation connecting them to the partial Bell polynomials, which plays a central role in combinatorial analysis and in the study of composite function derivatives [5, 13, 15, 30].

4.1 The (Exponential) Partial Bell Polynomials

Definition 4.1. *The exponential partial Bell polynomials, denoted by $B_{n,k}(x_1, \ldots, x_n)$ in the variables x_n, are sometimes also written as $B_{n,k}(x_j)$. They are defined by their generating function:*

$$\sum_{n=k}^{\infty} B_{n,k}\left(x_1, x_2, \ldots, x_n\right) \frac{z^n}{n!} = \frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right)^k. \tag{4.1}$$

Theorem 4.1. *The explicit formula for the (exponential) partial Bell polynomials given by:*

$$B_{n,k}\left(x_1, x_2, \ldots, x_n\right) = \sum_{\pi(n,k)} \frac{n!}{k_1! \ldots k_n!} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \cdots \left(\frac{x_n}{n!}\right)^{k_n}, \tag{4.2}$$

where $\pi\left(n, k\right)$, denote the set of all solutions in natural integers such that:

$$\pi\left(n, k\right) = \left\{ \; k_1, \ldots, k_n \in \mathbb{N}^n : \textstyle\sum_{i=1}^{n} k_i = k, \sum_{i=1}^{n} i k_i = k_1 + 2k_2 + 3k_3 \cdots = n \; \right\}.$$

Theorem 4.2. *The (exponential) partial Bell polynomials satisfy the following properties and recurrence relations:*

$$B_{n,k}\left(ax_1, ax_2, ax_3, \ldots\right) = a^k B_{n,k}\left(x_1, x_2, x_3, \ldots\right). \tag{4.3}$$

$$B_{n,k}\left(ax_1, a^2x_2, a^3x_3 \ldots\right) = a^n B_{n,k}\left(x_1, x_2, x_3, \ldots\right). \tag{4.4}$$

Proof. *1. Using the explicit formula of the (exponential) partial Bell polynomials (4.2)*

$$B_{n,k}(x_1, x_2, \ldots) = \sum_{\pi(n,k)} \frac{n!}{k_1! k_2! \cdots} \prod_{i \geq 1} \left(\frac{x_i}{i!}\right)^{k_i},$$

with

$$\sum_{i \geq 1} k_i = k, \qquad \sum_{i \geq 1} i k_i = n.$$

Replacing each x_i by ax_i, we obtain:

$$\begin{aligned} B_{n,k}(ax_1, ax_2, \dots) &= \sum_{\pi(n,k)} \frac{n!}{k_1!k_2!\cdots} \prod_{i\geq 1} \left(\frac{ax_i}{i!}\right)^{k_i} \\ &= \sum_{\pi(n,k)} \frac{n!}{k_1!k_2!\cdots} \prod_{i\geq 1} a^{k_i} \left(\frac{x_i}{i!}\right)^{k_i} \\ &= \sum_{\pi(n,k)} \frac{n!}{k_1!k_2!\cdots} \prod_{i\geq 1} a^{k_i} \prod_{i\geq 1} \left(\frac{x_i}{i!}\right)^{k_i}. \end{aligned}$$

Since

$$\sum_{i\geq 1} k_i = k,$$

we have:

$$\prod_{i\geq 1} a^{k_i} = a^{\sum k_i} = a^k.$$

Therefore,

$$\begin{aligned} B_{n,k}(ax_1, ax_2, \dots) &= a^k \sum_{\pi(n,k)} \frac{n!}{k_1!k_2!\cdots} \prod_{i\geq 1} \left(\frac{x_i}{i!}\right)^{k_i} \\ &= a^k B_{n,k}(x_1, x_2, \dots). \end{aligned}$$

Hence, the result follows from (4.3).

