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Entitled

ELLIPTIC AND PARABOLIC EQUATIONS WITH L^1 DATA

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CONTENTS

Acknowledgements **1**

Dedication **2**

Notations **3**

Introduction **5**

Abstract **6**

1 Models with L^1 Data **7**

1.1 Flow in Porous Media 7

1.2 Coupling of Electrical and Thermal Conductors 7

1.3 Heated Fluid 8

1.4 Turbulence Model in Climatology (k, ε Model) 9

1.5 Feline Leukemia Virus (FLV) Model 9

2 Preliminaries **11**

2.1 Weak Convergence, Weak-$\star$ Convergence 11

2.2 $L^p(\Omega)$ Spaces 12

2.3 Compact Applications 18

2.4 Fixed Point Theorems 19

2.5 Continuous Injections 20

2.6 Sobolev Spaces 21

3 The Laplacian with L^1 Data **25**

3.1 Variational Framework 25

3.2 Existence of Solutions in a Non-Variational Framework 27

3.3 Uniqueness 35

4 Elliptic equations in divergence form **40**

4.1 Entropy solutions and Uniqueness 42

5 Parabolic equations with L^1 Data **54**

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DEDICATION

To the source of safety from which I draw my strength.

To the one who was my support in life, to my late mother TOUMAOUI FERIHA.

To my brothers and sisters HAYETE and SAFIA Yanis

To my friende WAHIBA BOUAYEDE.

NOTATIONS

$\mathbb{R}^N$	euclidean space of dimension , n ≥ 1.
$\mathbb{R}^{m\times N}$	the space of real matrices.
Ω	an open bounded set of $\mathbb{R}^N$.
$\partial\Omega$	topological boundary of Ω.
$\|\Omega\|$	Lebesgue measure of the set Ω.
$x=(x_1,\cdots,x_N)$	generic point of $\mathbb{R}^N$.
$\mathrm{d}x=\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_N$	Lebesgue measure on Ω.
$\overline{\Omega}$	the closure of the set Ω.
$\rightharpoonup$	weak convergence.
$\longrightarrow$	strong convergence.
$\hookrightarrow$	Compact injection.
V	a Banach space.
$\langle\cdot,\cdot\rangle_{V'\times V}$	the duality bracket between the space V and its topological dual V'.
$\|.\|_V$	the norm on the Banach space V.
$[\cdot]$	designates the semi-norm.
$(\cdot,\cdot)$	the inner product on the Hilbert space H.
$\|.\|$	the norm on the Hilbert space H.
$\mathscr{C}^0(\mathbb{R})$	the space of continuous functions.
∂_i	The i-th partial derivative; $\partial_i=\frac{\partial}{\partial x_i}$.
∂^α	The derivative of order α; $\partial^\alpha=\frac{\partial^{\|\alpha\|}}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}$.
$\nabla u=\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\cdots,\frac{\partial u}{\partial x_N}\right)$	gradient of $v:\Omega\longrightarrow\mathbb{R}$.
Δu	The Laplacian of u.

$\operatorname{div} v$	The divergence of the vector $v = (v_1, ..., v_N)$. $\operatorname{div} v = \frac{\partial v_1}{\partial x_1} + ... + \frac{\partial v_N}{\partial x_N}$.
$[s]$	denotes the integer part of s.
a.e.	almost everywhere.
p'	The conjugate exponent of p, That is to say $p' = \frac{p}{p-1}$.

INTRODUCTION

Partial differential equations (PDEs) play a central role in modeling a wide range of physical, biological, and engineering phenomena. In particular, elliptic and parabolic equations arise in the study of steady-state and time-dependent models, respectively. The classical theory of such equations is well-developed under the assumption of sufficiently smooth or well-behaved data, typically belonging to function spaces like L^p with $p > 1$ is large enough.

However, in many real-world phenomena, the data may be merely integrable, i.e., belonging to L^1, or even measure-valued, which poses significant analytical challenges, as standard techniques and solution concepts may fail or require substantial modification. In this context, the notion of weak, entropy, or renormalized solutions becomes essential.

This dissertation investigates some elliptic and parabolic equations with L^1 data, focusing on the existence, uniqueness, and qualitative properties of solutions in this less regular setting. We explore the functional analytic tools and approximation techniques required to treat such problems and highlight the key differences between the L^1 framework and the classical theory with smoother data.

ABSTRACT

This dissertation addresses the study of elliptic and parabolic partial differential equations with data in the space $L^1(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain. Unlike the classical setting, which typically assumes sufficiently smooth, the L^1 framework poses significant analytical challenges due to the absence of compactness and regularity.

We investigate the existence, uniqueness, and regularity of solutions in the weak sense. The analysis requires a suitable functional framework for L^1 data, involving tools from the theory of Sobolev spaces, truncation techniques, and a priori estimates. These methods allow us to extend classical results and provide a rigorous foundation for studying PDEs under minimal regularity assumptions.

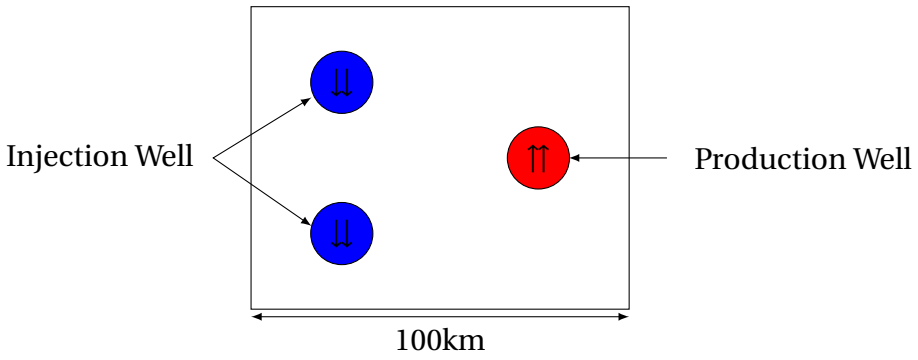
CHAPTER 1

Models with L^1 Data

We will present some problems that arise in various applications:

1.1 Flow in Porous Media

In numerical simulations of petroleum reservoirs, it is not possible to discretize points. Indeed, a petroleum reservoir is on the order of $100 \times 30 km^2$, while a well is on the order of 10 to 30 cm.



The wells thus appear as source terms in the form of Dirac masses. The pressure equation is:

$$-\triangle p = q\delta_0, \tag{1.1}$$

where δ_0 is the Dirac mass at 0 (a measure) representing the well, p is the pressure, and q is the well flow rate. This is a problem with measure data.

1.2 Coupling of Electrical and Thermal Conductors

When passing a current through a material that is both electrically and thermally conductive, we obtain a coupled system between temperature and electric potential. One of the

coupling terms arises from the Joule effect, namely the heat production by an electric current. The equations for temperature T and electric potential ϕ are written as:

$$\begin{cases} -\mathrm{div}\big(\sigma(x,T(x))\nabla\phi(x)\big)=0 \\ -\mathrm{div}(\lambda(x,T(x))\nabla T(x))=\sigma(x,T(x))|\nabla\phi|^2 \end{cases} \tag{1.2}$$

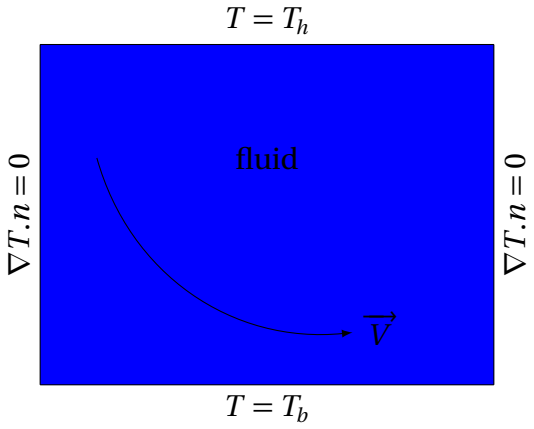
where λ and σ are the electrical and thermal conductivities, respectively.

Since we seek $\phi \in H_0^1(\Omega)$, then $\nabla\phi \in L^2(\Omega)$ and $|\nabla\phi|^2 \in L^1(\Omega)$. Thus, the temperature equation is a problem with L^1 data.

1.3 Heated Fluid

Consider a heated fluid; the energy (or temperature) equation coupled with the Navier-Stokes equations is written as:

$$\begin{cases} -\partial_t T(t,x)-\triangle T(t,x)+V.\nabla T(t,x)=|\nabla V+{}^t\nabla V|^2 \\ \mathrm{div}V=0; \quad \forall\ V\in H^1((0,\tau)\times\Omega) \end{cases} \tag{1.3}$$



where V is the given velocity (from the Navier-Stokes equations).

We have $V \in H^1$, so $\nabla V \in L^2$ and $|\nabla V+{}^t\nabla V|^2 \in L^1$. Thus, problem (1.3) is a parabolic problem with L^1 data.

1.4 Turbulence Model in Climatology (k, ε Model)

Consider an incompressible fluid; to estimate the velocity field $u(t,x)$, a widely used turbulence model consists of calculating the fluctuation relative to a mean velocity field $\overline{u}(t,x)$:

$$u(t,x) = \overline{u}(t,x) + u'(t,x),$$

where

$\overline{u}$: mean field (state over several days)

u': fluctuation

Let $k = |\overline{u'}|^2$ be the mean field of fluctuations. For small Reynolds numbers, the following model can be used as a simplification of the more general (k,ε) models:

$$-\partial_t k - \overline{u}.\nabla_x k - \mathrm{div}_x((\nu+\nu_t)\nabla_x k) + k^{\frac{3}{2}} = \nu_t |\nabla_x \overline{u} +^t \nabla_x \overline{u}|^2 \tag{1.4}$$

where ν and ν_t represent the fluid viscosity. If $\overline{u}'$ is given in $H^1((0,\tau)\times\Omega)$, then $|\nabla_x \overline{u} +^t \nabla_x \overline{u}|^2 \in L^1$. Thus, problem (1.4) is a parabolic problem with L^1 data.

1.5 Feline Leukemia Virus (FLV) Model

Consider a population (of cats) contaminated by a virus. The total population $p(t,x)$ at time t and point $x \in \Omega$ is structured by health status into three classes:

$u(t,x)$: healthy or susceptible individuals (number of individuals per m^2).

$v(t,x)$: infected individuals, i.e., those transmitting the disease to healthy individuals.

$w(t,x)$: immunized individuals, a part of the healthy population that cannot be infected.

The demographic and epidemiological parameters are denoted as:

b: fertility rate (birth rate).

m: natural mortality rate.

α: additional mortality rate due to infection.

We assume that infectious contacts depend on the social behavior of individuals. The FLV

model with spatial structure in an open set Ω of $\mathbb{R}^N$ is written as:

$$\begin{cases} -\partial_t u - \operatorname{div}(A_1 \nabla u + K_1 u) + r_1(u,v,w) = b(u+w) - m(t,x)u - \sigma(u,v,w) + f \\ -\partial_t v - \operatorname{div}(A_2 \nabla v + K_2 v) + r_2(u,v,w) = \pi\sigma(u,v,w) - (m+\alpha)v + g \\ -\partial_t w - \operatorname{div}(A_3 \nabla v + K_3 v) + r_3(t,x,u,v,w) = (1-\pi)\sigma(u,v,w) - m\,w + h; \end{cases} \tag{1.5}$$

where

$A_i(t,x)$:	diffusion matrix.
K_i:	transport vector.
$b(t,x)(u+w)$:	total birth rate of healthy or immunized individuals, considered healthy.
$m(t,x)z$:	$z = u, v, w$, mortality term.
$r_i(t,x,u,v,w)$:	self-regulation term.
$\sigma(t,x,u,v,w)$:	incidence function modeling the transition from one class of individuals to another, i.e., the number of new infections per unit time.
$\alpha(t,x)$:	mortality rate due to infection, present only in the equation for v.
f,g,h:	are data representing external input for each class.

It is natural here to consider the density of each class, i.e.,

$$\int_\Omega u(t,x)dx, \quad \int_\Omega v(t,x)dx \text{ and } \int_\Omega w(t,x)dx,$$

and thus to consider the initial data as densities, i.e., measurable functions:

$$\begin{aligned} u(0,x) &= u_0(x) \in L^1(\Omega), \\ v(0,x) &= v_0(x) \in L^1(\Omega), \\ w(0,x) &= w_0(x) \in L^1(\Omega). \end{aligned}$$

Also, the external inputs are measurable.

CHAPTER 2

Preliminaries

2.1 Weak Convergence, Weak-$\star$ Convergence

Definition 2.1 *Let E be a Banach space with dual E'.*

A sequence $(u_n)_n \in E$ is said to converge weakly to $u \in E$ if

$$\forall f \in E',\ f(u_n) = \langle f, u_n \rangle_{E',E} \longrightarrow \langle f, u \rangle_{E',E} = f(u).$$

A sequence $(f_n)_n \in E'$ converges weak-$\star$ to $f \in E'$ if

$$\forall u \in E,\ f_n(u) = \langle f_n, u \rangle_{E',E} \longrightarrow \langle f, u \rangle_{E',E} = f(u).$$

Theorem 2.1 .

- *Let E be a reflexive Banach space (i.e., E is isomorphic to E''). Then, from any bounded sequence in E, we can extract a subsequence that converges weakly in E.*
- *Let E be a separable Banach space (i.e., it admits a countable dense subset). Then, from any bounded sequence in E', we can extract a subsequence that converges weak-$\star$ in E'.*
- *Lower semi-continuity of the norm for the weak and weak-$\star$ topologies.*

  Let E be a Banach space and $(u_n)_n \in E$ such that $u_n \rightharpoonup u$ (weakly). Then $(u_n)_n$ is bounded in E, and we have

  $$\liminf_{n \to +\infty} \|u_n\|_E \geq \|u\|_E.$$

  Similarly, if $(u_n)_n \in E'$ such that $u_n \rightharpoonup u$ (weak-$\star$), then $(u_n)_n$ is bounded in E', and we

have

$$\liminf_{n\to+\infty}\|u_n\|_{E'} \geq \|u\|_{E'}.$$

2.2 $L^p(\Omega)$ Spaces

For all $1 \leq p \leq +\infty$, the conjugate exponent p' of p is defined by:

$$\frac{1}{p}+\frac{1}{p'}=1.$$

For $1 \leq p < +\infty$, we define:

$$L^p(\Omega)=\left\{ u:\Omega\to\mathbb{R};\ \int_\Omega |u(x)|^p < \infty \right\}.$$

And for $p=+\infty$:

$$L^\infty(\Omega)=\left\{ u:\Omega\to\mathbb{R};\ \sup_{x\in\Omega} ess|u(x)| < \infty \right\}.$$

2.2.1 Some properties of $L^p(\Omega)$ spaces

1) For $1 < p < +\infty$, the space $L^p(\Omega)$ is separable and reflexive, with dual $L^{p'}(\Omega)$. We then have the following result:

   Let $(u_n)_n$ be a bounded sequence in $L^p(\Omega)$. Then:

$$\exists (u_{n_k})_k,\ \exists u \in L^p(\Omega) \text{ such that } \forall \varphi \in L^{p'}(\Omega),\ \int_\Omega u_{n_k}\varphi\, dx \underset{k\to+\infty}{\longrightarrow} \int_\Omega u\varphi\, dx.$$

2) $L^1(\Omega)$ is separable but not reflexive, with dual $L^\infty(\Omega)$. We then have the following result:

   Let $(u_n)_n$ be a bounded sequence in $L^\infty(\Omega)=\left(L^1(\Omega)\right)'$. Then:

$$\exists (u_{n_k})_k,\ \exists u \in L^\infty(\Omega) \text{ such that } \forall \varphi \in L^1(\Omega),\ \int_\Omega u_{n_k}\varphi\, dx \underset{k\to+\infty}{\longrightarrow} \int_\Omega u\varphi\, dx.$$

3) $L^\infty(\Omega)$ is neither separable nor reflexive, and its dual $\left(L^1(\Omega)\right)'$ strictly contains $L^1(\Omega)$.

Nothing can be said about the convergence of a bounded sequence in $L^1(\Omega)$.

4) $D(\Omega)$ is dense in $L^p(\Omega)$, for $1 \leq p < +\infty$.

Here, $D(\Omega)$ is the set of $\mathscr{C}^\infty(\Omega)$ functions with compact support in Ω.

