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by
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The effect of adding a path between two vertices on the connected domination number

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Abstract

Let $G = (V, E)$ be a simple graph. a subset D of V is a dominating set of G, if every vertex in V is either an element of D or adjacent to an element in D. the minimum cardinality of a dominating set of G is called the domination number which is denoted by $\gamma(G)$. by adding additional constraints on the dominating set we get another types of domination, such as if we impose that the subgraph induced by the dominating set is edgless we get the independent domination of the domination number $i(G)$. and if the subgraph induced by the domination set is connected we get the connected domination of the parameter $\gamma_c(G)$. for any parameter of domination $\mu(G) \in \{i(G), \gamma(G), \gamma_c(G), ...\}$, a dominating set of cardinality $\mu(G)$, which verify the property is called a $\mu - set$ of G. a vertex $u \in V(G)$ is said to be μ–good if it belongs to some μ-sets of G, and μ-bad otherwise. the graph G is said to be μ-excellent if every vertex of $V(G)$ belongs to a $\mu - set$ of G. in this work we are interested in the behavior of domination number $\gamma(G)$ and the connected domination number $\gamma_c(G)$ of an arbitrary graph, when a change is made up by two operations, subdivising an edge and adding a path between two distinct vertices in the graph. we will show how both parameters are affected by those two operations and how does they change, and the effect of the characteristics of vertices on the changes with the relation between them.

Résumé

Soit $G = (V, E)$ un graph simple. un sous-ensemble D de V est dit dominant de G si tout sommet de $V - D$ possède au moins un voisin dans D. le cardinal minimum d'un ensemble dominant de G, appelé nombre de domination de G, est noté $\gamma(G)$. en ajoutant des conditions supplémentaires sur l'ensemble dominant, nous obtenons des autres types de domination, comme se nous imposions que le sous-graphe induit par l'ensemble dominant ne contient pas des arêtes (indépendante), on a la domination indépendante avec le nombre de domination $i(G)$. et si le sous graph induit par l'ensemble dominant est connexe, on a la domination connexe avec le nombre de domination connexe $\gamma_c(G)$. pour tout paramètre de domination $\mu(G) \in \{\gamma(G), i(G), \gamma_c(G), ...\}$, un ensemble dominant de cardinalité $\mu(G)$, qui vérifie que la propriété est appelée μ-ensemble de G. un sommet est dit μ-bien s'il appartient à certains μ-ensembles de G, et μ-mal sinon. le graphe est dit μ-excellent si chaque sommet de $V(G)$ appartient à un μ-ensemble. dans cette mémoire nous intéressons au l'attitude du nombre de domination $\gamma(G)$ et du nombre de domination connexe $\gamma_c(G)$ d'un graph arbitraire, lorsqu'un changement est composé de deux opérations, la subdivision, et l'ajout d'un chemin entre deux sommets distincts du graphe. nous montrerons comment il est affecté par ces deux opérations et comment changent-elles, et l'effet des caractéristiques des sommets de G sur les changements avec la relation entre eux.

ملخص

ليكن $G = (V, E)$ بيانا بسيطا. مجموعة فرعية ما D من V تدعى مجموعة مهيمنة للبيان G اذا كان كل رأس من V عنصرا في D او مجاورا لعنصر ما في D. اصغر عدد لعناصر مجموعة مهيمنة D يدعى بعدد الهيمنة و يرمز له ب $\gamma(G)$. بوضع شروط اضافية على المجموعة المهيمنة نتحصل على انواع اخرى من الهيمنة، مثل اذا فرضنا شرط ان البيان الفرعي المستحث بواسطة المجموعة المهيمنة لا يحتوي على حواف (مستقل)، نتحصل على الهيمنة المستقلة ذات العدد المهيمن $i(G)$. و اذا كان المخطط الفرعي المستحث بواسطة المجموعة المهيمنة مرتبط، نتحصل على الهيمنة المرتبطة ذات العدد المهيمن $\gamma_c(G)$. من اجل كل الأعداد $\mu(G) \ni \{..., i(G), \gamma_c(G), \gamma(G)\}$، مجموعة ما مهيمنة ذات عدد عناصر يساوي $\mu(G)$ بحيث تحقق احدى الخاصيات تدعى μ-مجموعة للبيان G. رأس $u \ni V(G)$ يدعى μ-جيد اذا كان ينتمي الى بعض μ-مجموعات البيان G، و μ-سيئ ما عدا ذالك. البيان G يدعى μ-ممتاز اذا كانت كل الرؤوس في $V(G)$ تنتمي الى μ-مجموعة للبيان G. في هذا العمل نحن مهتمون بدراسة سلوك العدد المهيمن $\gamma(G)$ و العدد المهيمن المرتبط $\gamma_c(G)$ في بيان عشوائي، عندما يحدث تغير ما فيه بواسطة عمليتين، تقسيم حافة و اضافة مسار بين رئسين مختلفين غير متجاورين في البيان، سوف نظهر تأثرهما بواسطة هاتين العمليتين و كيف يتغيران، و تأثير خاصيات الرؤوس على هذا التغير و العلاقة بينهما.

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Mohammed Lamine

Introduction

Every one of us have seen or at least heard about networks of communication, road plans, military radar network, or even a map of galaxies, also the meaning of a graph is known for a wide population in a typical way, however, between the common concept of a diagram which is formed by 'points' that are connected by 'paths', and the mathematical theory of graphs, there is a wide and deep elaboration of concepts.

Generally graphs theory is the study of a mathemathical structures used to model pairwise relations between objects, it is an important mathematical field nowadays, which links discrete mathematics and computer science, concrete problems can be modeled by using an abstract methods which are the core of this field.

Among the most studied branches in the graphs theory, the domination in graphs, with more than 120 types of domination and more than 1200 references, domination constitutes a flourishing field of research which contains an enormous number of problems which are until now open or have not even been studied yet, in this thesis we will use two types of domination, the standard one, and the connected domination, basically the domination problem was studied from the 1960s onwards by Claude Berge [1] and Oystein Ore [2], the domination sets are of practical interest in several areas, in wirless network, it used to find efficient routes within ad hoc mobile networks, they also been used in designing secure systems for electrical grids and a large group of concrete problems.

In this work we will use two operations to study the behavior of two domination types, in general, a subdivision is an expansion of 'points' and 'paths' of a graph [3], the concept of excellence in graphs was introduced by Fricke et al [4], also the concept of critical graphs was introduced by Sampathkumar et al [5] and Walikar et al [6].