2. *Using the explicit formula of the (exponential) partial Bell polynomials* (4.2)

$$B_{n,k}(x_1, x_2, \dots) = \sum_{\pi(n,k)} \frac{n!}{k_1!k_2!\cdots} \prod_{i\geq 1} \left(\frac{x_i}{i!}\right)^{k_i},$$

with

$$\sum_{i\geq 1} k_i = k, \qquad \sum_{i\geq 1} ik_i = n.$$

Replacing each x_i by $a^i x_i$, we obtain:

$$\begin{aligned} B_{n,k}(ax_1, a^2x_2, a^3x_3, \dots) &= \sum_{\pi(n,k)} \frac{n!}{k_1!k_2!\cdots} \prod_{i\geq 1} \left(\frac{a^i x_i}{i!}\right)^{k_i} \\ &= \sum_{\pi(n,k)} \frac{n!}{k_1!k_2!\cdots} \prod_{i\geq 1} a^{ik_i} \left(\frac{x_i}{i!}\right)^{k_i} \\ &= \sum_{\pi(n,k)} \frac{n!}{k_1!k_2!\cdots} \prod_{i\geq 1} a^{ik_i} \prod_{i\geq 1} \left(\frac{x_i}{i!}\right)^{k_i}. \end{aligned}$$

Since

$$\sum_{i\geq 1} ik_i = n,$$

we have:

$$\prod_{i\geq 1} a^{ik_i} = a^{\sum ik_i} = a^n.$$

Therefore,

$$\begin{aligned} B_{n,k}(ax_1, a^2x_2, a^3x_3, \dots) &= a^n \sum_{\pi(n,k)} \frac{n!}{k_1!k_2!\dots} \prod_{i\geq 1} \left(\frac{x_i}{i!}\right)^{k_i} \\ &= a^n B_{n,k}(x_1, x_2, x_3, \dots). \end{aligned}$$

Hence, the result follows from (4.4).

□

Theorem 4.3. *The (exponential) partial Bell polynomials admit the following expression.*

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \frac{n!}{k!} \sum_{n_1+\cdots+n_k=n-k} \frac{x_{n_1+1}\dots x_{n_k+1}}{(n_1+1)!\dots(n_k+1)!}, \quad n \geq k \geq 1.$$

4.2 The (Exponential) Complete Bell Polynomials

Definition 4.2. *The (exponential) complete Bell polynomials are defined by:*

$$\sum_{n=0}^{\infty} B_n(x_1, x_2, \dots, x_n) \frac{z^n}{n!} = \exp\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right). \tag{4.5}$$

Theorem 4.4. *The relation between the (exponential) complete Bell polynomials and the (exponential) partial Bell polynomials is given by:*

$$B_n(x_1, x_2, \dots, x_n) = \sum_{k=0}^{n} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}). \tag{4.6}$$

Proof. To prove that (4.6), we start from the exponential generating function of the (exponential) complete Bell polynomials (4.5), and using the Taylor expansion of the exponential function,

$$e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!}, \qquad with \qquad u = \sum_{m=1}^{\infty} x_m \frac{z^m}{m!},$$

we obtain

$$\exp\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right)^k.$$

By the definition of the (exponential) partial Bell polynomials (4.1),

$$\sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{z^n}{n!} = \frac{1}{k!} \left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right)^k,$$

hence (4.5), give us

$$\begin{aligned} \sum_{n=0}^{\infty} B_n(x_1, x_2, \dots, x_n) \frac{z^n}{n!} &= \exp\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right)^k \\ &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} B_{n,k}(x_1, x_2, \dots, x_{n-k+1})\right) \frac{z^n}{n!}. \end{aligned}$$

Comparing coefficients of $\frac{z^n}{n!}$ on the both sides, we conclude that:

$$B_n(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{n} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}).$$

□

Theorem 4.5. *The explicit formula for the (exponential) complete Bell polynomials is given by:*

$$B_n(x_1, x_2, \ldots, x_n) = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{n!}{k_1!\, k_2! \cdots k_n!} \prod_{i\geq 1} \left(\frac{x_i}{i!}\right)^{k_i}.$$