5) Young's Inequality:

Let $(x_i)_{i=1,n} \geq 0$ and $(p_i)_{i=1,n} \geq 0$ with:

$$\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_n} = 1,$$

then:

$$x_1 x_2 ... x_n \leq \frac{x_1^{p_1}}{p_1} + \frac{x_2^{p_2}}{p_2} + ... + \frac{x_n^{p_n}}{p_n}.$$

In particular, for $x \geq 0$, $y \geq 0$, and $\frac{1}{p} + \frac{1}{q} = 1$:

$$x\,y \leq \frac{1}{p}\varepsilon^p x^p + \frac{1}{q}\frac{y^q}{\varepsilon^q},\ \forall \varepsilon > 0.$$

6) Hölder's Inequality

Let $f_i \in L^{p_i}(\Omega)$, for $i = 1, n$. Then:

$$\|f_1 f_2 ... f_n\|_{L^r(\Omega)} \leq \|f_1\|_{L^{p_1}(\Omega)} \|f_2\|_{L^{p_2}(\Omega)} ... \|f_n\|_{L^{p_n}(\Omega)},$$

where:

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_n}.$$

7) Interpolation Between $L^p(\Omega)$ Spaces

Let Ω be an open set in $\mathbb{R}^d$ and $u \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \leq p, q \leq +\infty$. Then, for all r such that:

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},\ 0 \leq \theta \leq 1,$$

we have $u \in L^r(\Omega)$ and:

$$\|u\|_{L^r(\Omega)} \leq \|u\|^{\theta}_{L^p(\Omega)} \|u\|^{1-\theta}_{L^q(\Omega)}. \tag{2.1}$$

Indeed:

$$\int_\Omega |u|^r \, dx = \int_\Omega |u|^{r\theta} |u|^{r(1-\theta)} dx,$$

and:

$$1 = \frac{\theta r}{p} + \frac{(1-\theta)r}{q},$$

Applying Hölder's inequality, we obtain:

$$\int_\Omega |u|^r \, dx \leq \left(\int_\Omega |u|^{r\theta \frac{p}{\theta r}} dx \right)^{\frac{\theta r}{p}} \left(\int_\Omega |u|^{r(1-\theta)\frac{q}{r(1-\theta)}} dx \right)^{\frac{r(1-\theta)}{q}},$$

which is equivalent to (2.1).

8) Strong-Weak Convergence in $L^p(\Omega)$ Spaces

Let Ω be an open set in $\mathbb{R}^d$ and $u \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \leq p, q \leq +\infty$. Let $(u_n)_n$ be a sequence such that:

$$u_n \longrightarrow u \text{ strongly in } L^p(\Omega),$$
$$u_n \rightharpoonup u \text{ weakly in } L^q(\Omega) \text{ (or weak-}\star \text{ if } q = +\infty).$$

Then, for all r such that $p < r < q$, we have:

$$u_n \longrightarrow u \text{ strongly in } L^r(\Omega).$$

Indeed, if $p < r < q$, there exists $0 \leq \theta \leq 1$ such that $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$, and thus:

$$\|u_n - u\|_{L^r(\Omega)} \leq \|u_n - u\|^{\theta}_{L^p(\Omega)} \|u_n - u\|^{1-\theta}_{L^q(\Omega)},$$

Since $(u_n)_n$ converges weakly in $L^q(\Omega)$, it is bounded in $L^q(\Omega)$. Thus, $\|u_n - u\|^{1-\theta}_{L^q(\Omega)} \leq C$ independently of n. And since $(u_n)_n$ converges strongly to u in $L^p(\Omega)$, the result follows.

9) Strong-Weak Product Convergence in $L^p(\Omega)$ Spaces

Let $p, q, r \in [0; +\infty[$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. If:

$$u_n \longrightarrow u \text{ strongly in } L^p(\Omega),$$
$$v_n \rightharpoonup v \text{ weakly in } L^q(\Omega),$$

then:

$$u_n v_n \longrightarrow u v \text{ weakly in } L^r(\Omega).$$

Indeed, let $\varphi \in L^{r'}(\Omega)$ with $\frac{1}{r}+\frac{1}{r'}=1$. Then:

$$\int_\Omega (u_n v_n - u v)\varphi \, dx = \int_\Omega (u_n - u) v_n \varphi \, dx + \int_\Omega (v_n - v) u \varphi \, dx,$$

The sequence $(v_n)_n$ converges weakly and is thus bounded. Since $u_n \longrightarrow u$ strongly in $L^p(\Omega)$, the first integral converges to 0. The second integral also converges to 0 by the definition of weak convergence.

2.2.2 Convergence in $L^p(\Omega)$

Lemma 2.1 *(Fatou's Lemma) Let Ω be an open set and $(f_n)_n$ a sequence of measurable functions with values in $[0;+\infty]$. Then:*

$$\int_\Omega \liminf_{n\to+\infty} f_n \, dx \leq \liminf_{n\to+\infty} \int_\Omega f_n \, dx.$$

Definition 2.2 *A sequence $(h_n)_n$ of measurable functions on an open set Ω of $\mathbb{R}^d$ is said to be* ***equi-integrable*** *if and only if:*

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0,\ \forall E \subset \Omega;\ |E| < \delta, \text{ then } \int_E |h_n| dx \leq \varepsilon,\ \ \forall n.$$

Theorem 2.2 *(Egorov) Let $(h_n)_n$ be a sequence of measurable functions on a bounded open set Ω of $\mathbb{R}^d$ such that:*

$$h_n \to h \text{ a.e. on } \Omega.$$

Then:

$$\forall \varepsilon > 0,\ \exists K \text{ compact} \subset \Omega \text{ such that } |\Omega \setminus K| \leq \varepsilon \text{ and } \lim_{n\to+\infty} \int_K |h_n - h| dx = 0.$$

Theorem 2.3 *(Vitali) Let Ω be a bounded open set of $\mathbb{R}^d$ and let $(h_n)_n$ be an equi-integrable*

sequence such that $h_n \to h$ a.e.. Then:

$$h_n \to h \text{ strongly in } L^1(\Omega).$$

Proof. We have the following decomposition:

$$\int_\Omega |h_n - h| dx = \int_K |h_n - h| dx + \int_{\Omega \setminus K} |h_n - h| dx.$$

Using Egorov's theorem,

$$\forall \delta > 0,\ \exists K \text{ compact} \subset \Omega \text{ such that } |\Omega \setminus K| \leq \delta \text{ and } \lim_{n \to +\infty} \int_K |h_n - h| dx = 0.$$

By equi-integrability, we know that:

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0,\ \forall E \subset \Omega;\ |E| < \delta, \text{ then } \int_E |h_n| dx \leq \frac{\varepsilon}{3} \text{ and } \int_E |h_n| dx \leq \frac{\varepsilon}{3}.$$

For $\forall \varepsilon > 0$ and $\delta(\varepsilon) > 0$ chosen above, we have:

$$\int_\Omega |h_n - h| dx = \int_K |h_n - h| dx + \frac{2\varepsilon}{3},$$

and $\exists N \in \mathbb{N},\ \forall n \geq N,\ \int_\Omega |h_n - h| dx \leq \frac{\varepsilon}{3}$. Thus:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ \int_\Omega |h_n - h| dx \leq \varepsilon$$

■

Theorem 2.4 (Lebesgue's Dominated Convergence Theorem) *Let Ω be a bounded open set of $\mathbb{R}^d$ and $(f_n)_n$ a sequence of measurable functions that converge a.e. to f. Suppose there exists a sequence $(g_n)_n$ satisfying:*

- $\forall n \in \mathbb{N}, |f_n(x)| \leq g_n(x)$ *a.e. on* Ω,
- $g_n \to g$ *in* $L^1(\Omega)$.

then $f_n \to f$ *in* $L^1(\Omega)$.

Proof. We apply Vitali's theorem. To do this, we show that $(f_n)_n$ is equi-integrable. From the strong convergence of $(g_n)_n$, we have:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ \int_\Omega |g_n - g| dx \leq \frac{\varepsilon}{2}.$$

The function g is equi-integrable, so:

$$\forall \varepsilon > 0,\ \exists \sigma > 0,\ \forall E \subset \Omega;\ |E| < \delta,\ \text{ then } \int_E |g| dx \leq \frac{\varepsilon}{2},$$

By the hypothesis, we have:

$$\int_E |f_n| dx \leq \int_E |g_n| dx \leq \int_E |g_n - g| dx + \int_E |g| dx,$$

hence:

$$\forall \varepsilon > 0,\ \exists \sigma > 0, \exists N,\ \forall E \subset \Omega;\ |E| < \delta \text{ and } n \geq N \text{ then } \int_E |f_n| dx \leq \varepsilon.$$

Moreover, the family $\{(f_n)_{n<N}\}$ is finite, so the sequence $(f_n)_n$ is equi-integrable. ■

Example 1 *Let* $q : \mathbb{R} \longrightarrow \mathbb{R}_+$ *be a continuous and sublinear application, i.e.,*

$$\forall x \in \mathbb{R};\ q(x) \leq \alpha |x| + \beta.$$

Let Ω *be a bounded open set and* $(v_n)_n$ *a sequence of functions in* $L^1(\Omega)$ *such that* $v_n \longrightarrow v$ *in* $L^1(\Omega)$ *and a.e. Then* $q(v_n) \longrightarrow q(v)$ *in* $L^1(\Omega)$.

Theorem 2.5 (Partial Converse of the Dominated Convergence Theorem) *Let* $p \in [1, +\infty[$ *and* $(f_n)_n$ *a sequence of functions in* $L^p(\Omega)$ *converging strongly to* f *in* $L^p(\Omega)$. *Then there exists a subsequence* $(f_{n_k})_k$ *such that:*

- $f_{n_k} \to f$ *a.e. on* Ω,
- $|f_{n_k}(x)| \leq g(x)$, $\forall k$, *a.e. on* Ω, *with* $g \in L^p(\Omega)$.

2.3 Compact Applications

Definition 2.3 *Let E and F be two Banach spaces.*

An application $S : E \longrightarrow F$ is said to be compact if the image of any bounded set in E by S is a relatively compact set in F, i.e., a set whose closure is compact in F.

This result is used in the following form:

Let $(u_n)_n$ be a bounded sequence in E. If S is compact, then there exists a subsequence $(u_{n_k})_k$ such that $(Su_{n_k})_k$ converges in F. However, it is not known whether Su_{n_k} converges to Su. This limit can be identified in the case of linear and compact applications.

Example 2 *Let $(u_n)_n$ be a sequence in E converging weakly to u in E, and S a linear and compact application from E to F. Then, Su_{n_k} converges strongly to Su in F.*

To show this, first note that the sequence $(Su_{n_k})_k$ converges weakly to Su in F. Indeed, $\forall f \in F'$, we have:

$$\langle f, Su_n \rangle_{F',F} = \langle {}^tSf, u_n \rangle_{E',E}.$$

By the definition of the adjoint application ($S : E \longrightarrow F$ and ${}^tS : F' \longrightarrow E'$), then:

$$\langle f, Su_n \rangle_{F',F} = \langle {}^tSf, u_n \rangle_{E',E} \longrightarrow \langle {}^tSf, u \rangle_{E',E} = \langle f, Su \rangle_{F',F}.$$

On the other hand, $(u_n)_n$ converges weakly, so it is bounded in E, and thus $(u_n)_n$ belongs to a bounded subset B of E. Since S is compact, $\overline{S(B)}$ is compact, and there exists a subsequence $(u_{n_k})_k$ such that:

$$u_{n_k} \longrightarrow v \text{ strongly in } F.$$

In a compact set, a sequence converges if and only if it has a unique limit point. We have seen that $u_{n_k} \rightharpoonup Su$ weakly in F, and by the uniqueness of the weak limit, we have $Su = v$. Thus, Su is the unique limit point of $(Su_{n_k})_k$, which is therefore convergent.

2.4 Fixed Point Theorems

Theorem 2.6 (Banach's Fixed Point Theorem) *Let E be a Banach space and $T : E \longrightarrow E$ a strictly contractive application, i.e., there exists a constant $0 < k < 1$ such that:*

$$\forall u, v \in E,\ \|T(u) - T(v)\| \leq k\|u - v\|.$$

Then T admits a unique fixed point, $u = T(u)$.

Theorem 2.7 (Schauder's Fixed Point Theorem 1) *Let:*

- *E be a Banach space,*
- *C a convex compact subset of E,*
- *T an application from C to C (maps C into C),*
- *T is continuous.*

Then T admits at least one fixed point in C.

Theorem 2.8 (Schauder's Fixed Point Theorem 2) *Let:*

- *E be a Banach space,*
- *C a convex, closed, and bounded subset of E,*
- *T an application from C to C (maps C into C),*
- *T is continuous,*
- *T is compact.*

Then T admits at least one fixed point in C.

Theorem 2.9 (Schauder's Fixed Point 3) *Let*

- *E be a Banach space.*
- *T be an application from E to E.*
- *T is continuous.*
- *T is compact.*

- *There exists $r > 0$ such that if $x = \lambda T(x)$ with $\lambda \in]0,1[$, then $\|x\| \leq r$.*

Then T admits at least one fixed point in C.

2.5 Continuous Injections

Proposition 2.1 (Case of Banach spaces) *Let E and F be two Banach spaces. Assume that the injection from E into F is continuous and dense (i.e., the image of E is dense in F), then the following application*

$$T : f \in E' \longrightarrow T_f \in F',$$

defined by

$$\forall u \in E,\ \langle T_f, u\rangle_{E',E} = \langle f, u\rangle_{F',F},$$

is an injection from F' into E'. Moreover, if the injection from E into F is compact, then T is compact. Finally, if E is reflexive, then the image of T is dense in E'.

Proposition 2.2 (Hilbertian Case) *Let V and H be two Hilbert spaces such that the injection from V into H is continuous and dense, then the injection from H into V' defined by*

$$T : f \in H \longrightarrow T_f \in V',$$

defined by

$$\forall v \in V,\ \langle T_f, v\rangle_{V',V} = (f, v)_H,$$

is dense. Moreover, if the injection from V into H is compact, then the injection from H into V' is compact.

The space H is called the pivot space. According to the Riesz theorem, we identify H and its dual via the inner product of H. We identify $f \in H$ with its image $T_f \in V'$ and we will write

$$\langle f, v\rangle_{V',V} = (f, v)_H,\ \forall f \in H,\ \forall v \in V.$$

This result is frequently used by taking $V = H^1(\Omega)$ and $H = L^2(\Omega)$.

Theorem 2.10 (Lax-Milgram) *Let H be a Hilbert space, $a(.,.) : H \times H \to \mathbb{R}$ a bilinear form,*

and $L : H \to \mathbb{R}$ a linear form. Assume that a and L are continuous and a is coercive, that is,

$$\exists \alpha > 0,\ a(v,v) \geq \alpha \|v^2\|_H,\ \ \forall v \in H.$$

Then there exists a unique $u \in H$ such that:

$$a(u,v) = L(v),\ \forall v \in H.$$

2.6 Sobolev Spaces

Let Ω be an open subset of $\mathbb{R}^d$, $m \in \mathbb{N}$ and $1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(\Omega)$ is defined by:

$$W^{m,p}(\Omega) = \left\{ u \in \mathscr{D}'(\Omega), D^\mu u \in L^p(\Omega), \forall |\mu| \leq m \right\}$$

This space is a Banach space for the norm

$$\|u\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum\limits_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty \\ \max\limits_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{L^\infty(\Omega)}, & \text{if } p = \infty. \end{cases}$$

The space $W^{m,p}(\Omega)$ is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$.

For $p = 2$, we denote $W^{m,2}(\Omega) = H^m(\Omega)$. The space $H^m(\Omega)$ endowed with the inner product:

$$(u,v)_{H^m(\Omega)} = \sum_{|\mu| \leq m} (D^\mu u, D^\mu v)_{L^2(\Omega)},$$

is a Hilbert space.

We denote

$$W_0^{m,p}(\Omega) = \overline{\mathscr{D}(\Omega)}^{W^{m,p}(\Omega)}, \text{ for } 1 \leq p < \infty.$$

The dual of $W_0^{m,p}(\Omega)$ is denoted by $W^{-m,p'}(\Omega)$.

Definition 2.4 (Fractional Sobolev Spaces) *We can define a family of intermediate spaces*

between $L^p(\Omega)$ and $W^{1,p}(\Omega)$. More precisely, if $0 < s < 1$ ($s \in \mathbb{R}$) and $1 \leq p < +\infty$, we set

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega);\ \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{\|x-y\|^{d+sp}} dx\, dy < +\infty \right\},$$

this space endowed with the natural norm is a Banach space. We denote $H^s(\Omega) = W^{s,p}(\Omega)$.

2.6.1 Trace Theory

Theorem 2.11 (Trace Theorem) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $1 \leq p < +\infty$. The application*

$$\gamma_0 :\ u \in \mathscr{C}(\overline{\Omega}) \longrightarrow \gamma_0(u) = u_{|\partial\Omega} \in L^p(\partial\Omega),$$

is linear continuous and extends uniquely to a linear continuous application from $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$. We call the operator γ_0 thus obtained: the trace application. The operator γ_0 is not surjective from $W^{1,p}(\Omega)$ onto $L^p(\partial\Omega)$. The image of γ_0 is the fractional Sobolev space $W^{1,p-\frac{1}{p}}(\partial\Omega)$ and we have

$$\|u_{|\partial\Omega}\|_{W^{1,p-\frac{1}{p}}(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)},\ \forall u \in W^{1,p}(\Omega).$$

The kernel of the trace operator is $W_0^{1,p}(\Omega)$, that is

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega);\ \gamma_0 u = 0 \right\}.$$

Proposition 2.3 (Trace on $H^1(\Omega)$) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and let $u \in H^1(\Omega)$. Then*

1) $|u| \in H^1(\Omega)$.

2) $\gamma_0|u| = |\gamma_0 u|$.