The first chapter is divided into three sections, in the first one we will introduce the concept and foundations of graphs theory, and also we will define all the used notions during this thesis, second one is about the concept of subdivision and path addition operations on graphs, the third one is dedicated to a general presentation of the domination in graphs, we will start by a brief historical events about the studies on the dominating sets, with concrete illustrations, and at last, an explicite idea about the connected domination parametre.

The second chapter is dedicated to the studies of 'Changing and unchanging of the domination number of a graph: path addition numbers' of Mister Vladimir SAMODIVKIN [7], it will contain all the results about the behavior of the domination parametre, it is devided into three sections, the first one is about basic notations on the numbers of subdivision and path addition with respect to domination number, also some needed characteristics of vertices, and the concept of excellence in graphs, the second section is about the effect of subdivision on the domination number, generally we will study and show the effect of subdivision of an edge k times on the domiating set and domination number, such that $k \in \mathbb{N}^*$, we will demonstrate the results by provinding necessary and sufficient conditions, and illustrations, the third one is similar to the seconde section, we will study and show the behavior of the domination number when a path of length k is added between two distinct vertices in the graph, with $k \in \mathbb{N}$, again we will demonstrate the results by providing necessary and sufficient conditions, and illustrations, generally this chapter is all about the behavior of the domination number

when a change is made up in the graph by each operation.

The third chapter is my personal work on the behavior of the connected domination number, similarly to the seconde chapter, it contains the results about the changing and unchanging connected domination number, it is divided also into three sections, the first one is about basic notations on the numbers of subdivision and path addition with respect to connected domination number along with the characteristics of vertices that are included in the connected dominating sets, in the second one we will study and show the behavior of the connected domination number when an edge is subdivide k times, where $k \in \mathbb{N}^*$, we will also show the cases and conditions when the connected domination number is increased, decreased or unchanged, by demonstrating the necessary and sufficient conditions, and presenting illustration, in the third section, the behavior of the connected domination number will be studied and shown when a path of length k is added to the graph between two distinct vertices, with $k \in \mathbb{N}$, as same as the third section of the second chapter, we will demonstrate the results by providing necessary and sufficient conditions, and illustrations, for the changes and unchanges of the connected domination number, all the graphs in this thesis are undirected simple graphs.

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Chapter 1

Definitions and concepts

Historically the first paper about graphs theory was written by Leonhard Euler [8] on 'The seven bridges of konigsberg' and published in 1736, and the word 'graph' was first used in this sense by James Joseph Sylvester [9] in 1878, the first explicite textbook about it was written by Denes Konig [10], and published in 1936, for further infromations we refer the reader to the books of Claude Berge [1] and Oystein Ore [2].

1.1 Concept of a graph

An undirected graph G is defined by two sets E and V, such that E is the set of 'Edges' and V is the set of 'Vertices', it is an unordered pair $G = (V, E)$ [11], the vertices x and y are called the endpoints of the edge xy, the edge is said to join x and y to be incident on x and on y, any edge is associated with two distinct vertices, otherwise if the edge joins a vertex to it self is a 'loop', multiple edges are two or more edges that join the same two vertice, a simple graph is an undirected graph which have no multiple edges or loops, in other words every two distinct vertices of G are joint by a single edge, in simple graphs the sets E and V are denoted by:

- $V = \{v_1, v_2, v_3..., v_k\}$
- $E = \{v_i v_j \mid (v_i, v_j) \in V^2 \wedge i \neq j\}$

E and V are taken to be finite, moreover V is always assumed to be non empty set, but E is allowed to be an empty set, in this thesis all the graphs are considered to be simple graphs.

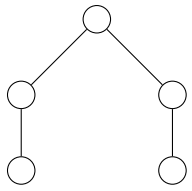


Figure 1.1: A simple graph with five vertices and four edges.

1.1.1 The order and the size of graph

- The order of a graph $G = (V, E)$ is the number of its vertices, denoted by $\mid V \mid = n$.
- The size of a graph $G = (V, E)$ is the number of its edges, denoted by $\mid E \mid = m$.

1.1.2 Degree of a vertex

In a simple graph $G = (V, E)$, the degree of a vertex v is the number of edges that are incident to the vertex, it is denoted by $deg(v)$ or $d_G(v)$.

- The maximum degree of a graph $G = (V, E)$, denoted by $\Delta(G)$ such that:
  $\Delta(G) = max\{d_G(v) \mid v \in V(G)\}$

- The minimum degree of a graph $G = (V, E)$, denoted by $\delta(G)$ such that:
  $\delta(G) = min\{d_G(v) \mid v \in V(G)\}$

Theorem 1.1 *Let $G = (V, E)$ be a simple graph, then $\sum_{v \in V} d_G(v) = 2 \mid E \mid$.*
Proof.
$\sum_{v \in V} d_G(v) = \sum_{v \in V} d_G(v) + \sum_{v \in V} d_G(v) = \mid E \mid + \mid E \mid = 2 \mid E \mid$.

1.1.3 Particular graphs

Induced subgraph

Let $S \in V$ be a subset of vertices of a simple graph G, the induced subgraph $G[S]$ of G is the graph whos vertices set is S and its edges set consist of all edges in E that have both endpoints in S.

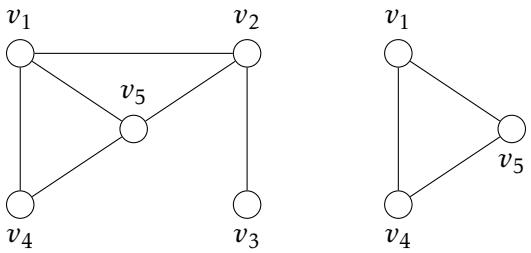


Figure 1.2: A graph G (left) and its induced subgraph $G[S]$ (right), $S = \{v_1, v_4, v_5\}$.

Path

In a simple graph $G = (V, E)$, a path is a finite sequence of edges and vertices, in which the edges joins the vertices, a path is denoted by $P_k = (u_0, e_1, u_1, e_2, u_2, ..., u_{k-1}, e_k, u_k)$, for all i, $1 \leqslant i \leqslant k$, such that $u_0 \neq u_k$ and the length k of the path equals the number of its edges.

Example 1.1.1 *In the graph of figure 1.3:*

- $P = (v_1, e_1, v_2, e_2, v_3, e_7, v_4, e_8, v_6, e_4, v_5)$ *is a path of length* $k = 5$.