4.3 Special Case

1. It is known that, by choosing an appropriate sequence $(x_n)_{n\geq 1}$, the partial Bell polynomials $B_{n,k}(x_1, x_1, \ldots, x_{n-k+1})$, can be expressed in terms of various classical combinatorial sequences. Some special cases are given below:

   a. Stirling numbers of the first kind

   $$B_{n,k}(((-1)^{i-1}(i-1)!)) = (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}$$

   with exponential generating function:

   $$\sum_{n\geq k} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{z^n}{n!} = \frac{1}{k!} \left(\sum_{m=1}^{\infty} (-1)^{m-1}(m-1)! \frac{z^m}{m!} \right)^k$$
   $$= \sum_{n=k}^{\infty} B_{n,k}(0!, -1!, 2!, \ldots) \frac{z^n}{n!}, \quad k \geq 0.$$

   - Using the exponential generating function of the partial Bell polynomials (4.1):

   $$\sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots) \frac{z^n}{n!} = \frac{1}{k!} \left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!} \right)^k.$$

   We put $x_m = (-1)^{m-1}(m-1)!$, substituting into the generating function, we obtain:

   $$\sum_{n=k}^{\infty} B_{n,k}(((-1)^{i-1}(i-1)!)) \frac{z^n}{n!} = \frac{1}{k!} \left(\sum_{m=1}^{\infty} (-1)^{m-1}(m-1)! \frac{z^m}{m!} \right)^k.$$

   Since

   $$\frac{(m-1)!}{m!} = \frac{1}{m!},$$

   the inner sum becomes:

   $$\sum_{n=k}^{\infty} B_{n,k}(((-1)^{i-1}(i-1)!)) \frac{z^n}{n!} = \frac{1}{k!} \left(\sum_{m=1}^{\infty} (-1)^{m-1} \frac{z^m}{m!} \right)^k$$
   $$= \frac{1}{k!} \left(\sum_{m=1}^{\infty} (-1)^{m-1} \frac{z^m}{m!} \right)^k$$
   $$= \frac{1}{k!} \left(\ln(1+z) \right)^k.$$

On the other hand, it is known that the exponential generating function of the Stirling numbers of the first kind is:

$$\sum_{n=k}^{\infty}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}\frac{z^n}{n!}=\frac{(\ln(1+z))^k}{k!}.$$

By identification of coefficients, we conclude that:

$$B_{n,k}(((-1)^{i-1}(i-1)!))=(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}.$$

b. Stirling numbers of the second kind

$$B_{n,k}\,(1,1,\dots)=\begin{Bmatrix}n\\k\end{Bmatrix}$$

with exponential generating function:

$$\begin{aligned}\sum_{n=k}^{\infty}\begin{Bmatrix}n\\k\end{Bmatrix}\frac{z^n}{n!}&=\frac{1}{k!}\left(\sum_{m=1}^{\infty}\frac{z^m}{m!}\right)^k\\&=\sum_{m=1}^{\infty}B_{n,k}(1,1,\dots)\frac{z^n}{n!},\quad k\geq 0.\end{aligned}$$

- Using the exponential generating function of the partial Bell polynomials (4.1):

$$\sum_{n=k}^{\infty}B_{n,k}(x_1,x_2,\dots)\frac{z^n}{n!}=\frac{1}{k!}\left(\sum_{m=1}^{\infty}x_m\frac{z^m}{m!}\right)^k.$$

We put $x_m=1\quad$ for all $m\geq 1$, substituting into the generating function, we obtain:

$$\sum_{n=k}^{\infty}B_{n,k}(1,1,\dots)\frac{z^n}{n!}=\frac{1}{k!}\left(\sum_{m=1}^{\infty}\frac{z^m}{m!}\right)^k.$$