3) If $m \leq \varphi \leq M$ a.e. on Ω, then $m \leq \gamma_0\varphi \leq M$ a.e. on $\partial\Omega$.

2.6.2 Poincaré Inequality

Theorem 2.12 *Let Ω be a bounded convex Lipschitz domain in $\mathbb{R}^d$ and $p \in [1, \infty[$. There exists a constant $C > 0$ independent of u such that:*

$$\forall u \in W_0^{1,p}(\Omega),\ \|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

As a consequence, the application $u \longmapsto \|\nabla u\|_{L^p(\Omega)}$ is a norm on $\|\nabla u\|_{L^p(\Omega)}$, equivalent to the usual norm.

Proposition 2.4 *Let Ω be a bounded convex Lipschitz domain in $\mathbb{R}^d$ with boundary Γ. Let $\Gamma_1 \subset \Gamma$ such that $mes(\Gamma_1) \neq 0$. Let*

$$H^1_{\Gamma_1}(\Omega) = \left\{ u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma_1 \right\}.$$

Then there exists a constant $C > 0$:

$$\forall u \in H^1_{\Gamma_1}(\Omega),\ \|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}.$$

2.6.3 Sobolev Injections

Theorem 2.13 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ ($N \geq 1$). Then*

- *For $1 \leq p < N$, the injection*

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega),\ \ \forall q \in [1; p^\star[,\ \frac{1}{p^\star} = \frac{1}{p} - \frac{1}{N},$$

  is compact. In particular, the injection $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact. The injection is continuous for $q \in [1; p^\star]$. The exponent $p^\star$ is called the critical Sobolev exponent.

- *For $1 \leq q < \infty$, the injection*

$$W^{1,N}(\Omega) \hookrightarrow L^p(\Omega),$$

  is compact.

- *For $p > N$, the injection*

$$W^{1,p}(\Omega) \hookrightarrow \mathscr{C}(\overline{\Omega}),$$

is compact.

Some examples of injections:

- For $N = 1$,

$$\begin{cases} W^{1,1}(\Omega) \hookrightarrow L^q(\Omega), & \forall q \in [1;+\infty[, \\ W^{1,1+\varepsilon}(\Omega) \hookrightarrow \mathscr{C}(\overline{\Omega}), & \forall \varepsilon > 0. \end{cases}$$

- For $N = 2$,

$$\begin{cases} W^{1,1}(\Omega) \hookrightarrow L^q(\Omega), & \forall q \in [1;2[, \\ H^1(\Omega) = W^{1,2}(\Omega) \hookrightarrow L^q(\Omega), & \forall q \in [1;+\infty[, \\ W^{1,2+\varepsilon}(\Omega) \hookrightarrow \mathscr{C}(\overline{\Omega}), & \forall \varepsilon > 0. \end{cases}$$

- For $N = 3$,

$$\begin{cases} W^{1,1}(\Omega) \hookrightarrow L^q(\Omega), & \forall q \in [1;\frac{3}{2}[, \\ H^1(\Omega) = W^{1,2}(\Omega) \hookrightarrow L^q(\Omega), & \forall q \in [1;6[. \end{cases}$$

Lemma 2.2 (Stampacchia) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lipschitz function such that $\varphi(0) = 0$, and $p \in]1,+\infty[$.*

1) If $u \in W_0^{1,p}(\Omega)$ then $\varphi \circ u \in W_0^{1,p}(\Omega)$.

2) If moreover φ' has a finite number of discontinuities, then for $1 \leq i \leq N$, $\frac{\partial \varphi(u)}{\partial x i} = \varphi'(u)\frac{\partial u}{\partial x i}$ almost everywhere in Ω.

CHAPTER 3

The Laplacian with L^1 Data

Let Ω be an open subset of $\mathbb{R}^N$, $(N \geq 1)$. We are interested in the following problem

$$\begin{cases} -\Delta u(x) = f(x), & \text{in } \Omega \\ u(x) = 0, & \text{on } \Omega \end{cases} \tag{3.1}$$

3.1 Variational Framework

If $f \in H^{-1}(\Omega)$, then there exists a unique solution of (3.1) according to the Lax-Milgram theorem in the following sense:

$$\begin{cases} \exists! u \in H_0^1(\Omega), \\ \int_\Omega \nabla u(x) \nabla v(x) dx = \langle f, v \rangle_{H^{-1}, H_0^1}, \ \forall v \in H_0^1(\Omega). \end{cases} \tag{3.2}$$

Remark 1 *In (3.2), we can replace $H_0^1(\Omega)$ by $\mathscr{D}(\Omega)$.*

Question: What happens if $f \in L^p(\Omega)$, $1 \leq p \leq \infty$?

We say we are in the variational framework if $H^{-1}(\Omega)$. We will distinguish according to the space dimension:

Case 1. In space dimension 1, $N = 1$. We have

$$H_0^1(\Omega) \hookrightarrow \mathscr{C}(\overline{\Omega}),$$

and by duality

$$\left(\mathscr{C}(\overline{\Omega})\right) \hookrightarrow H^{-1}(\Omega).$$

We denote $\mathscr{M}(\Omega) = \left(\mathscr{C}(\overline{\Omega})\right)'$, the set of Radon measures on Ω. Thus, in dimension 1,

for $f \in \mathscr{M}(\Omega)$ or $f \in L^1(\Omega)$ then $f \in H^{-1}(\Omega)$.

Case 3. In dimension $N \geq 3$. We have seen that:

$$H_0^1(\Omega) \hookrightarrow L^{2^\star}(\Omega),$$

where $2^\star = \frac{1}{2} - \frac{1}{N} = \frac{2N}{N-2}$, the critical Sobolev exponent.

By duality

$$\left(L^{2^\star}\right)(\Omega) \hookrightarrow H^{-1}(\Omega),$$

with $\overline{m} = (2^\star)'$ defined by

$$\begin{aligned} \tfrac{1}{\overline{m}} &= 1 - \tfrac{1}{2^\star} \\ &= \tfrac{N+2}{2N} \end{aligned}$$

thus $\overline{m} = \frac{2N}{N+2}$. Thus, we have two cases to distinguish:

- If $p \geq \overline{m} = \frac{2N}{N+2}$, we have $L^p(\Omega) \hookrightarrow H^{-1}(\Omega)$, so we are in the variational framework.
- Interesting framework if $f \in L^p(\Omega)$ with $1 \leq p \leq \overline{m}$ (or $f \in \mathscr{M}(\Omega)$).

Summary: Non-variational framework

- $N = 2$, $f \in L^1(\Omega)$.
- $N \geq 3$, $f \in L^p(\Omega)$ with $1 \leq p \leq \overline{m}$.

The consequence of this summary is that $f \in L^1(\Omega)$ is a separate study.

In the L^1 framework, what meaning can we give to "u solution of (3.1)"?

1) Give a meaning to $u = 0$ on the boundary.

   Theorem 2.11 for $p = 1$ states; if $u \in W^{1,1}(\Omega)$ then $u_{|\partial\Omega} \in L^1(\partial\Omega)$

2) Give a meaning to Δu.

   We have $\Delta u = \operatorname{div}(\nabla u)$ and we can take div in the sense of distributions provided that $\nabla u \in \left(L^1_{loc}\right)^N$.

Definition 3.1 *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$ and $f \in L^1(\Omega)$. We say that u is*

a weak solution of (3.1) if u satisfies:

$$\begin{cases} u \in W_0^{1,1}(\Omega), \\ \displaystyle\int_\Omega \nabla u(x)\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx, \ \forall \varphi \in \mathscr{D}(\Omega). \end{cases} \tag{3.3}$$

Questions: Given $f \in L^1(\Omega)$, what can we say about

- The existence of u solution of (3.3).
- The regularity of u.
- The uniqueness of u.

3.2 Existence of Solutions in a Non-Variational Framework

To prove the existence of solutions, we will use the approximation method. This method consists of approximating $f \in L^1(\Omega)$ by a sequence of more regular functions $(f_\varepsilon)_{\varepsilon>0}$. Then the idea is to obtain ε-independent estimates to pass to the limit.

Step 1: Regularized Problem

We have $\mathscr{D}(\Omega)$ is dense in $L^1(\Omega)$, that is,

$$\begin{aligned} \forall f \in L^1(\Omega), \quad & \exists (f_\varepsilon)_{\varepsilon>0} \in \mathscr{D}(\Omega) \text{ such that} \\ & f_\varepsilon \longrightarrow f \text{ strongly in } L^1(\Omega) \\ & \text{and } \|f_\varepsilon\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}, \ \forall \varepsilon > 0. \end{aligned}$$

For fixed $\varepsilon > 0$, $f_\varepsilon \in L^2(\Omega)$ but $\|f_\varepsilon\|_{L^2(\Omega)}$ depends on ε.

For fixed ε, $f_\varepsilon \in L^2(\Omega)$, we define u_ε solution of:

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega), \\ \displaystyle\int_\Omega \nabla u_\varepsilon(x)\nabla v(x)dx = \int_\Omega f_\varepsilon(x)v(x)dx, \ \forall v \in H_0^1(\Omega). \end{cases} \tag{3.4}$$

The existence of a unique solution $u_\varepsilon \in H_0^1(\Omega)$ of (3.4) is ensured by the Lax-Milgram theorem.

The difficulty now is to pass to the limit when $\varepsilon \longrightarrow 0$. A natural choice of function is to take $v(x) = u_\varepsilon(x) \in H_0^1(\Omega)$, thus we have

$$\int_\Omega |\nabla u_\varepsilon(x)|^2 dx = \int_\Omega f_\varepsilon(x) u_\varepsilon(x) dx.$$

To estimate the right-hand side and obtain an estimate on ∇u_ε, we use the Cauchy-Schwarz inequality, then the Poincaré inequality, it follows

$$\begin{aligned} \|\nabla u_\varepsilon(x)\|^2 &\leq \|f_\varepsilon\|_{L^2(\Omega)} \|u_\varepsilon\|_{L^2(\Omega)}, \\ &\leq C_p \|f_\varepsilon\|_{L^2(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)}. \end{aligned}$$

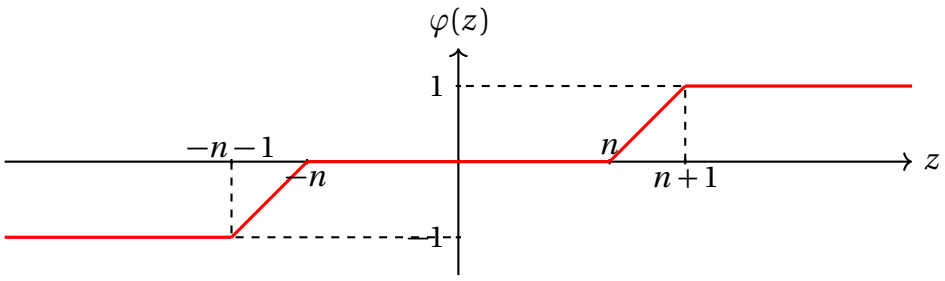
Thus,

$$\|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C_p \|f_\varepsilon\|_{L^2(\Omega)}. \tag{3.5}$$

However, $\|f_\varepsilon\|_{L^2(\Omega)}$ depends on ε and thus $\|\nabla u_\varepsilon\|_{L^2(\Omega)}$ also depends on it. This estimate, which seems useless, leads us to choose test functions v belonging to $H_0^1(\Omega) \cap L^\infty(\Omega)$ for two reasons; the first is that $v \in H_0^1(\Omega)$ in (3.4) and the second is that $v \in L^\infty(\Omega)$ to estimate the right-hand side since $\|f_\varepsilon\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$ independently of ε.

Example 3 (Wavelets) *Let φ be defined by:*

$$\varphi(z) = \begin{cases} 0, & |z| \leq n, \\ z - n, & n \leq z \leq n+1, \\ z + n, & n \leq z \leq n+1, \\ 1, & z \geq n+1, \\ -1, & z \leq -n-1. \end{cases}$$



We have φ is Lipschitz and $\varphi(0) = 0$, so by Stampacchia's lemma 2.2, if $u \in H_0^1(\Omega)$, then

$\varphi(u) \in H_0^1(\Omega)$ and $\nabla\varphi(u) = \varphi'(u)\nabla u$, thus

$$|\nabla\varphi(u(x))|^2 = |\nabla u(x)|^2 \mathbb{1}_{B_n}(x), \text{ a.e. on } \Omega,$$

with

$$B_n = \{x \in \Omega, n \leq |u(x)| \leq n+1\}.$$

On the other hand, by definition of φ, we have $|\varphi(u(x))| \leq 1$ and thus $\varphi(u) \in L^\infty(\Omega)$.

Thus, if $u \in H_0^1(\Omega)$, then $\varphi(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$. This choice of test function is then very interesting.

Lemma 3.1 (Estimate on the wavelets of u_ε solution of (3.4)) .
The solution u_ε of (3.4) satisfies:

$$\int_{B_n} |\nabla u_\varepsilon(x)|^2 dx \leq \|f\|_{L^1(\Omega)}. \tag{3.6}$$

Proof. Let $v = \varphi(u_\varepsilon) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a test function of (3.4), then

$$\begin{aligned}
\int_\Omega \nabla u_\varepsilon(x)\nabla\varphi(u_\varepsilon(x))dx &= \int_\Omega |\nabla u_\varepsilon(x)|^2 \mathbb{1}_{B_n}(x)dx, \\
&= \int_{B_n} |\nabla u_\varepsilon(x)|^2 dx, \\
&= \int_\Omega f_\varepsilon(x)\varphi(u_\varepsilon(x))dx, \\
&\leq \int_\Omega |f_\varepsilon(x)||\varphi(u_\varepsilon(x))|dx, \\
&\leq \sum_{n\geq 0}\int_\Omega |f_\varepsilon(x)|dx.
\end{aligned}$$

■

Remark 2 *We have $\Omega = \bigcup_{n\geq 0} B_n$, thus*

$$\begin{aligned}
\int_\Omega |\nabla u_\varepsilon(x)|^2 dx &= \sum_{n\geq 0}\int_{B_n} |\nabla u_\varepsilon(x)|^2 dx, \\
&\leq \sum_{n\geq 0}\int_\Omega |f_\varepsilon(x)|dx = +\infty.
\end{aligned}$$

The estimate on the wavelets B_n of the solution u_ε does not give an estimate in $H_0^1(\Omega)$.

Lemma 3.2 (Estimate on the solution u_ε in Ω independently of ε) .
Let $N \geq 2$ and let $(u_\varepsilon)_\varepsilon \in H_0^1(\Omega)$ satisfy

$$\int_{B_n} |\nabla u_\varepsilon(x)|^2 dx \leq a, \ \forall n \in \mathbb{N},$$

where a is a constant. Then, for all $1 \leq q < \frac{N}{N-1}$, there exists a constant $C(q,a)$ such that

$$\int_\Omega |\nabla u_\varepsilon(x)|^q dx \leq C(q,a).$$

This lemma is a fundamental result in L^1 theory as it gives an estimate on the solution independently of ε. This result shows that the sequence $(u_\varepsilon)_\varepsilon$ is bounded in $W_0^{1,q}(\Omega)$ for $1 \leq q < \frac{N}{N-1}$.

Proof. Since the solution is not bounded in $H_0^1(\Omega)$ independently of ε and $u_\varepsilon \in H_0^1(\Omega) = W_0^{1,2}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$ for $q < 2$, we seek to bound $(u_\varepsilon)_\varepsilon$ in $W_0^{1,q}(\Omega)$ with $q < 2$.

Let $1 \leq q < 2$, for $\frac{1}{\frac{2}{q}} + \frac{1}{q'} = 1$, i.e., $\frac{1}{q'} = 1 - \frac{2}{q}$, we apply Hölder's inequality to obtain

$$\begin{aligned}
\int_{B_n} |\nabla u_\varepsilon|^q dx &\leq \left(\int_{B_n} |\nabla u_\varepsilon|^{q \cdot \frac{2}{q}} dx \right)^{\frac{q}{2}} \left(\int_{B_n} (1)^{q'} dx \right)^{1-\frac{q}{2}}, \\
&\leq \left(\int_{B_n} |\nabla u_\varepsilon|^2 dx \right)^{\frac{q}{2}} (mes(B_n))^{1-\frac{q}{2}}, \\
&\leq a^{\frac{q}{2}} (mes(B_n))^{1-\frac{q}{2}}.
\end{aligned}$$

On the other hand, let n_0 be a strictly positive integer (to be chosen later), we have

$$\begin{aligned}
\int_\Omega |\nabla u_\varepsilon|^q dx &= \sum_{n \geq 0} \int_{B_n} |\nabla u_\varepsilon|^q dx, \\
&= \sum_{n=0}^{n_0} \int_{B_n} |\nabla u_\varepsilon|^q dx + \sum_{n=n_0+1}^{\infty} \int_{B_n} |\nabla u_\varepsilon|^q dx.
\end{aligned}$$

and we have

$$\begin{aligned}
\int_{B_n} |\nabla u_\varepsilon|^q dx &\leq a^{\frac{q}{2}} |B_n|^{1-\frac{q}{2}}, \\
&\leq a^{\frac{q}{2}} |\Omega|^{1-\frac{q}{2}},
\end{aligned}$$

and thus finally,

$$\int_\Omega |\nabla u_\varepsilon|^q \, dx \leq (n_0+1) a^{\frac{q}{2}} |\Omega|^{1-\frac{q}{2}} + \sum_{n=n_0+1}^{\infty} a^{\frac{q}{2}} |B_n|^{1-\frac{q}{2}}.$$

We denote $c = a^{\frac{q}{2}}$, $C_{n_0} = (n_0+1) a^{\frac{q}{2}} |\Omega|^{1-\frac{q}{2}}$ and we have $C_{n_0} \underset{n_0 \to +\infty}{\longrightarrow} +\infty$

thus

$$\int_\Omega |\nabla u_\varepsilon|^q \, dx \leq C_{n_0} + c \sum_{n=n_0+1}^{\infty} |B_n|^{1-\frac{q}{2}}. \tag{3.7}$$

On one hand, the injection $H_0^1(\Omega) = W_0^{1,2}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$, $\forall q < 2$ with continuous injection.