Cycle

In a simple graph $G = (V, E)$, a cycle is a non empty path in which its initial vertex is also its terminal vertex, such that $C_k = (v_0, e_1, v_1, e_2, ..., e_k, v_0)$, the length k of the cycle is equals to its number of edges, a cycle is odd or even if its length k is odd or even.

Example 1.1.2 *In the graph of figure 1.3:*

- $C = (v_1, e_1, v_2, e_2, v_3, e_3, v_6, e_4, v_5, e_5, v_1)$ *is even cycle of length* $k = 5$.

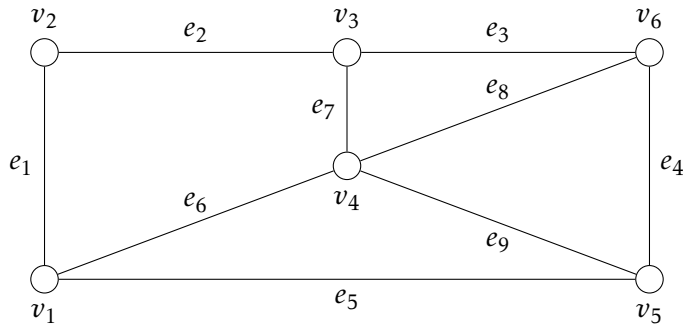


Figure 1.3: A simple graph with six vertices and nine edges.

Complete graph

A complete graph K_n is a simple graph of order $n \geq 3$ and size $m = n(n-1)/2$, such that $\forall u \in V(K_n)$, $d_{K_n}(u) = n-1$.

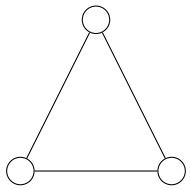


Figure 1.4: A complete graph K_3.

Complete bipartite graph

A complete bipartite graph $K_{p,q}$ is a simple graph of order $n = p+q$ and size $m = pq$ with $p, q \geq 2$, and $V(K_{p,q}) = X \cup Y$, such that $\forall u \in X$, $\forall v \in Y$, then $uv \in E(K_{p,q})$.

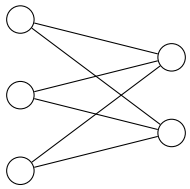


Figure 1.5: A complete bipartite graph $K_{3,2}$.

Wheel graph

A wheel W_n graph is a simple graph of order $n \geq 4$ and size $m = 2(n-1)$, is formed by a connected single universal vertex to $n-1$ vertex of a cycle C_{n-1}.

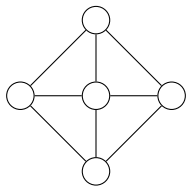


Figure 1.6: A wheel graph W_5.

Clique

A clique C, in a simple graph $G = (V, E)$ is a subset of vertices S of $V(G)$, such that $\forall u, v \in S$, $uv \in E(G)$, a maximal clique cannot be extended by adding one more adjacent vertex, a maximum clique of G is a clique which has the maximum number of vertices, the clique number $\omega(G)$ equals the number of vertices in the clique.

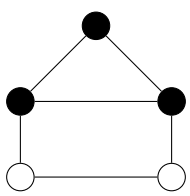


Figure 1.7: A clique of cardinality $\omega(G) = 3$.

Star

A star S_k is the complete bipartite graph $K_{1,k}$, of order $n = k+1$ and size $m = k$, such that $\exists w \in V(S_k)$, $d_{S_k}(w) = n-1$, and $\forall u \in V(S_k) - \{w\}$, $d_{S_k}(u) = 1$.

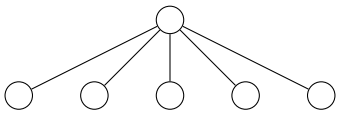


Figure 1.8: A star S_5.

Tree

A tree $T = (V, E)$ is an acyclic connected graph of order n and size $m = n-1$, and $\forall u, v \in V(T)$, there is a path between any two distinct vertices of T.

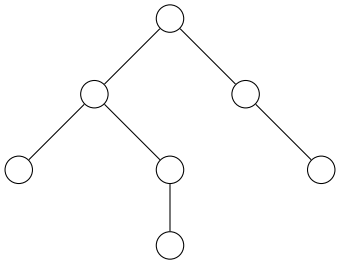


Figure 1.9: A tree of 7 vertices and 6 edges.

Edgless graph

An edgless graph $G = (V, E)$ is a simple graph of order n, such that $E(G) = \emptyset$.



Figure 1.10: An edgless graph of order $n = 3$.

Circulant graphs C(n,k)

A circulant graph $C(n,k)$ is a simple graph with a set of vertices $V = \{u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}\}$ and a set of edges $E = \{u_t u_m, u_t v_t, v_t v_r \mid t, m, r \in \mathbb{N}\}$, such that $t = i \bmod n$, $m = i+1 \bmod n$, $r = i+k \bmod n$, and $i \in \{0, 1, ..., n-1\}$ with $k \leq \lfloor n/2 \rfloor$.

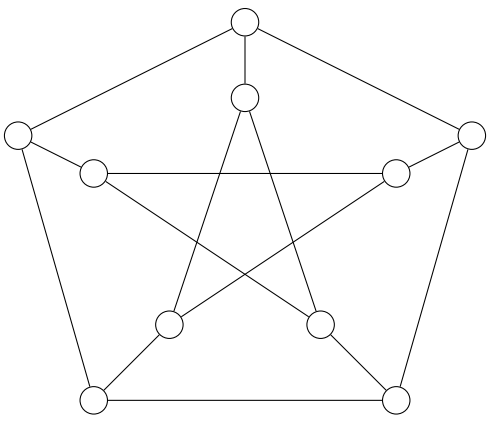


Figure 1.11: Petersen graph is a circulant graph of $C(5, 2)$.

1.1.4 Connectivity

A graph $G = (V, E)$ is said to be connected, if every pair of vertices in the graph is connected by a path (for every pair of vertices $u, v \in V(G)$, $u \backsim v$ means that G contains a path which connects u and v, and $\backsim$ is a relation of equivalence), a graph is connected when it has exactly one connected component, if there is at least one isolated vertex in the graph G, then G is a disconnected graph.

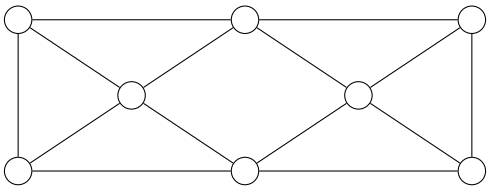


Figure 1.12: The graph G is a connected graph and has exactly one connected compnent.