Thus, we get:

$$\sum_{n=k}^{\infty}B_{n,k}(1,1,\dots)\frac{z^n}{n!}=\frac{1}{k!}(e^z-1)^k.$$

On the other hand, the exponential generating function of the Stirling numbers of the second kind is:

$$\sum_{n=k}^{\infty}\begin{Bmatrix}n\\k\end{Bmatrix}\frac{z^n}{n!}=\frac{1}{k!}(e^z-1)^k.$$

By comparing the coefficients, we conclude that:

$$B_{n,k}(1,1,\dots)=\begin{Bmatrix}n\\k\end{Bmatrix}.$$

c. Lah numbers

$$B_{n,k}(1!,2!,...)=L(n,k)$$

with exponential generating function:

$$\begin{aligned}\sum_{n=k}^{\infty}L(n,k)\frac{z^n}{n!}&=\frac{1}{k!}\left(\sum_{m=1}^{\infty}\frac{z^m}{m!}\right)^k\\&=\sum_{n=k}^{\infty}B_{n,k}(1!,2!,3!,\dots)\frac{z^n}{n!},\quad k\geq 0.\end{aligned}$$

- We start from the exponential generating function of the partial Bell polynomials (4.1):

$$\sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots)\frac{z^n}{n!} = \frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right)^k.$$

We put $x_m = i!$, substituting into the generating function, we obtain:

$$\begin{aligned}\sum_{n=k}^{\infty} B_{n,k}(1!, 2!, 3!, \dots)\frac{z^n}{n!} &= \frac{1}{k!}\left(\sum_{m=1}^{\infty} m!\frac{z^m}{m!}\right)^k \\ &= \frac{1}{k!}\left(\sum_{m=1}^{\infty} z^m\right)^k.\end{aligned}$$

Thus, we get:

$$\sum_{n=k}^{\infty} B_{n,k}(1!, 2!, 3!, \dots)\frac{z^n}{n!} = \frac{1}{k!}\left(\frac{z}{1-z}\right)^k.$$

On the other hand, the exponential generating function of the Lah numbers is:

$$\sum_{n=k}^{\infty} L(n,k)\frac{z^n}{n!} = \frac{1}{k!}\left(\frac{z}{1-z}\right)^k.$$

By identifying the coefficients, we conclude that:

$$B_{n,k}(i!) = B_{n,k}(1!, 2!, 3!, \dots) = L(n,k).$$

2. It is known that, by choosing an appropriate sequence $(x_n)_{n\geq 1}$, the complete Bell polynomials $B_n(x_1, x_2, \dots, x_n)$, can be expressed in terms of various classical combinatorial sequences. Some special cases are given below:

a. Bell numbers

$$B_n(1, 1, \dots) = \phi_n = \sum_{k=0}^{n} \left\{ n \atop k \right\}.$$

- Using the exponential generating function of the complete Bell polynomials (4.5):

$$\sum_{n=0}^{\infty} B_n(x_1, x_2, \dots, x_n)\,\frac{z^n}{n!} = \exp\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right),$$

We put $x_m = 1$ for all $m \geq 1$, substituting into the generating function, we obtain:

$$\sum_{n=0}^{\infty} B_n(1, 1, \dots)\frac{z^n}{n!} = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m!}\right).$$

Thus, we get:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = (e^z - 1).$$

Thus, we get:

$$\sum_{n=0}^{\infty} B_n(1, 1, \dots)\frac{z^n}{n!} = \exp(e^x - 1).$$

On the other hand, the exponential generating function of the Bell numbers of the second kind is (3.1):

$$\sum_{n=0}^{\infty} \phi_n \frac{x^n}{n!} = \exp(e^x - 1).$$

By comparing the coefficients, we get:

$$B_n(1, 1, \dots) = \phi_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

b. Complete Bell Polynomials with Alternating Factorials

$$B_n((-1)^{m-1}(m-1)!) = \begin{cases} 1, & n = 0, \\ 1, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