On the other hand, the Sobolev injection $W_0^{1,q}(\Omega) \hookrightarrow L^{q^\star}(\Omega)$ (continuous injection) with $q^\star$ is the Sobolev exponent, i.e., $\frac{1}{q^\star} = \frac{1}{q} - \frac{1}{N}$ ($q^\star = \frac{qN}{N-q}$) and $q < 2 \leq N$.

Thus, there exists a constant C_Ω depending on Ω such that

$$C_\Omega \left(\int_\Omega |u_\varepsilon|^{q^\star} \, dx \right)^{\frac{q}{q^\star}} \leq C_{n_0} + c \sum_{n=n_0+1}^{\infty} |B_n|^{1-\frac{q}{2}}. \tag{3.8}$$

We now estimate $|B_n|$, we have $\int_{B_n} |u_\varepsilon|^{q^\star} \, dx \geq n^{q^\star} |B_n|$ because $|u_\varepsilon| \geq n$ on B_n and thus

$$|B_n| \leq \frac{1}{n^{q^\star}} \int_{B_n} |u_\varepsilon|^{q^\star} \, dx.$$

The estimate (3.8) then gives

$$C_\Omega \left(\int_\Omega |u_\varepsilon|^{q^\star} \, dx \right)^{\frac{q}{q^\star}} \leq C_{n_0} + c \sum_{n=n_0+1}^{\infty} \frac{1}{n^{q^\star(1-\frac{q}{2})}} \left(\int_{B_n} |u_\varepsilon|^{q^\star} \, dx \right)^{1-\frac{q}{2}}.$$

The discrete Cauchy-Schwarz inequality is written:

$$\sum_i x_i y_i \leq \left(\sum_i x_i^\alpha \right)^{\frac{1}{\alpha}} \left(\sum_i y_i^\beta \right)^{\frac{1}{\beta}}, \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

To be able to write an integral over Ω, we choose

$$\alpha = \frac{1}{1-\frac{q}{2}} = \frac{2}{2-q} \text{ thus, } \frac{1}{\alpha} = 1-\frac{q}{2}$$

$$\frac{1}{\beta} = 1-\frac{1}{\alpha} = \frac{q}{2},$$

thus

$$\begin{aligned} C_\Omega \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{q}{q^\star}} &\leq C_{n_0} + c \left(\sum_{n=n_0+1}^{\infty} \frac{1}{n^{q^\star(1-\frac{q}{2})\frac{2}{q}}} \right)^{\frac{q}{2}} \left(\sum_{n=n_0+1}^{\infty} \left(\int_{B_n} |u_\varepsilon|^{q^\star} dx \right)^{(1-\frac{q}{2})\frac{1}{1-\frac{q}{2}}} \right)^{\frac{2-q}{2}}, \\ &\leq C_{n_0} + c \left(\sum_{n=n_0+1}^{\infty} \frac{1}{n^{q^\star(\frac{2-q}{q})}} \right)^{\frac{q}{2}} \left(\sum_{n=n_0+1}^{\infty} \int_{B_n} |u_\varepsilon|^{q^\star} dx \right)^{\frac{2-q}{2}}, \end{aligned}$$

thus

$$C_\Omega \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{q}{q^\star}} \leq C_{n_0} + c \left(\sum_{n=n_0+1}^{\infty} \frac{1}{n^{q^\star(\frac{2-q}{q})}} \right)^{\frac{q}{2}} \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{2-q}{2}}. \tag{3.9}$$

To estimate $\int_\Omega |u_\varepsilon|^{q^\star} dx$, we will give conditions on the exponents

1) If $q^\star\left(\frac{2-q}{q}\right) > 1$, then the series $\sum_{n=1}^{\infty} \frac{1}{n^{q^\star\left(\frac{2-q}{q}\right)}}$ is convergent and thus for n_0 large enough, we have $\sum_{n=n_0+1}^{\infty} \frac{1}{n^{q^\star\left(\frac{2-q}{q}\right)}} \leq \delta$ with δ small.

2) If $\frac{q}{q^\star} \geq \frac{2-q}{2}$, then the estimate (3.9) gives:

$$C_\Omega \|u_\varepsilon\|^q_{L^{q^\star}(\Omega)} \leq C_{n_0} + c\,\delta^{\frac{q}{2}} \|u_\varepsilon\|^{q^\star\frac{2-q}{2}}_{L^{q^\star}(\Omega)}. \tag{3.10}$$

If $\|u_\varepsilon\|_{L^{q^\star}(\Omega)} \leq 1$ then we're done, i.e. u_ε is bounded in $L^{q^\star}(\Omega)$.

The interesting case is when $\|u_\varepsilon\|_{L^{q^\star}(\Omega)} > 1$ and using condition 2, we then have

$$\|u_\varepsilon\|^{q^\star\frac{2-q}{2}}_{L^{q^\star}(\Omega)} \leq \|u_\varepsilon\|^{q^\star\frac{q}{q^\star}}_{L^{q^\star}(\Omega)} = \|u_\varepsilon\|^q_{L^{q^\star}(\Omega)},$$

and the estimate (3.10) gives

$$(C_\Omega - c\,\delta^{\frac{q}{2}}) \|u_\varepsilon\|^q_{L^{q^\star}(\Omega)} \leq C(q,a).$$

Finally, we take δ small enough so that $(C_\Omega - c\delta^{\frac{q}{2}}) = \frac{C_\Omega}{2} > 0$, i.e., for n_0 large enough we conclude that

$$\|u_\varepsilon\|_{L^{q^\star}(\Omega)} \leq C(q,a). \tag{3.11}$$

This estimate gives an estimate on the gradient of u_ε in $L^q(\Omega)$.

Indeed, we have shown that

$$\begin{aligned}
\int_\Omega |\nabla u_\varepsilon|^q \, dx &\leq C_{n_0} + c \sum_{n=n_0+1}^{\infty} |B_n|^{1-\frac{q}{2}}, \\
&\leq C_{n_0} + c \sum_{n=n_0+1}^{\infty} \frac{1}{n^{q^\star(1-\frac{q}{2})}} \left(\int_{B_n} |u_\varepsilon|^{q^\star} \, dx \right)^{1-\frac{q}{2}}, \\
&\leq C_{n_0} + c \left(\sum_{n=n_0+1}^{\infty} \frac{1}{n^{q^\star(\frac{2-q}{q})}} \right)^{\frac{q}{2}} \left(\int_\Omega |u_\varepsilon|^{q^\star} \, dx \right)^{\frac{2-q}{2}}.
\end{aligned}$$

From the estimate (3.12) and the fact that the series converges, we have

$$\|\nabla u_\varepsilon\|_{L^q(\Omega)} \leq \overline{C}(q,a). \tag{3.12}$$

Now let's return to the conditions on q, we have

$$q^\star = \frac{Nq}{N-q}, \; q < 2 \leq N, \text{ thus } \frac{q}{q^\star} = \frac{N-q}{N}$$

Condition 2: gives $\frac{N-q}{N} \geq \frac{2-q}{2}$,

- If $N = 2$, we have $\frac{2-q}{2} = \frac{2-q}{2}$, so condition 2 is satisfied.

- If $N > 2$, we have

$$\begin{aligned}
&\tfrac{N-q}{N} = \tfrac{2-q}{2}, \\
\Leftrightarrow \quad & 2(N-q) > N(2-q), \\
\Leftrightarrow \quad & -2q > -Nq, \\
\Leftrightarrow \quad & N > 2.
\end{aligned}$$

  In summary, condition 2 is always satisfied.

Condition 1: we have

$$\begin{aligned} & q^\star \tfrac{2-q}{q} > 1 \\ \Leftrightarrow \quad & \tfrac{Nq}{N-q}\tfrac{2-q}{q} > 1, \\ \Leftrightarrow \quad & N(2-q) > (N-q), \\ \Leftrightarrow \quad & N > (N-1)q, \\ \Leftrightarrow \quad & q < \tfrac{N}{N-1}. \end{aligned}$$

Conclusion If $1 \le q < \frac{N}{N-1}$, then

$$\|\nabla u_\varepsilon\|_{L^q(\Omega)} \le \overline{C}(q,a).$$

■

Theorem 3.1 (Existence) *Let $N \ge 2$ and $f \in L^1(\Omega)$. Then there exists u such that:*

$$\begin{cases} u \in W_0^{1,q}(\Omega), 1 \le q < \frac{N}{N-1} \\ \int_\Omega \nabla u(x)\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx, \ \forall \varphi \in \mathscr{D}(\Omega). \end{cases} \tag{3.13}$$

Proof. We have seen that for $f \in L^1(\Omega)$, there exists $(f_\varepsilon)_{\varepsilon>0} \in \mathscr{D}(\Omega)$ such that $f_\varepsilon \longrightarrow f$ strongly in $L^1(\Omega)$, and $\|f_\varepsilon\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}$, and there exists a unique solution $u_\varepsilon \in H_0^1(\Omega)$ to problem (3.4) satisfying

$$\|\nabla u_\varepsilon\|_{W_0^{1,q}(\Omega)} \le C(\|f\|_{L^1(\Omega)}, q, |\Omega|), \ 1 \le q < \frac{N}{N-1},$$

so we can extract a subsequence still denoted $(u_\varepsilon)_\varepsilon$ such that

$$\begin{aligned} & u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,q}(\Omega), \\ \text{and} \quad & u_\varepsilon \longrightarrow u \text{ strongly in } L^q(\Omega) \text{ and a.e. on } \Omega, \ 1 \le q < \tfrac{N}{N-1}. \end{aligned}$$

Passing to the limit,

$$\int_\Omega \underbrace{\nabla u_\varepsilon}_{\substack{\downarrow \\ \nabla u \\ \text{weak } L^q}} \underbrace{\nabla\varphi}_{\substack{\in L^\infty \\ \text{and in } L^{q'}}} dx = \int_\Omega \underbrace{f_\varepsilon}_{\substack{\downarrow \\ f \\ \text{strong } L^1}} \underbrace{\varphi}_{\in L^\infty} dx, \ \forall \varphi \in \mathscr{D}(\Omega).$$

Therefore $u \in W_0^{1,q}(\Omega)$ and satisfies

$$\int_\Omega \nabla u(x)\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx,\ \forall\varphi \in \mathscr{D}(\Omega)$$

■

Remark 3 *In (3.13), we can replace* $\mathscr{D}(\Omega)$ *by* $\bigcup_{r>N} W_0^{1,r}(\Omega)$.

Indeed, let $\varphi \in W_0^{1,r}(\Omega)$ *with* $r > N$ *then* $\varphi \in \mathscr{C}(\overline{\Omega})$, *by density there exists* $(\varphi_n)_n \in \mathscr{D}(\Omega)$ *such that*

$$\varphi_n \longrightarrow \varphi \text{ strongly in } W_0^{1,r}(\Omega)$$
$$\text{and} \quad \varphi_n \longrightarrow \varphi \text{ uniformly in } \mathscr{C}(\overline{\Omega})$$

we have

$$\int_\Omega \nabla u(x)\nabla\varphi_n(x)dx = \int_\Omega f(x)\varphi_n(x)dx,\ \forall\varphi \in \mathscr{D}(\Omega),$$

and since

$$\nabla\varphi_n \longrightarrow \nabla\varphi \text{ strongly in } L^r(\Omega),\ r > N \text{ and } \nabla u \in L^{r'}(\Omega) \text{ because } r' = \tfrac{r}{r-1} < \tfrac{N}{N-1}$$
$$\text{and} \quad \varphi_n \longrightarrow \varphi \text{ uniformly in } \mathscr{C}(\overline{\Omega}).$$

Then

$$\int_\Omega \nabla u(x)\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx,\ \forall\varphi \in \bigcup_{r>N} W_0^{1,r}(\Omega).$$

The problem (3.13) becomes:

$$\begin{cases} u \in W_0^{1,q}(\Omega), 1 \le q < \frac{N}{N-1} \\ \int_\Omega \nabla u(x)\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx,\ \forall\varphi \in \bigcup_{r>N} W_0^{1,r}(\Omega). \end{cases} \tag{3.14}$$

3.3 Uniqueness

For $f \in L^1(\Omega)$ there exists u solution of (3.14). This solution is unique if and only if the solution of (3.14) with $f = 0$ is zero since the operator is linear.

We need to show that the solution of

$$\begin{cases} u \in W_0^{1,q}(\Omega), 1 \le q < \frac{N}{N-1} \\ \int_\Omega \nabla u(x) \nabla \varphi(x) dx = 0, \ \forall \varphi \in \bigcup_{r>N} W_0^{1,r}(\Omega), \end{cases} \tag{3.15}$$

is zero.

Remark 4 .

1) *In the variational framework where φ and $u \in H_0^1(\Omega)$, it suffices to take $\varphi = u$ and thus $\int_\Omega |\nabla u(x)|^2 dx = 0$ implies that u is constant and since $u = 0$ on $\partial\Omega$, then $u = 0$ a.e. on Ω.*

2) *The difficulty is that $u \in W_0^{1,q}(\Omega), q < \frac{N}{N-1} \le 2$ and $u \in W_0^{1,r}(\Omega), r > N \ge 2$, so u and φ are not in the same space and we can never take $\varphi = u$.*

False proof of uniqueness for solution of (3.14)

Let B be a subset of Ω and consider the problem

$$\begin{cases} -\Delta v = \mathbb{1}_B \text{ in } \Omega \\ v = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.16}$$

Problem (3.16) has a unique solution in the sense:

$$\begin{cases} \exists! v \in H_0^1(\Omega), \text{ such that} \\ \int_\Omega \nabla v \nabla \psi dx = \int_B \psi(x) dx, \ \forall \psi \in H_0^1(\Omega). \end{cases} \tag{3.17}$$

Let $\psi = u$ in (3.17), then

$$\int_\Omega \nabla v \nabla u dx = \int_B u dx,$$

and let $\varphi = v$ in (3.15), then

$$\int_\Omega \nabla u \nabla v dx = 0,$$

thus

$$\int_B u dx = 0, \ \forall B \subset \Omega.$$

For

$$B = \{x \in \Omega,\ u(x) > 0\},$$

and writing

$$u^+ = \max(u,0),\ u^- = -\min(u,0) \text{ with } u(x) = u^+(x) + u^-(x).$$

we have

$$\begin{aligned} \int_B u\,dx &= \int_B (u^+ + u^-)dx \\ &= \int_B u^+ dx \\ &= 0 \end{aligned}$$

so $u^+ = 0$ a.e. in Ω. Similarly for $B = \{x \in \Omega,\ u(x) < 0\}$, we obtain $u^- = 0$ a.e. in Ω. Finally, $u = 0$ a.e. in Ω.

This proof is false due to the choice of test functions. Indeed, we cannot take $\psi = u$ because $\psi \in H_0^1(\Omega)$ and $u \in W_0^{1,q}(\Omega)$, $q < 2$, and we have $H_0^1(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$.

For $\varphi = v$, we have $\varphi \in W_0^{1,r}(\Omega)$, $r > N$ and $v \in H_0^1(\Omega) = W_0^{1,2}(\Omega)$ ($W_0^{1,r}(\Omega) \hookrightarrow H_0^1(\Omega), \forall r > N$ and $N \geq 2$).

How to fix this false uniqueness proof?

Case 1. If we seek a solution u of (3.15) in $H_0^1(\Omega)$, i.e., if the solution of (3.15) is more regular than what we found, then in this case we take

$$\psi = u \text{ in (3.15)},$$
$$\varphi = v \text{ in (3.17)}.$$

This is an easy case but difficult to achieve.

Case 2. If we show that v solution of (3.17) is in $W_0^{1,r}(\Omega)$, $r > N$, then the false solution above becomes correct.