Connected components

A connected component is a maximal connected subgraph by inclination of G, each vertex belongs to exactly one connected component.



Figure 1.13: The graph G is disconnected, and contains two connected compnents.

1.1.5 Neighborhood

In a graph $G = (V, E)$, two vertices u and v are adjacent, then u and v are connected by an edge uv, the neighborhood of a vertex v is the subgraph of G induced by all vertices adjacent to v, there are two types of neighborhoods:

- Open neighborhood of v, denoted by $N_G(v)$, such that: $N_G(v) = \{u \in V, uv \in E\}$, it includes all the adjacent veritces of v in G, but not v it self.
- Closed neighborhood of v, denoted by $N_G[v]$, such that: $N_G[v] = N_G(v) \cup \{v\}$, it includes all the adjacent veritces of v in G, and v it self.

Neighborhood of a set

For a set A of vertices, the neighborhood of A is the union of all neighborhoods of its vertices, denoted by $N_G(A) = \cup_{v \in A} N_G(v)$, and $N_G[A] = N_G(A) \cup A$.
The private neighbor set of a vertex u with respect to a set S, denoted by $pn[v, S]$, that is $pn[v, S] = N[v] - N[S - \{v\}]$, and each $u \in pn[v, S]$ is called a private neighbor of v with respect to S.

Example 1.1.3 *In the graph G of figure 1.14: Let be* $A = \{v_4, v_6\}$

- $N_G(v_1) = \{v_2, v_3, v_4\}$.
- $N_G(v_7) = \emptyset$.
- $N_G[v_1] = \{v_1, v_2, v_3, v_4\}$.
- $N_G[v_7] = \{v_7\}$.

- $N_G(A) = \{v_1, v_3\}$.
- $N_G[A] = \{v_1, v_3, v_4, v_6\}$.
- *Let be* $S = \{v_1, v_2\}$, $pn[v_1, S] = \{v_4\}$.

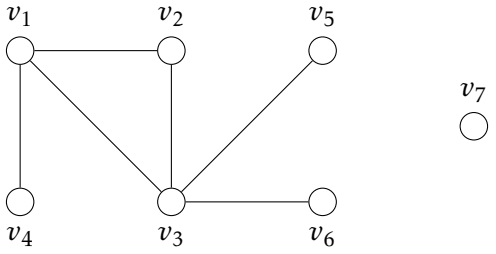


Figure 1.14: A graph G with seven vertices and six edges.

1.1.6 Graph parameters

Independence

Definition 1.1 *Let G be a graph, a subset S of $V(G)$ is an independent set if the subgraph $G[S]$ contains no edges ($\Delta(G[S]) = 0$), or is K_2-free.*

Definition 1.2 *An independent set S of a graph G is said to be maximal if for each vertex $u \in (V - S)$, $S \cup \{u\}$ is not an independent set.*

Definition 1.3 *The independence number $\alpha(G)$ of a graph G, is the maximum cardinality of a maximal independent set of G, the lower independence number $i(G)$ equals the minimum cardinality of a maximal independent set of G, an independent set of maximum cardinality is called $\alpha -$ set of G, notice that a graph G can have more than one $\alpha -$ set.*

Theorem 1.2 *An independent set S of a graph G is said to be maximal if and only if for every vertex $u \in (V - S)$, u is adjacent at least to one vertex in S.*

Matching

Given a graph $G = (V, E)$, a matching M of G is a set of pairwise non adjacent edges, that is, M is an independent set of edges of G, a maximum matching is matching of maximal size, $\beta(G)$ is the matching number, it equals the maximum cardinality of a maximal matching set of G, a matching set M is said to be maximal and $\mid M \mid = \beta(G)$ if and only if for any edge $e \in E - M$, $M \cup \{e\}$ is not a matching set, a vertex is matched (or saturated) if it is an endpoint of one of the edges in the matching set M, otherwise, the vertex is unmatched, if all vertices of a graph G are saturated by a matching set M, then M is a perfect matching with $\beta(G) = n/2$.

Vertex cover

A vertex covering of a graph $G = (V, E)$ is a set T of vertices which covers all the edges of G, in other words, every vertex in T is incident to at least one edge in G, a minimal set of vertex covering is of a minimum cardinality $\tau(G)$, and T is minimal if and only if for every vertex $v \in T$, $T - \{v\}$ is not a vertex covering set, the complement set $V - T$ of a vertex covering set T is an independent set.

1.2 Operations on graphs

1.2.1 Edge subdivision

Let $G = (V, E)$ be a simple graph, the edge subdivision operation [3] for an edge $e = uv \in E(G)$, is the removal of the edge e from $E(G)$, with the addition of w to $V(G)$ and the two edges $e' = uw$ and $e'' = wv$ to $E(G)$, this operation generates a new graph H, such that $H = (V \cup \{w\}, (E - \{e\}) \cup (\{e', e''\}))$, formally, the new obtained graph from the subdivision of an edge in G is denoted by $G_{u,v,k}$, where k is the number of the vertices of the subdivision.

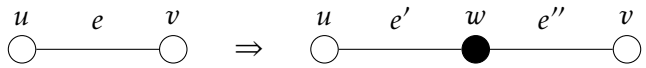


Figure 1.15: The subdivision of the edge e in the graph P_2.

1.2.2 Path addition

In a graph $G = (V, E)$ path addition is an operation that adds to a graph G a disjoint path between two distinct non adjacent vertices in G, it can be considered as a generalisation of the edge addition operation, the new optained graph has a supplementary edges and vertices, formally, let u and v be two distincet non adjacent vertices in a graph G, and let be the path $P_k : x_1, x_2, ..., x_k$ for $k \geq 0$, where the vertices u and v are the endpoints of the path, the new obtained graph from the disjoint union of G and P_k is denoted by $G_{u,v,k}$, where k is the number of the vertices in the added path, and P_0 is an edge e.

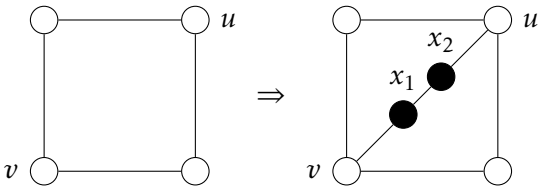


Figure 1.16: For $k = 2$, the graph $G_{u,v,2}$ is obtained from the disjoint union of P_2 and C_4.