- Using the exponential generating function of the complete Bell polynomials (4.5):

$$\sum_{n=0}^{\infty} B_n(x_1, x_2, \dots, x_n) \frac{z^n}{n!} = \exp\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!} \right),$$

We put $x_m = (-1)^{m-1}(m-1)!$, substituting into the generating function, we obtain:

$$\sum_{n=0}^{\infty} B_n(((-1)^{m-1}(m-1)!)) \frac{z^n}{n!} = \exp\left(\sum_{m=1}^{\infty} (-1)^{m-1}(m-1)! \frac{z^m}{m!} \right).$$

Since

$$\frac{(m-1)!}{m!} = \frac{1}{m!},$$

the inner sum becomes:

$$\begin{aligned} \sum_{n=0}^{\infty} B_n((-1)^{m-1}(m-1)!) \frac{z^n}{n!} &= \exp\left(\sum_{m=1}^{\infty} (-1)^{m-1} \frac{z^m}{m!} \right) \\ &= \exp\left(\ln(1+z) \right). \end{aligned}$$

So the generating function becomes:

$$\sum_{n=0}^{\infty} B_n(((-1)^{m-1}(m-1)!)) \frac{z^n}{n!} = (1+z).$$

We now expand both sides as power series in z, writing $(1+z)$ in the form $\sum_{n=0}^{\infty} c_n \frac{z^n}{n!}$ gives:

$$(1+z) = 1 \cdot \frac{z^0}{0!} + 1 \cdot \frac{z^1}{1!} + \sum_{n=2}^{\infty} 0 \cdot \frac{z^n}{n!},$$

so the coefficients are $c_0 = 1$, $c_1 = 1$, and $c_n = 0$ for all $n \geq 2$, equating coefficients of $\frac{z^n}{n!}$ on both sides yields:

$$B_n((-1)^{m-1}(m-1)!) = \begin{cases} 1, & n = 0, \\ 1, & n = 1, \\ 0, & n \geq 2. \end{cases}$$

c. Lah numbers

$$B_n(1!,\, 2!,\, \ldots,\, n!) \;=\; \sum_{k=1}^{n} L(n,k).$$

- We start from the exponential generating function of the complete Bell polynomials (4.5):

$$\sum_{n=0}^{\infty} B_n(x_1, x_2, \ldots, x_n)\, \frac{z^n}{n!} = \exp\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right).$$

We put $x_m = i!$, substituting into the generating function, we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(1!, 2!, 3!, \ldots)\frac{z^n}{n!} &= \exp\left(\sum_{m=1}^{\infty} m! \frac{z^m}{m!}\right) \\ &= \exp\left(\sum_{m=1}^{\infty} z^m\right) \\ &= \exp\left(\frac{z}{1-z}\right).\end{aligned}$$

Thus, we use the Taylor series definition $e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!}$ with $u = \frac{z}{1-z}$

$$\exp\left(\frac{z}{1-z}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{z}{1-z}\right)^k = \sum_{k=0}^{\infty} \frac{z^k}{k!\,(1-z)^k}.$$

We expand $\frac{1}{(1-z)^k}$ via Newton's generalized binomial series, we get:

$$\frac{1}{(1-z)^k} = \sum_{j=0}^{\infty} \binom{k+j-1}{j} z^j.$$

Substituting back yields the double sum:

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(1!, 2!, 3!, \ldots)\frac{z^n}{n!} &= \exp\left(\frac{z}{1-z}\right) \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{j=0}^{\infty} \binom{k+j-1}{j} z^j \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{k+j-1}{j} \frac{z^{k+j}}{k!}.\end{aligned}$$

We set $j = n - k$ to collect all pairs (k, j) with $k + j = n$:

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(1!, 2!, 3!, \ldots)\frac{z^n}{n!} &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{k+(n-k)-1}{(n-k)} \frac{z^{k+n-k}}{k!} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \binom{n-1}{n-k} \frac{z^n}{k!}.\end{aligned}$$