Indeed, we now have $v \in W_0^{1,r}(\Omega)$, $r > N$, solution of

$$\int_\Omega \nabla v \nabla \psi\,dx = \int_B \psi(x)dx,\ \forall \psi \in H_0^1(\Omega)$$

and $H_0^1(\Omega) \hookrightarrow W_0^{1,r'}(\Omega)$ (dense embedding) with $r' = \frac{r}{r-1} < \frac{N}{N-1}$. By density

$$\forall \psi \in W_0^{1,r'}(\Omega),\ \exists (\psi_n)_n \in H_0^1(\Omega),\ \psi_n \longrightarrow \psi \text{ strongly in } W_0^{1,r'}(\Omega).$$

Problem (3.17) becomes

$$\begin{cases} v \in W_0^{1,r}(\Omega),\ r > N, \text{ solution of} \\ \int_\Omega \nabla v \nabla \psi \, dx = \int_B \psi(x) dx,\ \forall \psi \in W_0^{1,r'}(\Omega),\ r' < \frac{N}{N-1}. \end{cases} \tag{3.18}$$

thus, we can take $\psi = u$ in (3.18) because $u \in \bigcap_{q < \frac{N}{N-1}} W_0^{1,q}(\Omega)$ and $\varphi = v$ in (3.15) because $v \in W_0^{1,r}(\Omega)$, $r > N$, (i.e $v \in \bigcup_{r>N} W_0^{1,r}(\Omega)$).

Summary: The uniqueness proof is correct if $\forall B \subset \Omega$, the solution of (3.17) is in $W_0^{1,r}(\Omega)$ for $r > N$.

3.3.1 Regularity of Laplacian and uniqueness

The solution of (3.18) is regular, more precisely, $\forall B \subset \Omega$, $g(x) = \mathbb{1}_B(x) \in L^\infty(\Omega) \cap L^\infty(\Omega)$ and v satisfies

$$\begin{cases} v \in H_0^1(\Omega), \text{ such that} \\ \int_\Omega \nabla v \nabla \psi \, dx = \int_B \psi(x) dx,\ \forall \psi \in H_0^1(\Omega). \end{cases}$$

According to a result from [Agmon-Douglis-Nirenberg](vol.17, page 35-92, 1964), if Ω is regular, then $v \in W_0^{2,p}(\Omega)$, $\forall p < \infty$ and thus $v \in W_0^{1,\infty}(\Omega)$, and by density of $H_0^1(\Omega)$ in $W_0^{1,1}(\Omega)$. Problem (3.17) becomes

$$\begin{cases} v \in W_0^{1,\infty}(\Omega), \text{ solution of} \\ \int_\Omega \nabla v \nabla \psi \, dx = \int_B \psi(x) dx,\ \forall \psi \in W_0^{1,1}(\Omega). \end{cases} \tag{3.19}$$

Theorem 3.2 (Uniqueness for Laplacian) *Let Ω be a bounded open set of $\mathbb{R}^N, N \geq 2$ and $f \in L^1(\Omega)$, there exists one and only one solution of (3.3) in the sense:*

$$\begin{cases} u \in W_0^{1,1}(\Omega), \\ \int_\Omega \nabla u(x) \nabla \varphi(x) dx = \int_\Omega f(x) \varphi(x) dx,\ \forall \varphi \in W_0^{1,\infty}(\Omega). \end{cases} \tag{3.20}$$

Proof. By linearity, let $u = u_1 - u_2$ with u_1 and u_2 solutions of (3.20), we have

$$\begin{cases} u \in W_0^{1,1}(\Omega), \\ \int\limits_\Omega \nabla u(x) \nabla \varphi(x) dx = 0, \ \forall \varphi \in W_0^{1,\infty}(\Omega). \end{cases} \tag{3.21}$$

we can take

$$\psi = u \text{ in (3.19) because } \psi \text{ and } u \in W_0^{1,1}(\Omega),$$

$$\text{and} \quad \varphi = v \text{ in (3.21) because } \psi \text{ and } u \in W_0^{1,1}(\Omega),$$

and we conclude as before. ■

CHAPTER 4

Elliptic equations in divergence form

Let Ω be a bounded open set of $\mathbb{R}^N$, $A(x)=\left(a_{ij}(x)\right)_{1\le i,j\le N}$ an operator defined by coefficients $a_{ij}\in L^\infty(\Omega),\ i,j=1,N$, and satisfying the ellipticity hypothesis:

$$\exists\alpha>0,\ A(x)\xi\xi=\sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j\ge\alpha|\xi|^2=\alpha\sum_{i=1}^{N}|\xi_i|^2,\ \forall\xi\in\mathbb{R}^N,\ \text{a.e. } x\in\Omega.$$

Since $a_{ij}\in L^\infty(\Omega)$, there exists $\beta>0$ such that

$$|A(x)\xi\xi|\le\beta|\xi|^2,\ \forall\xi\in\mathbb{R}^N,\ \text{a.e. } x\in\Omega.$$

We are interested in the following problem

$$\begin{cases}-\operatorname{div}(A(x)\nabla u(x))=f(x) & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega.\end{cases}$$

where

$$\operatorname{div}(A(x)\nabla u(x))=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}(x)\right).$$

Theorem 4.1 (Existence) *Let $f\in L^1(\Omega)$ and $N\ge 2$, there exists u such that*

$$\begin{cases}u\in W_0^{1,q}(\Omega),\forall 1\le q<\frac{N}{N-1},\\ \int_\Omega A(x)\nabla u(x)\nabla v(x)dx=\sum_{i,j=1}^{N}\int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dx=\int_\Omega f(x)v(x)dx,\\ \forall v\in\bigcup_{r>N}W_0^{1,r}(\Omega)\ or\ \forall v\in\mathscr{D}(\Omega).\end{cases}\tag{4.1}$$

Proof. The proof is similar to the Laplacian case

$$\begin{aligned}\forall f \in L^1(\Omega), \quad & \exists (f_\varepsilon)_{\varepsilon>0} \in \mathscr{D}(\Omega) \text{ such that} \\ & f_\varepsilon \longrightarrow f \text{ strongly in } L^1(\Omega) \\ & \text{and } \|f_\varepsilon\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)},\ \forall \varepsilon > 0.\end{aligned}$$

For fixed ε, we have

$$\begin{cases} \exists! u_\varepsilon \in H_0^1(\Omega), \\ \int_\Omega A(x)\nabla u_\varepsilon(x)\nabla v(x)dx = \int_\Omega f_\varepsilon(x)v(x)dx,\ \forall v \in H_0^1(\Omega). \end{cases} \tag{4.2}$$

Taking $v = \varphi(u_\varepsilon)$, we deduce that

$$\alpha \int_{B_n} |\nabla u_\varepsilon(x)|^2 dx \leq \|f\|_{L^1(\Omega)},$$

and thus, we have seen that there exists $C > 0$ independent of ε such that

$$\|u_\varepsilon\|_{W_0^{1,q}(\Omega)} \leq C(\|f\|_{L^1(\Omega)}, q, |\Omega|),\ \forall 1 \leq q < \frac{N}{N-1},$$

we can assume, up to extracting a subsequence, that

$$\begin{aligned} & u_\varepsilon \rightharpoonup u \text{ weakly in } W_0^{1,q}(\Omega), \\ \text{and} \quad & u_\varepsilon \longrightarrow u \text{ strongly in } L^q(\Omega) \text{ and a.e. on } \Omega,\ 1 \leq q < \tfrac{N}{N-1}. \end{aligned}$$

and we deduce that (4.1) admits a solution by passing to the limit. ∎

The question of uniqueness is more delicate because the coefficients a_{ij} are not continuous. Essentially, we are led to study the singularity of the problem

$$\begin{cases} v \in H_0^1(\Omega), \text{ such that} \\ \sum_{i,j=1}^N \int_\Omega a_{ij}(x)\frac{\partial v}{\partial x_j}\frac{\partial \psi}{\partial x_i} dx = \int_B \psi(x)dx,\ \forall \psi \in H_0^1(\Omega). \end{cases} \tag{4.3}$$

and $a_{ij} \in L^\infty(\Omega)$?

In dimension 2 ($N = 2$), Meyers' theorem [13] allows us to conclude in the following sense:

There exists $p_0 > 2$, such that the mapping

$$\begin{array}{ccc} W^{-1,p}(\Omega) & \longrightarrow & W^{1,p}(\Omega) \\ f & \longmapsto & v \end{array}$$

is linear continuous for all $p \in [2, p_0]$, and there exists a constant $C_p > 0$ such that $\|v\|_{W^{1,p}(\Omega)} \leq C_p > 0 \|v\|_{W^{1,p}(\Omega)}$.

We cannot use Meyers' theorem in dimension 3 because we have seen that to conclude uniqueness it suffices that $W_0^{1,r}(\Omega),\ r > N$? But Meyers' theorem only gives $r > N$ and for $N = 2$ it works.

Serrin [16] constructed a counterexample of non-uniqueness of the solution even in the case $N = 2$! and in dimension $N > 2$ by A.Prignet [14].

This leads us to the notion of entropy solutions.

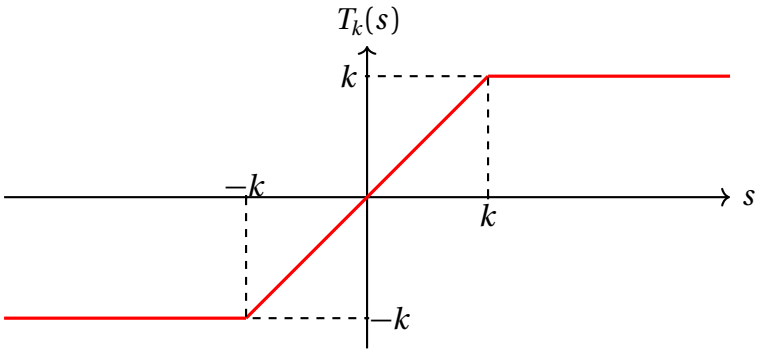
4.1 Entropy solutions and Uniqueness

We first define the truncations:

Let $k \in \mathbb{R}^+$, $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$T_k(s) = \begin{cases} s, & -k \leq s \leq k \\ k, & s > k \\ -k, & s < k \end{cases}$$

The function T_k is used, for example, to show the weak maximum principle for elliptic



equations. We have $|T_k(s)| \leq k,\ \forall s$, and since T_k is Lipschitz and $T_k(0) = 0$, Stampacchia's

lemma 2.2 asserts that for $u \in W_0^{1,p}(\Omega)$, then $T_k(u) \in W_0^{1,p}(\Omega)$ $p > 1$ and

$$\nabla T_k(u) = T_k'(u)\nabla u.$$

Theorem 4.2 (Entropy solutions) *Let $f \in L^1(\Omega)$, the solution u of (4.1) satisfies:*

$$\begin{cases} u \in L^1(\Omega),\ T_k(u) \in H_0^1(\Omega)\ \forall k > 0, \\ \int_\Omega A\nabla u \nabla T_k(u-\varphi)dx \leq \int_\Omega f(x) T_k(u-\varphi)dx,\ \forall \varphi \in \mathscr{D}(\Omega). \end{cases} \tag{4.4}$$

The goal is to show that the entropy solution is unique.

Theorem 4.3 *Let $f \in L^1(\Omega)$, $N \geq 2$. Then there exists at most one solution of (4.4).*

Remark 5 .

1) *Problem (4.4) makes sense because:*

   $f \in L^1(\Omega)$ and $T_k(u-\varphi) \in L^\infty(\Omega)$, so $f\, T_k(u-\varphi) \in L^1(\Omega)$.

2) *How to give meaning to $\nabla u \nabla T_k(u-\varphi)$?*

   We have $A \in L^\infty(\Omega)$, $\nabla u \in L^p(\Omega)$, $p < 2$ and $T_k(u-\varphi) \in L^2(\Omega)$, a priori this quantity is a function?

Let's look more closely at this quantity:

$$A\nabla u \nabla T_k(u-\varphi) = A\mathbb{1}_{\{|u-\varphi| \leq k\}} \nabla u \nabla (u-\varphi)$$

where we used Stampacchia's lemma. Let $M = k + \|\varphi\|_\infty$, and thus on the set $\{|u-\varphi| \leq k\}$, we have

$$|u| = |u - \varphi + \varphi| \leq k + \|\varphi\|_\infty = M,$$

because $\varphi \in L^\infty(\Omega)$. Thus,

$$A\nabla u \nabla T_k(u-\varphi) = A\mathbb{1}_{\{|u-\varphi| \leq k\}} \nabla T_M(u)(\nabla T_M(u) - \nabla \varphi),$$

but $T_M(u) \in H_0^1(\Omega)$ and thus $A\nabla u \nabla T_k(u-\varphi) \in L^1(\Omega)$.

Finally, we can give meaning to the entropy formulation as soon as $T_k(u) \in H_0^1(\Omega)$ and ($\varphi \in$

$L^\infty(\Omega)\cap H_0^1(\Omega)$ or $\mathscr{D}(\Omega)$).

Proof. (Theorem 4.2) We will see u as the limit of the sequence $(u_\varepsilon)_\varepsilon$, where $(u_\varepsilon)_\varepsilon$ defined by (4.2) satisfies (4.4).

We have seen that $(u_\varepsilon)_\varepsilon$ satisfies:

$$\int_{B_n} |\nabla u_\varepsilon(x)|^2 dx \leq \frac{\|f\|_{L^1(\Omega)}}{\alpha},$$

with

$$B_n = \{x \in \Omega, n \leq |u(x)| \leq n+1\}.$$

Let now,

$$G_k = \{x \in \Omega, 0 \leq |u_\varepsilon(x)| \leq k\}.$$

We have

$$\begin{aligned}
\int_\Omega |\nabla T_k(u_\varepsilon)|^2 dx &= \int_\Omega \mathbb{1}_{\{|u_\varepsilon|\leq k\}} |\nabla u_\varepsilon|^2 dx, \\
&= \int_{G_k} |\nabla u_\varepsilon|^2 dx, \\
&= \sum_{i=1}^{k-1} \int_{B_i} |\nabla u_\varepsilon|^2 dx, \\
&\leq k \frac{\|f\|_{L^1(\Omega)}}{\alpha}.
\end{aligned}$$

Thus, $\forall k > 0$, we have

$$\begin{cases}
\nabla T_k(u_\varepsilon) \in H_0^1(\Omega), \\
\|T_k(u_\varepsilon)\|_{H_0^1(\Omega)} \leq k \frac{\|f\|_{L^1(\Omega)}}{\alpha}. \text{ independently of } \varepsilon \text{ and depends on } k,
\end{cases}$$

we can deduce (up to a subsequence) that

$$T_k(u_\varepsilon) \rightharpoonup \xi \text{ weakly in } H_0^1(\Omega) \text{ when } \varepsilon \longrightarrow 0.$$

On the other hand,

$$u_\varepsilon \longrightarrow u \text{ a.e. on } \Omega,$$

and since T_k is continuous, then

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ a.e. on } \Omega.$$

Thus $\xi = T_k(u)$ and

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } H_0^1(\Omega).$$

We also have by Fatou's lemma

$$\|T_k(u)\|_{H_0^1(\Omega)} \leq k\frac{\|f\|_{L^1(\Omega)}}{\alpha}.$$

■

4.1.0.1 Showing the entropy inequality

From (4.2),

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega), \\ \int_\Omega A(x)\nabla u_\varepsilon(x)\nabla\varphi(x)dx = \int_\Omega f_\varepsilon(x)\varphi(x)dx, \ \forall \varphi \in H_0^1(\Omega). \end{cases}$$

Let $\varphi = T_k(u_\varepsilon - \psi)$ with $\psi \in \mathscr{D}(\Omega)$ given and $k > 0$, we have

$$\varphi \in H_0^1(\Omega) \text{ and } \nabla\varphi = \nabla T_k(u_\varepsilon - \psi)\mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}} \text{ a.e.}$$

Thus,

$$\int_\Omega A(x)\nabla u_\varepsilon \nabla T_k(u_\varepsilon - \psi)\mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}}dx = \int_\Omega f_\varepsilon T_k(u_\varepsilon - \psi)dx.$$

Or equivalently,

$$\int_\Omega A(x)\nabla u_\varepsilon \nabla u_\varepsilon \mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}}dx - \int_\Omega A(x)\nabla u_\varepsilon \nabla\psi \mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}}dx = \int_\Omega f_\varepsilon T_k(u_\varepsilon - \psi)dx,$$

which we denote as,

$$a_\varepsilon + b_\varepsilon = c_\varepsilon.$$

- We have

$$c_\varepsilon = \int_\Omega f_\varepsilon T_k(u_\varepsilon - \psi)dx \longrightarrow \int_\Omega f\, T_k(u - \psi)dx.$$

  Because, we have

$$f_\varepsilon T_k(u_\varepsilon - \psi) \longrightarrow f\, T_k(u - \psi) \text{ a.e. on } \Omega.$$

and

$$\begin{cases} |f_\varepsilon T_k(u_\varepsilon - \psi)| \leq k|f_\varepsilon| \text{ a.e. on } \Omega, \\ k.f_\varepsilon \longrightarrow k.f \text{ in } L^1(\Omega). \end{cases}$$

By Lebesgue's dominated convergence theorem,

$$f_\varepsilon T_k(u_\varepsilon - \psi) \longrightarrow f\, T_k(u-\psi) \text{ in } L^1(\Omega).$$

- We have

$$-\int_\Omega A(x)\nabla u_\varepsilon \nabla \psi \mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}} dx = -\int_\Omega A(x)\nabla T_M(u_\varepsilon)\nabla \psi \mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}} dx,$$

with $M = k + \|\psi\|_\infty$.