1.3 Domination in graphs

The origins of domination topics backs to a chess problem, generally it is about dominating the chess pieces of the opponent by the minimum number of your pieces, the first idea is from the 16th century in india [12], in the 1860s, chess enthusiasts in Europe considered the problem of determing the minimum number of queens that can be placed on a standard 8^2 chessboard so that all the squares are either can be attacked or occupied by a queen, Dragan [13] studied the minimum number of queens for an n^2 chessboard and it is unknown untill the day, the study of dominating sets in graphs began basically in the 1960s by Claude Berge [1] and Oystein Ore [2], they established and defined many new concepts for the first time.

Definition 1.4 *Let $G = (V, E)$ be a simple graph, a set $D \subset V$ of vertices is called a dominating set, if every vertex $v \in V$ is either an element of D, or is adjacent to an element of D.*

Definition 1.5 *Let $G = (V, E)$ be a simple graph, a dominating set D to be minimal, if there is no proper subset $S \subset D$, such that S is a dominating set.*

There are different ways to define the concept of a dominating set, in a simple graph $G = (V, E)$, a set $D \subset V$ of vertices is a dominating set if and only if:

- For every vertex $v \in V$, $| N[v] \cap D | \geq 1$.

- For every vertex $v \in V - D$, $| N(v) \cap D | \geq 1$.
- $| N[D] | = V$

Theorem 1.3 [2] *Let D be a dominating set of a simple graph $G = (V, E)$, then D is a minimal dominating set if and only if for each vertex $u \in D$, one of the following two conditions holds:*

- *u is an isolate of D.*
- *There exists a vertex $v \in V - D$, for which $N(v) \cap D = \{u\}$.*

Theorem 1.4 [2] *If $G = (V, E)$ is a simple graph with no isolated vertex, of order $n \geq 2$, and D is a minimal dominating set of G, then $V - D$ is a dominating set of G.*

Definition 1.6 *The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G, a dominating set of a minimum cardinality is called a $\gamma - set$ of G, notice that a graph G can have more than one $\gamma - set$.*

Theorem 1.5 [14] *Every simple graph $G = (V, E)$ without an isolated vertex, has a minimum dominating set D ($\gamma - set$), in which for every vertex $u \in D$, there exists a vertex $w \in V - D$, such that $N(w) \cap D = \{u\}$.*

Remarks

- A subset $M \subset D$ is not necessarily a dominating set of G, but any super set $S \supseteq D$ is a dominating set of G.
- D contains all the isolated vertices of G.
- $V(G)$ can be considered as a dominating set of G, and any dominating set contains at least one vertex, then $1 \leq \gamma(G) \leq | V |$.

Theorem 1.6 [1] *Let $G = (V, E)$ be a simple graph, then an independent set S of G is a maximal set if and only if S is a dominating set of G.*

Theorem 1.7 [1] *Every independent set of a maximum cardinality of G is a minimal dominating set.*

1.3.1 Bounds on the domination number

The bounds are useful to determine the minimum cardinality for the domination number $\gamma(G)$ for arbitrary graph, because the set of vertices $V(G)$ can be considered as a dominating set of cardinality n, and any dominating set includs at least one vertex, hence $1 \leq \gamma(G) \leq n$ for any graph, the high quality bounding helps to narrow the interval of the domination number possible values.

Theorem 1.8 [1] *For any graph G of order n and a maximum degree $\Delta(G)$, $\gamma(G) \leq n - \Delta(G)$.*

Theorem 1.9 [2] *If a graph G has no isolated vertices, then $\gamma(G) \leq n/2$.*

Theorem 1.10 [15] *Let G be a graph of order n, size m and a maximum degree Δ, $\gamma(G) \geq \frac{2m - n(\Delta - 1)}{\Delta + 1}$.*

Theorem 1.11 [5] *Let T be a tree, $\gamma(G) = n - \Delta(T)$ if and only if T is a wonded spider.*

Theorem 1.12 [16] *Let $k = \lfloor (log(n) - 2log(log(n)) + log(log(e))) \rfloor$, then for almost every graph $k + 1 \leq \gamma(G) \leq k + 2$.*

1.3.2 Conditions on the dominating set

The conditional domination number $\gamma_{P_i}(G)$ was defined by Harary and Haynes [17] to be the smallest cardinality of a dominating set $D \subseteq V$, such that the subgraph $G[D]$ induced by D satisfies the property P_i, those parameters are difined by imposing an additional constraint on the dominating set of G.

- P_1: $G[D]$ has no edges (independent domination $i(G)$).
- P_2: $G[D]$ has no isolated vertices (total domination $\gamma_t(G)$).
- P_3: $G[D]$ is connected (connected domination $\gamma_c(G)$).

Obviously $\gamma(G) \leq \gamma_{P_i}(G)$ for any property P_i, notice that this propreties do not exist for all graphs.

Independent domination

Definition 1.7 *In a simple graph $G = (V, E)$, an independent dominating set S is a dominating set of the smallest cardinality, such that $G[S]$ has no edges (all vertices of S are isolated), where the cardinality of such a set is equal to $i(G)$, this set is called an $i-$set of G.*

Theorem 1.13 [14] *If a graph G does not contain the star $K_{1,k+1}$, $k \geq 2$, as an induced subgraph, then $i(G) \leq (k-1)\gamma(G) - k + 2$.*

Theorem 1.14 [14] *If a graph G has no isolated vertices, then*
$i(G) \leq n - \gamma(G) + 1 - \lceil (n - \gamma(G))\gamma(G) \rceil$.