By the symmetry rule $\binom{m}{r} = \binom{m}{m-r}$, setting $m = n-1$ and $r = n-k$:

$$\binom{n-1}{n-k} = \binom{n-1}{(n-1)-(n-k)} = \binom{n-1}{k-1},$$

where we used the symmetry $\binom{n-1}{n-k} = \binom{n-1}{k-1}$. Since our generating function is $\sum \frac{z^n}{n!}$, we have:

$$\begin{aligned}[z^n] &= \frac{B_n(1!, 2!, \ldots, n!)}{n!} \\ &= \sum_{k=1}^{n} \binom{n-1}{n-k} \frac{1}{k!}.\end{aligned}$$

We multiplay both sides by $n!$, we get:

$$B_n(1!, 2!, 3!, \ldots) = \sum_{k=1}^{n} \binom{n-1}{n-k} \frac{n!}{k!}.$$

Recogning the Lah numbers $L(n,k) = \binom{n-1}{k-1} \frac{n!}{k!}$, we get:

$$B_n(1!,\, 2!,\, \ldots,\, n!) \;=\; \sum_{k=1}^{n} L(n,k).$$

Theorem 4.6. *For $n \geq k \geq 1$, we have:*

$$\sum_{j=1}^{n} \binom{n}{j} x_j B_{n-j,k-1}(x_1, x_2, \ldots) = k B_{n,k}(x_1, x_2, \ldots). \tag{4.7}$$

Example 4.1.

$$\begin{aligned}B_{3,1}(a_1, a_2, a_3) &= a_3, \\ B_{3,2}(a_1, a_2) &= 3a_1 a_2, \\ B_{3,3}(a_1) &= a_1^3.\end{aligned}$$

Example 4.2.

$$\begin{aligned}B_{5,1}(a_1, a_2, a_3, a_4, a_5) &= a_5, \\ B_{5,2}(a_1, a_2, a_3, a_4) &= 5a_1 a_4 + 10 a_2 a_3, \\ B_{5,3}(a_1, a_2, a_3) &= a_1^2 a_3 + 15 a_1 a_2, \\ B_{5,4}(a_1, a_2) &= 10 a_1^3 a_2^2, \\ B_{5,5}(a_1) &= a_1^5.\end{aligned}$$

Combinatorial Interpretation

The (exponential) partial Bell polynomials admit a combinatorial interpretation related to colored set partitions. More precisely, let $a_n \in \mathbb{N}$, $n \geq 1$. The (exponential) partial Bell polynomial:

$$B_{n,k}(a_1, a_2, \ldots, a_{n-k+1})$$

represents the number of colorings of the partitions of a set with n elements into k blocks, where:

- Blocks of size 1 have a_1 colors,
- Blocks of size 2 have a_2 colors,
- Blocks of size 3 have a_3 colors, and so on.

Example 4.3. *$B_{5,2}(2,2,1,1)$ is the number of colorings of the partitions of the set $\{1,2,3,4,5\}$ into 2 blocks, where:*

- *Blocks of size 1 have 2 colors, c_1^1 and c_1^2,*
- *Blocks of size 2 have 2 colors, c_2^1 and c_2^2,*
- *Blocks of size 3 have 1 color c_3,*
- *Blocks of size 4 have 1 color c_4.*

The set $\{1,2,3,4,5\}$ has 5 partitions into two blocks of sizes 1 and 4, and 10 partitions into two blocks of sizes 2 and 3.