We have, on one hand,

$$\nabla T_M(u_\varepsilon) \rightharpoonup \nabla T_k(u) \text{ weakly in } \left(L^2(\Omega)\right)^N,$$

and on the other hand,

$$\begin{cases} A\nabla\psi \mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}} \longrightarrow A\nabla\psi \mathbb{1}_{\{|u-\psi| \leq k\}} \text{ a.e. on } \Omega, \\ |A\nabla\psi \mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}}| \leq |A\nabla\psi| \text{ a.e. on } \Omega, \end{cases}$$

by Lebesgue's dominated convergence theorem,

$$A\nabla\psi \mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}} \longrightarrow A\nabla\psi \mathbb{1}_{\{|u-\psi| \leq k\}} \text{ strongly in } \left(L^2(\Omega)\right)^N.$$

Thus,

$$b_\varepsilon \longrightarrow b = -\int_\Omega A(x)\nabla u \nabla \psi \mathbb{1}_{\{|u-\psi| \leq k\}} dx.$$

- The term a_ε:

$$a_\varepsilon = \int_\Omega A\nabla u_\varepsilon \nabla u_\varepsilon \mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}} dx = \int_\Omega A\nabla T_M(u_\varepsilon)\nabla T_M(u_\varepsilon)\mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}} dx,$$

The difficulty here is that we end up with a product of two weak convergences! To over-

come this, either we show that $T_k(u_\varepsilon) \longrightarrow T_k(u)$ strongly in $H_0^1(\Omega)$ (this is a result by Murat-Lions) and in this case we will have an inequality. Here, we will establish the following lemma:

Lemma 4.1

$$\liminf_{\varepsilon \longrightarrow 0} a_\varepsilon \geq \int_\Omega A \nabla T_M(u) \nabla T_M(u) \mathbb{1}_{\{|u-\psi| \leq k\}} dx.$$

Proof. Let $w_\varepsilon = \mathbb{1}_{\{|u_\varepsilon - \psi| \leq k\}}$, we have $0 \leq w_\varepsilon \leq 1$ and

$$w_\varepsilon \longrightarrow \mathbb{1}_{\{|u-\psi| \leq k\}} \text{ a.e. on } \Omega.$$

Let $v_\varepsilon = T_M(u_\varepsilon)$, we have $v_\varepsilon \in H_0^1(\Omega)$ and

$$v_\varepsilon \longrightarrow v = T_M(u) \text{ weakly in } H_0^1(\Omega).$$

Finally, we write $a_\varepsilon = d_\varepsilon(v_\varepsilon, v_\varepsilon)$ with

$$d_\varepsilon(\varphi, \xi) = \int_\Omega A \nabla \varphi \nabla \xi w_\varepsilon dx.$$

We have, by the ellipticity condition

$$d_\varepsilon(\varphi - \lambda\xi, \varphi - \lambda\xi) \geq 0, \ \forall \varepsilon > 0, \forall \lambda \in \mathbb{R},$$

thus,

$$d_\varepsilon(\varphi, \varphi) - \lambda[d_\varepsilon(\varphi, \xi) + d_\varepsilon(\xi, \varphi)] + \lambda^2 d_\varepsilon(\xi, \xi) \geq 0, \ \forall \lambda \in \mathbb{R}.$$

We distinguish two cases:

Case where $A(x)$ is symmetric: i.e. $d_\varepsilon(.,.)$ is symmetric Then

$$d_\varepsilon(\varphi, \varphi) - 2\lambda d_\varepsilon(\varphi, \xi) + \lambda^2 d_\varepsilon(\xi, \xi) \geq 0, \ \forall \lambda \in \mathbb{R},$$

we choose $\lambda = \frac{d_\varepsilon(\varphi, \xi)}{d_\varepsilon(\xi, \xi)}$, thus

$$d_\varepsilon(\varphi, \varphi) - 2\frac{(d_\varepsilon(\varphi, \xi))^2}{d_\varepsilon(\xi, \xi)} + \frac{(d_\varepsilon(\varphi, \xi))^2}{d_\varepsilon(\xi, \xi)} \geq 0,$$

hence

$$(d_\varepsilon(\varphi,\xi))^2 \le d_\varepsilon(\varphi,\varphi).d_\varepsilon(\xi,\xi),$$

take this inequality with $\varphi = v_\varepsilon$ and $\xi = v$, so

$$(d_\varepsilon(v_\varepsilon,v))^2 \le d_\varepsilon(v_\varepsilon,v_\varepsilon).d_\varepsilon(v,v),$$

taking the limit inferior, we obtain

$$\begin{aligned}(d_\varepsilon(v,v))^2 &= \liminf_{\varepsilon\longrightarrow 0}(d_\varepsilon(v_\varepsilon,v))^2\\ &\le \liminf_{\varepsilon\longrightarrow 0} d_\varepsilon(v_\varepsilon,v_\varepsilon).d(v,v),\end{aligned}$$

thus

$$\begin{aligned}\liminf_{\varepsilon\longrightarrow 0} a_\varepsilon &= \liminf_{\varepsilon\longrightarrow 0} d_\varepsilon(v_\varepsilon,v_\varepsilon)\\ &\ge d(v,v) = \int_\Omega A\nabla T_M(u)\nabla T_M(u)\mathbb{1}_{\{|u-\psi|\le k\}}dx\end{aligned}$$

Case where $A(x)$ is non-symmetric: we can show (more involved and we'll do it in the nonlinear case) that $\nabla u_\varepsilon \longrightarrow \nabla u$ a.e. and since $\nabla T_k(u_\varepsilon)$ is bounded in $\left(L^2(\Omega)\right)^N$, (for all $k>0$), then

$$\nabla T_k(u_\varepsilon)\longrightarrow \nabla T_k(u) \text{ a.e. on } \Omega,$$

thus

$$A\nabla T_M(u_\varepsilon)\nabla T_M(u_\varepsilon)\mathbb{1}_{\{|u_\varepsilon-\psi|\le k\}} \longrightarrow A\nabla T_M(u)\nabla T_M(u)\mathbb{1}_{\{|u-\psi|\le k\}} \text{ a.e. on } \Omega.$$

And by Fatou's lemma, we have

$$\int_\Omega A\nabla T_M(u)\nabla T_M(u)\mathbb{1}_{\{|u-\psi|\le k\}}dx \le \liminf_{\varepsilon\longrightarrow 0} a_\varepsilon < +\infty,$$

since $T_M(u_\varepsilon)$ is bounded in $H_0^1(\Omega)$. ■

4.1.1 Proof of Theorem 4.3

Let u and v be two solutions of (4.1) satisfying:

$$\begin{cases} u\in L^1(\Omega),\ T_k(u)\in H_0^1(\Omega)\ \forall k>0,\\ \int_\Omega A\nabla u\nabla T_k(u-\varphi)\mathbb{1}_{\{|u-\varphi|\le k\}}dx \le \int_\Omega f(x)T_k(u-\varphi)dx,\ \forall\varphi\in\mathscr{D}(\Omega).\end{cases} \tag{4.5}$$

The proof is done in three steps:

Step 1 We can take $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Indeed, for $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, there exists $(\varphi_n)_n \in \mathscr{D}(\Omega)$ such that

$$\begin{aligned} &\varphi_n \longrightarrow \varphi \text{ strongly in } H_0^1(\Omega) \\ &\varphi_n \longrightarrow \varphi \text{ a.e. on } \Omega. \\ &\text{and } \|\varphi_n\|_{L^\infty(\Omega)} \leq \|\varphi\|_{L^\infty(\Omega)},\ \forall n \in \mathbb{N}. \end{aligned}$$

Let

$$M = k + \|\varphi\|_{L^\infty(\Omega)} \geq M_n = k + \|\varphi_n\|_{L^\infty(\Omega)}.$$

Writing (4.5) with $\varphi = \varphi_n$, we obtain

$$\int_\Omega A(x)\nabla T_M(u)\nabla T_M(u)\mathbb{1}_{\{|u-\varphi_n|\leq k\}}dx - \int_\Omega A(x)\nabla T_M(u)\nabla \varphi_n \mathbb{1}_{\{|u-\varphi_n|\leq k\}}dx \leq \int_\Omega f\, T_k(u-\varphi_n)dx.$$

By Lebesgue's dominated convergence theorem,

$$\int_\Omega f\, T_k(u-\varphi_n)dx \longrightarrow \int_\Omega f\, T_k(u-\varphi)dx,$$

we also have

$$A\nabla T_M(u)\psi\mathbb{1}_{\{|u-\varphi_n|\leq k\}} \longrightarrow A\nabla T_M(u)\psi\mathbb{1}_{\{|u-\varphi|\leq k\}} \text{ strongly in } \left(L^2(\Omega)\right)^N.$$

and

$$\nabla\varphi_n \longrightarrow \nabla\varphi \text{ strongly in } \left(L^2(\Omega)\right)^N.$$

Thus the second term converges and obviously the first term also converges.

Step 2 Let k and $h \in \mathbb{R}^+$, take $T_k(u - T_h v)$ as a test function in (4.5) for the equation in u (i.e., $\varphi = T_h v$) and $T_k(v - T_h u)$ in (4.5) for the equation in v (i.e., $\varphi = T_h u$).

Start with the inequality for u

$$\int_\Omega A\nabla u\nabla T_k(u - T_h(v))dx \leq \int_\Omega f(x)T_k(u - T_h(v))dx,$$

and for v,

$$\int_\Omega A\nabla v\nabla T_k(v-T_h(u))dx \le \int_\Omega f(x)T_k(v-T_h(u))dx.$$

Adding these two inequalities, we obtain:

$$A_1+A_2+A_3+A_4+A_5+A_6 \le B_1+B_2+B_3+B_4,$$

where

$$\int_\Omega A(x)\nabla u\nabla T_k(u-T_h(v))dx = A_1+A_2+A_3,$$

with

$$A_1 = \int_{\{|v|\le h,\ |u|\le h,\ |u-v|\le k\}} A(x)\nabla u\nabla(u-v)dx,$$

$$A_2 = \int_{\{|v|\ge h,\ |u-v|\le k\}} A(x)\nabla u\nabla u\,dx \ge 0,$$

$$A_3 = A_3^1 - A_3^2 = \int_{\{|v|\le h,\ |u|\ge h,\ |u-v|\le k\}} A(x)\nabla u\nabla(u-v)dx,$$

with

$$A_3^1 = \int_{\{|v|\le h,\ |u|\ge h,\ |u-v|\le k\}} A(x)\nabla u\nabla u\,dx \ge 0.$$

Thus the term A_3^1 is not problematic. Let's show that the term $A_3^2 \underset{h\to+\infty}{\longrightarrow} 0$.

Indeed, we have $|v|\le h$, $|u|\ge h$ and $|u-v|\le k$ implies that $h\le|u|\le h+k$ and $h-k\le|v|\le h$.

We deduce that

$$\begin{aligned}|A_3^2| &= \Big|\int_{\{|v|\le h,\ |u|\ge h,\ |u-v|\le k\}} A(x)\nabla u\nabla v\,dx\Big| \\ &\le \|A\|_\infty\left[\int_{\{h\le|u|\le h+k\}}|\nabla u|^2dx\int_{\{h-k\le|v|\le h\}}|\nabla u|^2dx\right]^{\frac12},\end{aligned}$$

we show in step 3 that

$$\int_{\{h\le|u|\le h+k\}}|\nabla u|^2dx \underset{h\to+\infty}{\longrightarrow} 0,\ \forall k.$$

and thus, we indeed have $A_3^2 \underset{h\to+\infty}{\longrightarrow} 0$ for any fixed k.

Now we write the term in v:

$$\int_\Omega A(x)\nabla v\nabla T_k(v-T_h(u))dx = A_4 + A_5 + A_6,$$

with

$$A_4 = \int_{\{|v|\le h,\ |u|\le h,\ |u-v|\le k\}} A(x)\nabla v\nabla(v-u)dx,$$

$$A_5 = \int_{\{|u|\ge h,\ |v-u|\le k\}} A(x)\nabla v\nabla v\,dx \ge 0,$$

$$A_6 = A_6^1 - A_6^2 = \int_{\{|u|\le h,\ |v|\ge h,\ |v-u|\le k\}} A(x)\nabla u\nabla(v-u)dx,$$

with

$$A_6^1 = \int_{\{|u|\le h,\ |v|\ge h,\ |v-u|\le k\}} A(x)\nabla v\nabla v\,dx \ge 0.$$

$$|A_6^2| = \int_{\{|u|\le h,\ |v|\ge h,\ |v-u|\le k\}} A(x)\nabla v\nabla u\,dx \underset{h\to+\infty}{\longrightarrow} 0,\ \forall k \text{ fixed}.$$

Now forming the sum, we get:

$$\sum_{i=1}^{6} A_i = \int_{\{|v|\le h,\ |u|\le h,\ |u-v|\le k\}} A(x)\nabla(u-v)\nabla(u-v)dx + D_1 + D_2.$$

with $D_1 \ge 0$ and $D_2 x \underset{h\to+\infty}{\longrightarrow} 0,\ \forall k$ fixed.

For fixed k, we have

$$\begin{aligned}\liminf_{h\longrightarrow\infty}\sum_{i=1}^{4} B_i &\ge \liminf_{h\longrightarrow\infty}\sum_{i=1}^{6} A_i \\ &\ge \int_{\{|u-v|\le k\}} A(x)\nabla(u-v)\nabla(u-v)dx \\ &\ge \alpha \int_{\{|u-v|\le k\}} |\nabla(u-v)|^2 dx.\end{aligned}$$

Now let's examine the right-hand side:

$$\int_\Omega f(x)T_k(u-T_h(v))dx = B_1 + B_2,$$

with

$$B_1 = \int_{\{|v|\le h,\ |u|\le h\}} f(x)T_k(u-v)dx,\ B_2 = \int_{\{|v|\ge h \text{ or } |u|\ge h\}} f(x)T_k(u-T_h(v))dx.$$

We have

$$|B_2| \le k\int_{\{|v|\ge h\}} |f|dx + k\int_{\{|u|\ge h\}} |f|dx \underset{h\to+\infty}{\longrightarrow} 0,\ \forall k \text{ fixed.}$$

we used the dominated convergence theorem. We also decompose

$$\int_\Omega f(x)T_k(v-T_h(u))dx = B_3 + B_4,$$

with

$$B_3 = \int_{\{|u|\le h,\ |v|\le h\}} f(x)T_k(v-u)dx,\ B_4 = \int_{\{|v|\ge h \text{ or } |u|\ge h\}} f(x)T_k(v-T_h(u))dx,$$

similarly

$$B_4 \underset{h\to+\infty}{\longrightarrow} 0,\ \forall k \text{ fixed,}$$

and we have

$$B_1 + B_3 = \int_{\{|u|\le h,\ |v|\le h\}} f(x)(T_k(u-v)+T_k(v-u))\,dx = 0,$$

since T_k is an odd function.

Finally, we deduce that

$$\begin{aligned}\int_{\{|u-v|\le k\}} |\nabla(u-v)|^2 dx &= \int_\Omega |\nabla(u-v)|^2 \mathbb{1}_{\{|u-v|\le k\}} dx \\ &= \int_\Omega |\nabla T_k(u-v)|^2 dx \\ &= 0\end{aligned}$$

We have

$$\int_\Omega |\nabla(u-v)|^2 \mathbb{1}_{\{|u-v|\le k\}} dx \underset{k\to+\infty}{\longrightarrow} \int_\Omega |\nabla(u-v)|^2 dx,$$

thus $\int_\Omega |\nabla(u-v)|^2 dx = 0$, hence $u - v = c$ a.e. on Ω.

We assumed that $T_k(u) = T_k(v+c) \in H_0^1(\Omega)$ and $T_k(v) \in H_0^1(\Omega)$, then $c = 0$ and $u = v$ a.e. on Ω.