Theorem 1.15 [18] *If a graph G is claw-free, then $\gamma(G) = i(G)$.*

Corollary 1.1 *For any path P_n or cycle C_n, $\gamma(G) = i(G)$, since both are claw-free graphs.*

Theorem 1.16 [19] *If G is a graph of order n and a maximum degree $\Delta(G)$, then $i(G) < n - \Delta(G)$.*

Theorem 1.17 [20] *For any tree T of order $n \geq 3$, with $l(T)$ leavs, then $i(G) \leq \dfrac{n + l(T)}{3}$.*

Total domination

Definition 1.8 *In a graph G, a subset D of V is said to be a total dominating of G if every vertex in V has a neighbor in D ($N_G(D) = V(G)$), in other words the subgraph induced by D has no isolated vertices, the total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G, the total domination can be only in connected graphs.*

Theorem 1.18 [21] *If a graph G is connected and $\Delta(G) \leq n - 1$, then $\gamma_t(G) \leq n - \Delta(G)$.*

Theorem 1.19 [21] *Let G be a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq \dfrac{2n}{3}$.*

Theorem 1.20 [22] *If each component of a graph G has at least three vertices, then $\gamma_t(G) + i(G) \leq n$.*

Corollary 1.2 *For every connected graph G, $\gamma(G) \leq \gamma_t(G) \leq 2\gamma_t(G)$.*

Connected domination

Definition 1.9 *In a graph $G = (V, E)$, a connected dominating set D of G is a dominating set which its induced subgraph $G[D]$ is connected, D contains at least one vertex, it follows that only connected graphs has a connected dominating set, the minimum cardinality of a connected dominating set is the connected domination number $\gamma_c(G)$, obviously $\gamma(G) \leq \gamma_c(G)$, and if $\gamma(G) = 1$, then $\gamma(G) = \gamma_c(G) = i(G) = 1$, any connected dominating set is also a total dominating set where $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$.*

Theorem 1.21 [23] *If H is a connected spanning subgraph of G, then $\gamma_c(G) \leq \gamma_c(H)$.*

Corollary 1.3 [23] *Let T be a connected tree of order $n \geq 3$, then $\gamma_c(G) = n - \varepsilon_T$.*

Theorem 1.22 [23] *Let G be a connected graph of order n, size m and a maximum degree $\Delta(G)$, then $\frac{n}{\Delta+1} \leq \gamma_c(G) \leq 2m - n$.*

Theorem 1.23 [24] *If G is a connected graph of order $n \geq 3$, then $\gamma_c(G) = n - \varepsilon_T \leq n - 2$. where ε_T denote the maximum number of leavs in a spanning tree of G.*

Theorem 1.24 [24] *For every connected graph G of order n and a maximum degree $\Delta(G)$, $\gamma_c(G) \leq n - \Delta(G)$.*

Chapter 2

Changing and unchanging domination

An important research about graphs for knowing how the domination parametrer $\gamma(G)$ is affected when a change is made up in the graph, for instance edge subdivision and path addition operations, as an example this study shows the ability of a network to provide service even when it contains a faulty components, the behavior of the network in the presence of a fault can be analyzed by determining the effect that the edge subdivision or path addition has on its performance, in other words we will show the behavior of the domination number γ when a change is made up in an arbitrary graph.

2.1 Definitions and notations

We are interested in the minimum and the maximum values of the number k with respect to γ, which denoted by $pa_\gamma(u,v)$, such that $\gamma(G) \neq \gamma(G_{u,v,k})$, for any graph G of order $n \geq 2$, furthermore, we will study the behavior of the domination number of an arbitrary graph G, by determing the minimum and the maximum values of k for two different cases, when a subdivision is made up in one of its edges, and when a path P_k is added between two of its distinct non adjacent vertices.

Let $epa_\gamma(G)$ and $\overline{e}pa_\gamma(G)$ denotes the minimum values of $pa_\gamma(u,v)$ for a subdivision and a path addition respectively, such that:

- $epa_\gamma(G) = min\{pa_\gamma(u,v) \mid u,v \in V(G), uv \in E(G)\}$.
- $\overline{e}pa_\gamma(G) = min\{pa_\gamma(u,v) \mid u,v \in V(G), uv \notin E(G)\}$.

Let $Epa_\gamma(G)$ and $\overline{E}pa_\gamma(G)$ denotes the maximum values (also called the upper values) of $pa_\gamma(u,v)$ for a subdivision and a path addition respectively, such that:

- $Epa_\gamma(G) = max\{pa_\gamma(u,v) \mid u,v \in V(G), uv \in E(G)\}$.
- $\overline{E}pa_\gamma(G) = max\{pa_\gamma(u,v) \mid u,v \in V(G), uv \notin E(G)\}$.

Consider the partition of $V(G)$ into three sets according to how their removal affects $\gamma(G)$, so let $V = V^o \cup V^+ \cup V^-$, such that:

- $V^o = \{u \in V(G) \mid \gamma(G-u) = \gamma(G)\}$.
- $V^+ = \{u \in V(G) \mid \gamma(G-u) > \gamma(G)\}$.
- $V^- = \{u \in V(G) \mid \gamma(G-u) < \gamma(G)\}$, and V^- is called the set of crtical vertices of G.

If G is a complete graph, then $\overline{e}pa_\gamma(G) = \overline{E}pa_\gamma(G) = \infty$, and if G is an edgless graph, then $epa_\gamma(G) = Epa_\gamma(G) = \infty$.

Let D be a $\gamma - set$ of G, then the removal of any vertex in $V - D$ cannot increase the domination number, so $\mid V^+ \mid \leq \gamma(G)$, it is obvious that every isolated vertex is in V^-, in what follows we show the characterisation of vertices of the three sets, Bauer et al characterized the vertices in V^+ in the following theorem.

Theorem 2.1 [25] *A vertex $u \in V^+$ if and only if:*

- *u is not an isolated vertex in G and it is in every $\gamma-$set of G.*
- *No subset $M \subseteq V - N[u]$ of cardinality $\gamma(G)$ dominates $G-u$.*

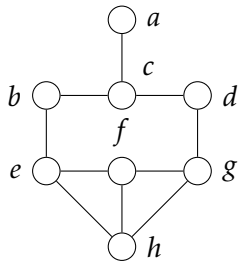


Figure 2.1: The graph G has two $\gamma-set$, $D_1 = \{c, f\}$ and $D_2 = \{c, h\}$ with $V - N[c] = \{e, f, g, h\}$, thus $c \in V^+$, hence $\gamma(G-c) = 3 > \gamma(G) = 2$ where $D_{G-c} = \{e, a, g\}$.

Sampathkumar and Neeralagi characterized the vertices of V^- in the following theorem.

Theorem 2.2 [5] *A vertex $u \in V^-$ if and only if $pn[u, D] = \{u\}$ for some $\gamma-$set D containing u.*

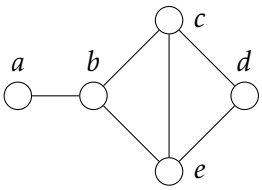


Figure 2.2: The graph G has six $\gamma-set$, where a is contained in three of them, it is obvious that a is isolated in all of those sets, thus $a \in V^-$, then $\gamma(G-a) = 1 < \gamma(G) = 2$.