$$\begin{array}{l}\{1\}\cup\{2,3,4,5\},\ \{2\}\cup\{1,3,4,5\},\ \{3\}\cup\{1,2,4,5\},\{4\}\cup\{1,2,3,5\},\\ \{5\}\cup\{1,2,3,4\},\ \{1,2\}\cup\{3,4,5\},\ \{1,3\}\cup\{2,4,5\},\ \{1,4\}\cup\{2,3,5\},\\ \{1,5\}\cup\{2,3,4\},\ \{2,3\}\cup\{1,4,5\},\ \{2,4\}\cup\{1,3,5\},\ \{2,5\}\cup\{1,3,4\},\\ \{3,4\}\cup\{1,2,5\},\ \{3,5\}\cup\{1,2,4\},\ \{4,5\}\cup\{1,2,3\}.\end{array}$$

With this coloring, we obtain the 30 partitions:

$$\begin{array}{llll}
\{1\}^{c_1^1}\cup\{2,3,4,5\}^{c_4}, & \{1\}^{c_1^2}\cup\{2,3,4,5\}^{c_4}, & \{2\}^{c_1^1}\cup\{1,3,4,5\}^{c_4}, & \{2\}^{c_1^2}\cup\{1,3,4,5\}^{c_4},\\
\{3\}^{c_1^1}\cup\{1,2,4,5\}^{c_4}, & \{3\}^{c_1^2}\cup\{1,2,4,5\}^{c_4}, & \{4\}^{c_1^1}\cup\{1,2,3,5\}^{c_4}, & \{4\}^{c_1^2}\cup\{1,2,3,5\}^{c_4},\\
\{5\}^{c_1^1}\cup\{1,2,3,4\}^{c_4}, & \{5\}^{c_1^2}\cup\{1,2,3,4\}^{c_4}, & \{1,2\}^{c_2^1}\cup\{3,4,5\}^{c_3}, & \{1,2\}^{c_2^2}\cup\{3,4,5\}^{c_3},\\
\{1,3\}^{c_2^1}\cup\{2,4,5\}^{c_3}, & \{1,3\}^{c_2^2}\cup\{2,4,5\}^{c_3}, & \{1,4\}^{c_2^1}\cup\{2,3,5\}^{c_3}, & \{1,4\}^{c_2^2}\cup\{2,3,5\}^{c_3},\\
\{1,5\}^{c_2^1}\cup\{2,3,4\}^{c_3}, & \{1,5\}^{c_2^2}\cup\{2,3,4\}^{c_3}, & \{2,3\}^{c_2^1}\cup\{1,4,5\}^{c_3}, & \{2,3\}^{c_2^2}\cup\{1,4,5\}^{c_3},\\
\{2,4\}^{c_2^1}\cup\{1,3,5\}^{c_3}, & \{2,4\}^{c_2^2}\cup\{1,3,5\}^{c_3}, & \{2,5\}^{c_2^1}\cup\{1,3,4\}^{c_3}, & \{2,5\}^{c_2^2}\cup\{1,3,4\}^{c_3},\\
\{3,4\}^{c_2^1}\cup\{1,2,5\}^{c_3}, & \{3,4\}^{c_2^2}\cup\{1,2,5\}^{c_3}, & \{3,5\}^{c_2^1}\cup\{1,2,4\}^{c_3}, & \{3,5\}^{c_2^2}\cup\{1,2,4\}^{c_3},\\
\{4,5\}^{c_2^1}\cup\{1,2,3\}^{c_3}, & \{4,5\}^{c_2^2}\cup\{1,2,3\}^{c_3}. & &
\end{array}$$

Hence,

$$B_{5,2}(2,2,1,1)=30.$$

Conclusion

Conclusion

In this thesis, we have studied several important families of special polynomials, namely Bell polynomials and their generalizations, including p-Bell, poly-Bell, and r-Bell polynomials, as well as partial and complete Bell polynomials. For each family, we established their generating functions, explicit formulas, and recurrence relations, and highlighted their deep connections with Stirling numbers and binomial coefficients.

The results established throughout this work pave the way for future investigations, whether by considering further generalizations of the families studied here, or by exploring deeper connections with Stirling numbers, hypergeometric functions, and their potential applications in combinatorics and mathematical analysis.

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