Step 3 Let $\varphi = T_h(u)$ in (4.5) for the equation in u

$$\int_\Omega A\nabla u \nabla T_k(u - T_h(u)) dx \leq \int_\Omega f(x) T_k(u - T_h(u)) dx,$$

we have

$$\int_{\{|u-T_h(u)|\leq k\}} A\nabla u \nabla T_k(u-T_h(u)) dx = \int_{\{|u|\leq h,\ |u-u|\leq k\}} A\nabla u \nabla(u-u) dx + \int_{\{|u|\geq h,\ |u-h|\leq k\}} A\nabla u \nabla u dx.$$

We have $|u| \geq h$ and $|u-h| \leq k$ implies $h \leq |u| \leq h+k$

$$\begin{aligned}
\int_{\{h\leq|u|\leq h+k\}} A\nabla u \nabla u dx &\leq \int_{\{|u|\leq h\}} f(x) T_k(0) dx + \int_{\{|u|\geq h\}} f(x) T_k(u-h) dx \\
&\leq k \int_{\{|v|\geq h\}} |f| dx \\
&\leq k \int_\Omega \mathbb{1}_{\{|v|\geq h\}}(x) |f| dx \underset{h\to+\infty}{\longrightarrow} 0
\end{aligned}$$

CHAPTER 5

Parabolic equations with L^1 Data

We are interested in the following parabolic problem:

$$\begin{cases} \partial_t u(t,x) - \operatorname{div}(A(t,x)\nabla u(t,x)) = f(t,x), & (t,x) \in Q_T =]0;T[\times\Omega, \\ u(t,x) = 0, & x \in \partial\Omega,\ t \in]0;T[, \\ u(0,x) = u_0(x), & x \in \partial\Omega \end{cases} \tag{5.1}$$

with the following hypotheses:

Hypothesis **(H1)**

- Ω a bounded open subset of $\mathbb{R}^N$, $N \geq 2$; with Lipschitz boundary.
- $A(t,x) = \left(a_{ij}(t,x)\right)_{i,j=1,N}$ where $a_{ij} \in L^\infty(]0;T[\times\Omega)$.
- A is coercive, i.e. $\exists \alpha > 0$ such that

$$A(t,x)\xi\xi \geq \alpha|\xi|^2 \text{ a.e. } (t,x) \in Q_T.$$

Hypothesis **(H2)**

$$f \in L^1(Q_T) \text{ and } u_0 \in L^1(\Omega).$$

Definition 5.1 (Weak solution of (5.1)) *We say that u is a weak solution of (5.1) if: $u \in L^q\left(0;T;W_0^{1,q}(\Omega)\right) \cap \mathscr{C}^0\left(0;T;L^1(\Omega)\right)$ satisfying:*

$$\int_\Omega u\phi(T,x)dx - \int_\Omega u_0\phi(0,x)dx - \int_{Q_T} u\partial_t\phi(t,x)dtdx + \int_{Q_T} A\nabla u\nabla\phi\, dtdx = \int_{Q_T} f\phi(t,x)dtdx,$$

for all $T > 0$, $\phi \in \mathscr{C}^0\left(0;T;W_0^{1,q'}(\Omega)\right) \cap \mathscr{C}^1\left(0;T;L^{q'}(\Omega)\right)$ such that $1 \leq q \leq \frac{N+2}{N+1}$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Remark 6 *Each term makes sense by duality (L^q, $L^{q'}$).*

Theorem 5.1 (Existence of solution) *Assume (H1)-(H2). Then there exists a weak solution of (5.1) according to Definition 5.1.*

Proof. The proof is done in several steps:

Step 1: The first step consists in regularizing the data. By density

$\exists (u_{0,\varepsilon})_{\varepsilon>0} \in \mathscr{D}(\Omega)$ and $(f_\varepsilon)_{\varepsilon>0} \in \mathscr{D}(Q_T)$ such that
$u_{0,\varepsilon} \longrightarrow u_0$ strongly in $L^1(\Omega)$ and $f_\varepsilon \longrightarrow f$ strongly in $L^1(Q_T)$
with $\|u_{0,\varepsilon}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}$ and $\|f_\varepsilon\|_{L^1(Q_T)} \leq \|f\|_{L^1(Q_T)}$.

We then consider the regularized problem:

$$\begin{cases} \partial_t u_\varepsilon - \operatorname{div}(A\nabla u_\varepsilon) = f_\varepsilon, & (t,x) \in Q_T, \\ u_\varepsilon(t,x) = 0, & x \in \partial\Omega,\ t \in]0;T[, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), & x \in \partial\Omega \end{cases} \tag{5.2}$$

■

Proposition 5.1 *For all $\varepsilon > 0$, there exists a unique solution u_ε of (5.2) in the following sense: $u_\varepsilon \in \mathscr{C}^0\left(0;T;L^2(\Omega)\right) \cap L^2\left(0;T;H_0^1(\Omega)\right)$, $\partial_t u_\varepsilon \in L^2\left(0;T;H^{-1}(\Omega)\right)$ such that:*

$$\int_0^T \langle \partial_t u_\varepsilon, \phi \rangle_{H^{-1},H_0^1} dt + \int_{Q_T} A\nabla u_\varepsilon \nabla \phi \, dt\, dx = \int_{Q_T} f_\varepsilon \phi \, dt\, dx, \tag{5.3}$$

for all $T > 0$, $\phi \in L^2\left(0;T;H_0^1(\Omega)\right)$.

Remark 7 *$u \in L^2(0;T;E)$ with E a Banach space, i.e., $u(t,.) \in E$ a.e. in t and $\int_0^T \|u(t,.)\|_E^2 dt < \infty$.*

Quick proof

The problem (5.3) being linear, we can use the Galerkin method to show the existence of a solution.

Let $\{\varphi_k\}_{k=1,\infty}$ be a Hilbert basis of $H_0^1(\Omega)$, we denote $V_n = \langle \varphi_1, ..., \varphi_n \rangle$ the vector space generated by n-eigenfunctions of the Laplacian.

We look for $u_n = \sum_{k=1}^{n} \alpha_i(t)\varphi_i(x)$ solution of

$$\langle \partial_t u_n, \xi \rangle_{H^{-1},H_0^1} + \int_\Omega A\nabla u_n \nabla \xi \, dx = \int_\Omega f\xi \, dx, \tag{5.4}$$

with $f = f_\varepsilon \in L^2(Q_T)$.

Choosing $\xi = \varphi_j$, thus (5.4) is equivalent to solving the system of differential equations

$$\frac{d}{dt}\alpha(t) = F(t), \text{ with } \alpha(t) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

There exists T_n such that the solution $\alpha \in \mathscr{C}^0([0;T_n])$. The solution is global, i.e. $T_n = T$, if u_n is bounded.

For $\xi = u_n$,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_n|^2 dx + \int_\Omega A\nabla u_n \nabla u_n dx = \int_\Omega f u_n dx,$$

by integration,

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\int_\Omega |u_n|^2 dx + \alpha\|\nabla u_n\|^2_{L^2(\Omega)} &\leq \|f\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)} \\ &\leq C_p\|f\|_{L^2(\Omega)}\|\nabla u_n\|_{L^2(\Omega)}, \end{aligned}$$

thus, u_n is bounded in $L^\infty\big(0;T;L^2(\Omega)\big) \cap L^2\big(0;T;H_0^1(\Omega)\big)$,

and we have

$$\begin{aligned} |\langle \partial_t u_n, \xi \rangle_{H^{-1},H_0^1}| &\leq \|\nabla u_n\|_{L^2(\Omega)}\|\nabla \xi\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)} \\ &\leq C\|\xi\|_{L^2\left(0;T;H_0^1(\Omega)\right)}, \end{aligned}$$

thus $\partial_t u_n$ is bounded in $L^2\big(0;T;H^{-1}(\Omega)\big)$. ■

Step 2: ε-independent estimates

Proposition 5.2 *There exist C_0, C_1 depending only on $\|u_0\|_{L^1(\Omega)}$, $\|f\|_{L^1(Q_T)}$, $|\Omega|$ and T such that*

the sequence $(u_\varepsilon)_\varepsilon$ *solution of (5.1) satisfies:*

$$\sup_{t\in(0,T)} \|u_\varepsilon(t)\|_{L^1(\Omega)} \leq C_0, \tag{5.5}$$

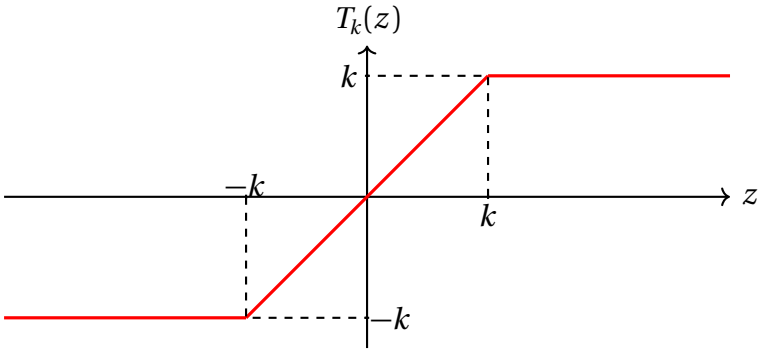
$$\int_{B_n} |\nabla u_\varepsilon|^2 \leq C_1, \tag{5.6}$$

with $B_n = \{(t,x)\in Q_T;\ n \leq |u_\varepsilon(t,x)| \leq n+1\}$.

Proof. To obtain ε-independent estimates, we are led to take test functions in $L^\infty(Q_T)\cap L^2\left(0,T,H_0^1(\Omega)\right)$

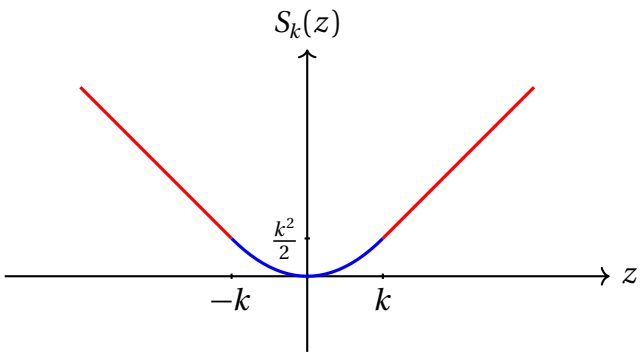
Let

$$T_k(z) = \begin{cases} z, & |z|\leq k \\ k, & z>k \\ -k, & z<k \end{cases}$$



T_k is Lipschitz, if $u\in L^2\left(0,T,H_0^1(\Omega)\right)$ then $T_k(u)\in L^2\left(0,T,H_0^1(\Omega)\right)$.

We denote $S_k(z) = \int_0^z T_k(\tau)d\tau$.



$$S_k(z)=\begin{cases}\frac{z^2}{2}, & |z|\le k\\ -\frac{k^2}{2}+kz, & z>k\\ -\frac{k^2}{2}-kz, & z<-k\end{cases}$$

In particular, we have $|z|\le S_1(z)+\frac{1}{2}$.

Indeed,

$$\begin{array}{lll}\text{If } |z|\le 1 & \text{implies that} & |z|\le \frac{1}{2}|z|^2+\frac{1}{2}=S_1(z)+\frac{1}{2} \text{ (Young's inequality)}\\ \text{If } |z|>1 & \text{implies that} & z=|z|=(z-\frac{1}{2})+\frac{1}{2}=S_1(z)+\frac{1}{2}\\ \text{If } |z|<-1 & \text{implies that} & |z|=-z=(-z-\frac{1}{2})+\frac{1}{2}=S_1(z)+\frac{1}{2}\end{array}$$

Let $\phi=T_k(u_\varepsilon)\in L^2\left(0,T,H_0^1(\Omega)\right)$ be a test function, we obtain

$$\frac{d}{dt}\int_\Omega S_k(u_\varepsilon)dx+\underbrace{\int_\Omega A(t,x)\nabla u_\varepsilon\nabla T_k(u_\varepsilon)dx}_{\ge 0}=\int_\Omega f\,T_k(u_\varepsilon)dx,$$

and by integrating in time, we have

$$\begin{aligned}\int_\Omega S_k(u_\varepsilon(t,x))dx-\int_\Omega S_k(u_0)dx &\le k\int_0^t\int_\Omega f(t,x)dt\,dx,\\ &\le k\|f\|_{L^1(Q_T)}.\end{aligned}$$

Therefore, for $k=1$

$$\int_\Omega S_1(u_\varepsilon(t,x))dx\le\int_\Omega S_1(u_0)dx+\|f\|_{L^1(Q_T)}.$$

We have $S_1(z)=\int_0^z T_1(\tau)d\tau$ so $S_1(z)\le|z|$ because $|T_1(z)|\le 1$, hence

$$\int_\Omega S_1(u_\varepsilon(t,x))dx\le\|u_0\|_{L^1(\Omega)}+\|f\|_{L^1(Q_T)}.$$

Finally, to conclude, we have

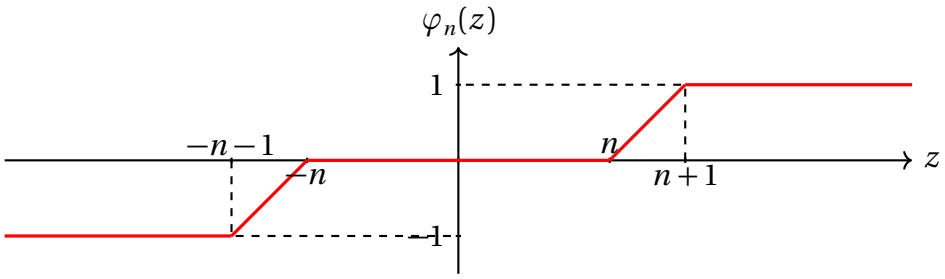
$$\begin{aligned}\int_\Omega|u_\varepsilon(t,x)|dx &\le\int_\Omega S_1(u_\varepsilon(t,x))dx+\tfrac{1}{2}|\Omega|\\ &\le\|u_0\|_{L^1(\Omega)}+\|f\|_{L^1(Q_T)}+\tfrac{1}{2}|\Omega|.\end{aligned}$$

Thus,

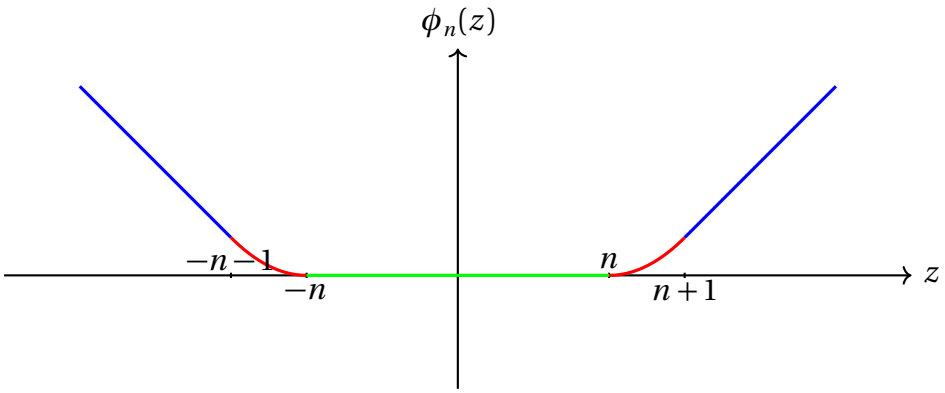
$$\sup_{t\in(0,T)} \|u_\varepsilon(t)\|_{L^1(\Omega)} \le C_0.$$

We now show the estimate on the wavelets:

$$\varphi_n(z) = \begin{cases} 0, & |z| \le n, \\ z-n, & n \le z \le n+1, \\ z+n, & -n-1 \le z \le -n, \\ 1, & z \ge n+1, \\ -1, & z \le -n-1. \end{cases}$$



Let $\phi_n(z) = \int_0^z \varphi_n(\tau)d\tau$, we have $|\phi_n(z)| \le |z|$.



Let $\phi = \varphi_n(u_\varepsilon) \in L^2\left(0,T,H_0^1(\Omega)\right)$ be a test function, we obtain

$$\underbrace{\int_\Omega \phi_n(u_\varepsilon(t,x))dx}_{\ge 0} - \int_\Omega \phi_n(u_{0,\varepsilon}(x))dx + \int_{Q_T} A(t,x)\nabla u_\varepsilon \nabla \varphi_n(u_\varepsilon)dt\,dx = \int_{Q_T} f\varphi_n(u_\varepsilon)dt\,dx,$$

thus

$$\begin{aligned} \alpha \int_{B_n} |\nabla u_\varepsilon|^2 dt\,dx &\le \int_\Omega \phi_n(u_{0,\varepsilon}(x))dx + \|f\|_{L^1(Q_T)} \\ &\le \|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(Q_T)}, \end{aligned}$$

hence the estimate (5.6). ■

Step 3: Compactness result

Proposition 5.3 *Let $u_\varepsilon \in L^2\big(0,T,H_0^1(\Omega)\big)$ satisfying:*

$$\|u_\varepsilon\|_{L^\infty(0,T,L^1(\Omega))} \le C_0, \tag{5.7}$$

$$\int_{B_n} |\nabla u_\varepsilon|^2 \le C_1, \tag{5.8}$$

where $B_n = \{(t,x)\in Q_T;\ n \le |u_\varepsilon(t,x)| \le n+1\}$.

Then, for all $1 \le q < \frac{N+2}{N+1}$, there exists $C>0$ depending on C_0, C_1, $|\Omega|$ and q such that

$$\|u_\varepsilon\|_{L^q\left(0,T,W_0^{1,q}(\Omega)\right)} \le C. \tag{5.9}$$

Proof. We would like to proceed as in the elliptic case and obtain an estimate on ∇u_ε in $L^q(Q_T)$.