Carrington et al determined properties of the vertices in V^+ and V^-, and proved the conditions for $\gamma(G-u) < \gamma(G)$ in the following theorem.

Theorem 2.3 [26] *For any graph G, and let D be any $\gamma-$set of G,*
(i) If $u \in V^+$, then for every $\gamma-$set D, $u \in D$ and $pn[u, D]$ contains at least two non adjacent vertices.
(ii) If $x \in V^+$ and $y \in V^-$, then x and y are not adjacent.
(iii) $\mid V^o \mid \geq 2 \mid V^+ \mid$.
(iv) $\gamma(G) \neq \gamma(G-u)$ for all $u \in V(G)$ if and only if $V = V^-$, and G is γ-critical graph.
(v) If $u \in V^-$ and u is not an isolated vertex in G, then there exists a $\gamma-$set D of G, such that $u \notin D$.

Since we are interested in path addition operation, the addition of P_0 between two distinct non adjacent vertices of G is considered to be an edge addition $e \in E(\overline{G})$, Walikar and Acharya [15] proved that the addition of $e \in E(\overline{G})$ can decrease the domination number of G by at most one, hence, $\gamma(G) - 1 \leq \gamma(G+e) \leq \gamma(G)$.

The concept of μ-good and μ-bad vertices was introduced by Fricke et al [4], in this thesis $\mu = \gamma$, a vertex is called γ-good if it is contained in some $\gamma-sets$ of G, and γ-bad otherwise, also they proved some resaults that we will use, in a graph G, for $u \in V(G)$:

- If u is γ-bad, then $\gamma(G-u) = \gamma(G)$.
- If u is γ-critical, then all of its neighbors are γ-bad vertices of $G-u$.

Brigham et al [27] showed that in a graph G for $u \in V(G)$, u is γ-critical if and only if $\gamma(G-u) = \gamma(G) - 1$.

2.2 The effect of subdivision on the domination number

In this section we will prove that $pa_\gamma(G) \in \{1,2,3\}$ by providing necessary and sufficient conditions for $1 \le epa_\gamma(G) \le 3$ and $2 \le Epa_\gamma(G) \le 3$, where $uv \in E(G)$ for any arbitrary graph G.

Theorem 2.4 [28] *Let G be an arbitrary graph and H is any graph obtained from G by subdivising some edges of G, then $\gamma(H) \ge \gamma(G)$.*

Theorem 2.5 [7] *Let u and v be adjacent vertices of an arbitrary graph G, then $\gamma(G) \le \gamma(G_{u,v,k}) \le \gamma(G_{u,v,k+1})$, for $k \ge 1$.*

Proof. *Let D be any $\gamma-$set of $G_{u,v,1}$, both $D_u = (D - \{x_1\}) \cup \{u\}$ and $D_v = (D - \{x_1\}) \cup \{v\}$ are dominating sets of G, and at least one of them is a $\gamma-$set of $G_{u,v,1}$, hence $\gamma(G) \le min\{|\, D_u \,|, |\, D_v \,|\} = \gamma(G_{u,v,1})$, and according to **Theorem 2.4**, $\gamma(G_{u,v,k}) \le \gamma(G_{u,v,k+1})$.*

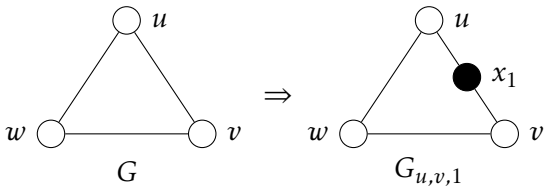


Figure 2.3: $\gamma(G) = 1$, $\gamma(G_{u,v,1}) = min\{|\, D' \,|, |\, D_u \,|, |\, D_v \,|\} = 2$ with $D_u = \{u, w\}$, $D_v = \{v, w\}$.

Theorem 2.6 [7] *Let u and v be adjacent vertices of a graph G, then*
$\gamma(G) \le \gamma(G_{u,v,1}) \le \gamma(G) + 1$, furthermore,
(a) $\gamma(G_{u,v,1}) = \gamma(G)$ if and only if at least one of u and v is a γ-good vertex of G.
(b) $\gamma(G_{u,v,1}) = \gamma(G) + 1$ if and only if both u and v are γ-bad vertices of G.

Proof. *Let D be any $\gamma-$set of G, then $D \cup \{x_1\}$ is a dominating set of $G_{u,v,1}$ with $|\, D \cup \{x_1\} \,| = \gamma(G) + 1$, which implies that $\gamma(G_{u,v,1}) \le \gamma(G) + 1$, and by **Theorem 2.5**, $\gamma(G) \le \gamma(G_{u,v,1})$, for $k = 1$, hence, the inequality $\gamma(G) \le \gamma(G_{u,v,1}) \le \gamma(G) + 1$.*

(a)$\Rightarrow$ Let $\gamma(G_{u,v,1}) = \gamma(G)$ holds, and let D be any $\gamma-$set of G, then D is also a $\gamma-$set of $G_{u,v,1}$, this implies that at least one of u and v is in D, hence, at least on of u and v is γ-good vertex of G.
(a)$\Leftarrow$ Let at least one of u and v belongs to some $\gamma-$set of G, then x_1 is dominated by D, hence, D is a $\gamma-$set of $G_{u,v,1}$, and this implies that $\gamma(G) = \gamma(G_{u,v,1})$.

(b)$\Rightarrow$ Let $\gamma(G_{u,v,1}) = \gamma(G) + 1$ holds, and let D be any $\gamma-$set of G, this implies that $D' = D \cup \{x_1\}$ with $|\, D' \,| = \gamma(G) + 1$ is a $\gamma-$set of $G_{u,v,1}$, thus, u and v are γ-bad vertices of G.
(b)$\Leftarrow$ Let both u and v be γ-bad vertices of G, and suppose that $\gamma(G_{u,v,1}) = \gamma(G)$, now for any $\gamma-$set D of $G_{u,v,1}$, $u, v \notin D$, clearly $x_1 \in D$, but in this case $(D - \{x_1\}) \cup \{u\}$ and $(D - \{x_1\}) \cup \{v\}$ are both $\gamma-$sets of G and $G_{u,v,1}$, a contradiction, hence $\gamma(G_{u,v,1}) = \gamma(G) + 1$.