Let $1 \le q < 2$, we have

$$\begin{aligned}\int_{Q_T} |\nabla u_\varepsilon|^q \,dt\,dx &= \sum_{n\ge 0}\int_{B_n} |\nabla u_\varepsilon|^q \,dt\,dx,\\ &= \sum_{n=0}^{n_0-1}\int_{B_n} |\nabla u_\varepsilon|^q \,dt\,dx + \sum_{n=n_0}^{\infty}\int_{B_n} |\nabla u_\varepsilon|^q \,dt\,dx.\end{aligned}$$

For $\frac{1}{\frac{2}{q}} + \frac{1}{q'} = 1$, i.e. $\frac{1}{q'} = 1 - \frac{2}{q}$, we apply Hölder's inequality to obtain

$$\begin{aligned}\int_{B_n} |\nabla u_\varepsilon|^q \,dt\,dx &\le \left(\int_{B_n} |\nabla u_\varepsilon|^{q\cdot\frac{2}{q}} \,dt\,dx\right)^{\frac{q}{2}} \left(\int_{B_n} (1)^{q'} \,dt\,dx\right)^{1-\frac{q}{2}},\\ &\le \left(\int_{B_n} |\nabla u_\varepsilon|^2 \,dt\,dx\right)^{\frac{q}{2}} |B_n|^{1-\frac{q}{2}},\\ &\le C_1^{\frac{q}{2}} |B_n|^{1-\frac{q}{2}},\\ &\le C_1^{\frac{q}{2}} (T\times|\Omega|)^{1-\frac{q}{2}}.\end{aligned}$$

Thus,

$$\int_{Q_T} |\nabla u_\varepsilon|^q \,dt\,dx \le C_3(n_0) + C_4 \sum_{n=n_0}^{\infty} |B_n|^{1-\frac{q}{2}}, \tag{5.10}$$

where $C_3(n_0) = C_1^{\frac{q}{2}}(T \times |\Omega|)^{1-\frac{q}{2}}(n_0)$ and $C_4 = C_1^{\frac{q}{2}}$.

We now use the Sobolev embedding $W_0^{1,q}(\Omega) \hookrightarrow L^{q^\star}(\Omega)$ (continuous embedding) with $q^\star$ being the Sobolev exponent, i.e., $\frac{1}{q^\star} = \frac{1}{q} - \frac{1}{N}$ ($q^\star = \frac{qN}{N-q}$) and $q < 2 \leq N$.

Therefore, there exists a constant C_Ω depending on Ω such that

$$C_\Omega \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{q}{q^\star}} \leq \int_\Omega |\nabla u_\varepsilon|^q dx, \tag{5.11}$$

and by integrating in time (5.11), we obtain

$$C_\Omega \int_0^T \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{q}{q^\star}} dt \leq \int_0^T \int_\Omega |\nabla u_\varepsilon|^q dt dx, \tag{5.12}$$

we then note that for all $r \geq 1$, we have $\int_{B_n} |u_\varepsilon|^r dt dx \geq n^r |B_n|$ because $|u_\varepsilon| \geq n$ on B_n and thus

$$|B_n| \leq \frac{1}{n^r} \int_{B_n} |u_\varepsilon|^r dt dx.$$

From estimates (5.10), (5.11) and (5.12), we deduce that

$$C_\Omega \int_0^T \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{q}{q^\star}} dt \leq C_3(n_0) + C_4 \sum_{n=n_0}^{\infty} \frac{1}{n^{r(1-\frac{q}{2})}} \left(\int_{B_n} |u_\varepsilon|^r dt dx \right)^{1-\frac{q}{2}}.$$

We apply Hölder's inequality to the series with

$$p = \frac{1}{1-\frac{q}{2}} = \frac{2}{2-q} \text{ thus, } \frac{1}{p} = 1 - \frac{q}{2}$$

$$\frac{1}{p'} = 1 - \frac{1}{p} = \frac{q}{2},$$

hence

$$\begin{aligned} C_\Omega \int_0^T \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{q}{q^\star}} dt &\leq C_3(n_0) + C_4 \left(\sum_{n=n_0}^{\infty} \frac{1}{n^{r(1-\frac{q}{2})\frac{2}{q}}} \right)^{\frac{q}{2}} \left(\sum_{n=n_0}^{\infty} \left(\int_{B_n} |u_\varepsilon|^r dt dx \right)^{(1-\frac{q}{2})\frac{1}{1-\frac{q}{2}}} \right)^{\frac{2-q}{2}}, \\ &\leq C_3(n_0) + C_4 \left(\sum_{n=n_0}^{\infty} \frac{1}{n^{r(\frac{2-q}{q})}} \right)^{\frac{q}{2}} \left(\sum_{n=n_0}^{\infty} \int_{B_n} |u_\varepsilon|^r dt dx \right)^{\frac{2-q}{2}}. \end{aligned}$$

For the series on the right to converge, we need

$$(C1) \quad ... \quad r(\frac{2-q}{q})>1,$$

we denote

$$\sum_{n=n_0}^{\infty} \frac{1}{n^{r(\frac{2-q}{q})}} = \delta(n_0) \underset{n_0\to+\infty}{\longrightarrow} 0.$$

Thus,

$$C_\Omega \int_0^T \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{q}{q^\star}} dt \le C_3(n_0) + \delta(n_0) \left(\int_0^T \int_\Omega |u_\varepsilon|^r dt dx \right)^{\frac{2-q}{2}}, \tag{5.13}$$

with $\delta(n_0) \underset{n_0\to+\infty}{\longrightarrow} 0$ and $C_3(n_0) \underset{n_0\to+\infty}{\longrightarrow} +\infty$.

We need to make $\int_\Omega |u_\varepsilon|^{q^\star} dx$ appear, for this we choose $1<r<q^\star$, we have $u_\varepsilon(t,.)\in L^1(\Omega)\cap L^{q^\star}(\Omega)$ and by interpolation

$$\frac{1}{r} = \frac{\theta}{1} + \frac{1-\theta}{q^\star},$$

we have

$$\|u_\varepsilon(t,.)\|^r_{L^r(\Omega)} \le \|u_\varepsilon(t,.)\|^{r\theta}_{L^1(\Omega)} . \|u(t,.)\|^{r(1-\theta)}_{L^{q^\star}(\Omega)},$$

since u_ε satisfies (5.7), we have

$$\|u_\varepsilon(t,.)\|^r_{L^r(\Omega)} \le C_0^{r\theta} . \|u_\varepsilon(t,.)\|^{r(1-\theta)}_{L^{q^\star}(\Omega)}.$$

Thus, by integrating in time,

$$\int_0^T \int_\Omega |u_\varepsilon|^r dt dx \le \int_0^T \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{r(1-\theta)}{q^\star}} dt.$$

To estimate the term on the left of (5.13), we take

$$(C2) \quad ... \quad \frac{r(1-\theta)}{q^\star} = \frac{q}{q^\star},$$

which gives:

$$C_\Omega \int_0^T \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{q}{q^\star}} dt \leq C_3(n_0) + \delta(n_0) \left[\int_0^T \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{q}{q^\star}} dt \right]^{\frac{2-q}{2}}, \tag{5.14}$$

we have $\frac{2-q}{2} \leq 1$. For n_0 large enough, that is, ($\delta(n_0)$ small enough so that $C_\Omega - \delta(n_0) = \frac{C_\Omega}{2} > 0$, we conclude that

$$\int_0^T \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{q}{q^\star}} dt = \|u_\varepsilon\|_{L^q\left(0;T,L^{q^\star}(\Omega)\right)} \leq C.$$

And thus from estimate (5.10) and the choice of r,

$$\begin{aligned} \int_{Q_T} |\nabla u_\varepsilon|^q dt dx &\leq C_3(n_0) + \delta(n_0 \left[\int_0^T \left(\int_\Omega |u_\varepsilon|^{q^\star} dx \right)^{\frac{q}{q^\star}} dt \right]^{\frac{2-q}{2}} \\ &\leq C \end{aligned}$$

and

$$\|\nabla u_\varepsilon\|_{L^q(0;T,L^q(\Omega))} \leq C.$$

Let us now return to the conditions on q, we have

Condition C2: gives $r(1-\theta) = q$ and $\frac{1}{r} = \theta + \frac{1-\theta}{q^\star} = \theta\left(1 - \frac{1}{q^\star}\right) - \frac{1}{q^\star}$.

we have $1 = r\theta + \frac{r(1-\theta)}{q^\star}$, so $r\theta = 1 - \frac{q}{q^\star}$, and finally,

$$\begin{aligned} r &= q + r\theta \\ &= q + 1 - \frac{q}{q^\star} \\ &= q + 1 - \frac{q}{\frac{qN}{N-q}} \\ &= q + 1 - \frac{N-q}{N} \\ &= q\frac{N+1}{N} \end{aligned}$$

Condition C1: now writes

$$\begin{aligned} & q\frac{N+1}{N}\frac{2-q}{q} > 1 \\ \Leftrightarrow \quad & \frac{N+1}{N}(2-q) > 1, \\ \Leftrightarrow \quad & 2\frac{N+1}{N} - 1 > q\frac{N+1}{N}, \\ \Leftrightarrow \quad & q < \frac{N+2}{N+1}. \end{aligned}$$

Conclusion Under the sole condition $1 \le q < \frac{N+2}{N+1}$,

$$\|u_\varepsilon\|_{L^q\left(0;T,W_0^{1,q}(\Omega)\right)} \le C.$$

■

Proposition 5.4 (Estimate on $(\partial_t u_\varepsilon)$) .
$(\partial_t u_\varepsilon)_\varepsilon$ is bounded in $L^1\left(0;T,W^{-1,q}(\Omega)\right)$ independently of ε.

Proof. To obtain an estimate on $\partial_t u_\varepsilon$, we consider a function $\varphi \in L^\infty(Q_T) \cap L^{q'}\left(0;T,W_0^{1,q'}(\Omega)\right)$

$$\begin{aligned}
|\langle \partial_t u_\varepsilon, \varphi\rangle| &\le |\int_{Q_T} A\nabla u_\varepsilon \nabla d\,t\,d\,x| + \int_{Q_T} |f_\varepsilon||\varphi| d\,t\,d\,x \\
&\le \|\nabla u_\varepsilon\|_{L^q(0;T,L^q(\Omega))} \|\nabla\varphi\|_{L^{q'}\left(0;T,L^{q'}(\Omega)\right)} + \|f_\varepsilon\|_{L^1(Q_T)} \|\varphi\|_{L^\infty(\Omega)} \\
&\le C\left(\|\nabla\varphi\|_{L^{q'}\left(0;T,L^{q'}(\Omega)\right)} + \|\varphi\|_{L^\infty(\Omega)}\right).
\end{aligned}$$

Thus $(\partial_t u_\varepsilon)_\varepsilon$ is bounded in $\left(L^\infty(Q_T) \cap L^{q'}\left(0;T,W_0^{1,q'}(\Omega)\right)\right)'$.

We have

$$L^\infty\left(0;T,W_0^{1,q'}(\Omega)\right) \subset L^\infty(Q_T) \cap L^{q'}\left(0;T,W_0^{1,q'}(\Omega)\right),$$

since $q' > N$, thus

$$\begin{aligned}
\left(L^\infty(Q_T) \cap L^{q'}\left(0;T,W_0^{1,q'}(\Omega)\right)\right)' &\subset \left(L^\infty\left(0;T,W_0^{1,q'}(\Omega)\right)\right)' \\
&\subset L^1\left(0;T,W^{-1,q}(\Omega)\right)
\end{aligned}$$

■

Convergence: the sequence $(u_\varepsilon)_\varepsilon$ is bounded in

$$E = \left\{u_\varepsilon \in L^q\left(0;T,W_0^{1,q}(\Omega)\right),\ \partial_t u_\varepsilon \in L^1\left(0;T,W^{-1,q}(\Omega)\right)\right\}$$

and $W_0^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$ with compact embedding. Then $(u_\varepsilon)_\varepsilon$ is relatively compact in $L^1(0;T,L^q(\Omega))$. Thus, up to extracting a subsequence still denoted $(u_\varepsilon)_\varepsilon$, we have

$$\begin{aligned}
&u_\varepsilon \longrightarrow u \text{ strongly in } L^1(0;T,L^q(\Omega)), \\
\text{and} \quad &u_\varepsilon \rightharpoonup u \text{ weakly in } L^q\left(0;T,W_0^{1,q}(\Omega)\right).
\end{aligned}$$

The remaining problem is the continuity of $u \in \mathscr{C}^0\left(0;T,L^1(\Omega)\right)$.

Proposition 5.5 *The sequence* $(u_\varepsilon)_\varepsilon$ *is Cauchy in* $\mathscr{C}^0\big(0;T,L^1(\Omega)\big)$.

Proof. Let $(u_\varepsilon)_\varepsilon$ be the solution of (5.3) associated with $u_{0,\varepsilon}$ and f_ε and $(u_{\varepsilon'})_{\varepsilon'}$ be the solution of (5.3) associated with $u_{0,\varepsilon'}$ and $f_{\varepsilon'}$

Let $w_{\varepsilon,\varepsilon'} = u_\varepsilon - u_{\varepsilon'}$ satisfy:

$$\partial_t w_{\varepsilon,\varepsilon'} - \operatorname{div}\big(A\nabla w_{\varepsilon,\varepsilon'}\big) = F_{\varepsilon,\varepsilon'} = f_\varepsilon - f_{\varepsilon'},$$

we multiply by $T_1(w_{\varepsilon,\varepsilon'})$,

$$\frac{d}{dt}\int_\Omega S_1(w_{\varepsilon,\varepsilon'})dx + \underbrace{\int_\Omega A(t,x)\nabla w_{\varepsilon,\varepsilon'}\nabla T_1(w_{\varepsilon,\varepsilon'})dx}_{\geq 0} = \int_\Omega F_{\varepsilon,\varepsilon'}T_1(w_{\varepsilon,\varepsilon'})dx,$$

integrating in time,

$$\int_\Omega S_1(w_{\varepsilon,\varepsilon'})dx \leq \int_\Omega S_1(w^0_{\varepsilon,\varepsilon'})dx + \int_{Q_T}|F_{\varepsilon,\varepsilon'}|dx.$$

with $w^0_{\varepsilon,\varepsilon'} = u_{0,\varepsilon} - u_{0,\varepsilon'}$. We have $S_1(z) = \int_0^z T_1(\tau)d\tau$ so $S_1(z) \leq |z|$, hence

$$\int_\Omega S_1(w_{\varepsilon,\varepsilon'})dx \leq \int_\Omega |u_{0,\varepsilon} - u_{0,\varepsilon'}|dx + \int_{Q_T}|f_\varepsilon - f_{\varepsilon'}|dx.$$

The sequences $(u_{0,\varepsilon})_\varepsilon$ and $(f_\varepsilon)_\varepsilon$ are convergent in $L^1(\Omega)$ and $L^1(Q_T)$ respectively, hence they are Cauchy, so

$$\int_\Omega S_1(w_{\varepsilon,\varepsilon'})dx \underset{\varepsilon,\varepsilon'\to 0}{\longrightarrow} 0.$$

Finally, we have

$$\begin{aligned}\int_\Omega |w_{\varepsilon,\varepsilon'}|dx &= \int_{|w_{\varepsilon,\varepsilon'}|\leq 1}|w_{\varepsilon,\varepsilon'}|dx + \int_{|w_{\varepsilon,\varepsilon'}|>1}|w_{\varepsilon,\varepsilon'}|dx \\ &\leq \left(\int_{|w_{\varepsilon,\varepsilon'}|\leq 1}|w_{\varepsilon,\varepsilon'}|^2dx\right)^{\frac{1}{2}}\left(\int_{|w_{\varepsilon,\varepsilon'}|\leq 1}|1dx\right)^{\frac{1}{2}} + \int_{|w_{\varepsilon,\varepsilon'}|>1}|w_{\varepsilon,\varepsilon'}|dx.\end{aligned}$$

We have

$$\frac{|z|}{2}\mathbb{1}_{|z|>1} \leq \left(\frac{|z|}{2} + \frac{|z|-1}{2}\right)\mathbb{1}_{|z|>1} = S_1(z)\mathbb{1}_{|z|>1},$$

and

$$\frac{|z|}{2}\mathbb{1}_{|z|\leq 1} = S_1(z)\mathbb{1}_{|z|\leq 1},$$

thus,

$$\int_\Omega |w_{\varepsilon,\varepsilon'}|dx \leq \sqrt{|\Omega|}\left(\int_{|w_{\varepsilon,\varepsilon'}|\leq 1} 2S_1(w_{\varepsilon,\varepsilon'})dx\right)^{\frac{1}{2}} + \int_{|w_{\varepsilon,\varepsilon'}|>1} 2S_1(w_{\varepsilon,\varepsilon'})dx \underset{\varepsilon,\varepsilon'\to 0}{\longrightarrow} 0.$$

Finally, we pass to the limit as $\varepsilon \longrightarrow 0$ in the formulation:

$$\int_\Omega u_\varepsilon(T,x)\phi(T,x)dx - \int_\Omega u_{0,\varepsilon}\phi(0,x)dx - \int_{Q_T} u_\varepsilon\partial_t\phi(t,x)dtdx + \int_{Q_T} A\nabla u_\varepsilon\nabla\phi\, dtdx = \int_{Q_T} f_\varepsilon\phi(t,x)dtdx,$$

with $\phi \in \mathscr{C}^0\left(0;T;W_0^{1,q'}(\Omega)\right)\cap\mathscr{C}^1\left(0;T;L^{q'}(\Omega)\right)$

■

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