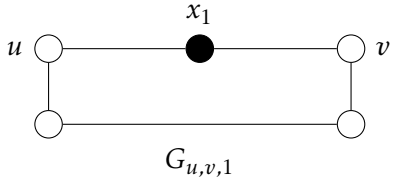


Figure 2.4: $\gamma(G_{u,v,1}) = \gamma(G) = 2$, both u and v are γ-good vertices of G.

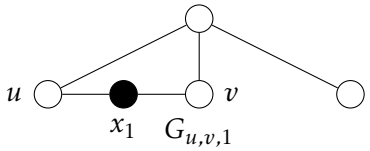


Figure 2.5: $\gamma(G) = 1$ and $\gamma(G_{u,v,1}) = 2$, both u and v are γ-bad vertices of G.

Theorem 2.7 [7] *Let u and v be adjacent vertices of a graph G, then*
$\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G)+1$, moreover,
(I) *$\gamma(G_{u,v,2}) = \gamma(G)+1$ if and only if one of the following holds:*
(a) Both u and v are γ-bad vertices of G.
(b) At least one of u and v is a γ-good vertex, $u,v \notin V^-(G)$, and each $\gamma-$set of G contains at most one of u and v.
(II) *$\gamma(G_{u,v,2}) = \gamma(G)$ if and only if at least one of the following holds:*
(c) There exists a $\gamma-$set of G which contains both u and v.
(d) At least one of u and v is in $V^-(G)$.

Proof. *Let D be any $\gamma-$set of G, then $D\cup\{x_1\}$ is a dominating set of $G_{u,v,2}$ with $\mid D\cup\{x_1\}\mid = \gamma(G)+1$, which implies $\gamma(G_{u,v,2}) \leq \gamma(G)+1$, and by **Theorem 2.5**, $\gamma(G) \leq \gamma(G_{u,v,2})$, for $k=2$, hence, the inequality $\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G)+1$.*

*(a)$\Rightarrow$ Let $\gamma(G_{u,v,2}) = \gamma(G)+1$ holds, let D be any $\gamma-$set of G, according to **Theorem 2.6**, if $\gamma(G_{u,v,1}) = \gamma(G)+1$ then both u and v are γ-bad vertices of G, now in $G_{u,v,2}$, $D\cup\{x_1\}$ is also a dominating set since u and v are dominated by D and x_2 by x_1.*
(a)$\Leftarrow$ Let both u and v be γ-bad vertices of G, and let D be any $\gamma-$set of G, with $u,v \notin D$, hence, $D\cup\{x_1\}$ and $D\cup\{x_2\}$ are both $\gamma-$sets of $G_{u,v,2}$ with $\gamma(G_{u,v,2}) =\mid D\cup\{x_1\}\mid=\mid D\cup\{x_2\}\mid = \gamma(G)+1$.

*(b)$\Rightarrow$ Let $\gamma(G_{u,v,2}) = \gamma(G)+1$ holds, according to **Theorem 2.6**, if $\gamma(G_{u,v,1}) = \gamma(G)$ then at least one of u and v is a γ-good vertex of G, since $\gamma(G_{u,v,2}) = \gamma(G)+1$ it is clear that there is no $\gamma-$set of G which contains both u and v (otherwise $\gamma(G_{u,v,2}) = \gamma(G)$), now without loss of generality, if $u \in V^-(G)$ and U is a $\gamma-$set of $G-u$ with $\mid U\mid = \gamma(G)-1$, then the set $U\cup\{x_1\}$ is a $\gamma-$set of $G_{u,v,2}$ and $\mid U\cup\{x_1\}\mid = \gamma(G)$, a contradiction, thus $u,v \notin V^-(G)$.*
(b)$\Leftarrow$ Let at least one of u and v be γ-good vertex, with $u,v \notin V^-(G)$ and each $\gamma-$set of G contains at most one of u and v, now let D be a $\gamma-$set of $G_{u,v,2}$, if $x_1,x_2 \notin D$, then $u,v \in D$ which leads to $\gamma(G_{u,v,2}) > \gamma(G)$, if $x_1,x_2 \in D$ then $(D-\{x_1,x_2\})\cup\{u,v\}$ is a $\gamma-$set of G with $\mid (D-\{x_1,x_2\})\cup\{u,v\}\mid > \gamma(G)$, now without loss of generality let $x_1 \in D$, and $x_2 \notin D$, if $D-\{x_1\}$ is a $\gamma-$set of G with $\mid D-\{x_1\}\mid = \gamma(G)$, then $\gamma(G)+1 \leq \mid D\mid = \gamma(G_{u,v,2}) \leq \gamma(G)+1$, and since $u \notin V^-(G)$, then $\gamma(G) \leq \gamma(G-u) \leq \mid D-\{x_1\}\mid < \gamma(G_{u,v,2}) = \gamma(G)+1$.

(II)$\Rightarrow$ *Let $\gamma(G_{u,v,2}) = \gamma(G)$, suppose that $u,v \notin V^-(G)$ and no $\gamma-$set of G contains both u and v, but then one of (a) and (b) holds, then $\gamma(G_{u,v,2}) = \gamma(G)+1$, a contradiction.*
(II)$\Leftarrow$ *Let at least one of (c) and (d) holds, then (a) and (b) are not valides, and $\gamma(G_{u,v,2}) \neq \gamma(G)+1$, since $\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G)+1$, then $\gamma(G_{u,v,2}) = \gamma(G)$.*

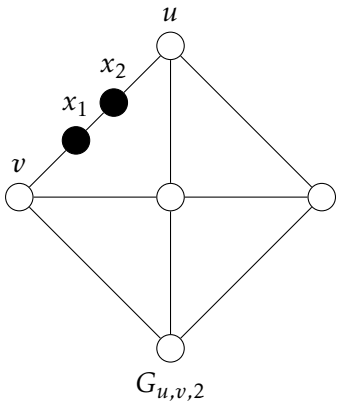


Figure 2.6: An illustration for (a), where $\gamma(G) = 1$ and $\gamma(G_{u,v,2}) = 2$.

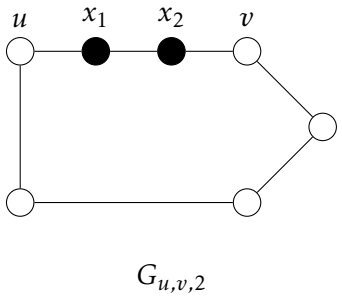


Figure 2.7: An illustration for (b), where $\gamma(G) = 2$ and $\gamma(G_{u,v,2}) = 3$.

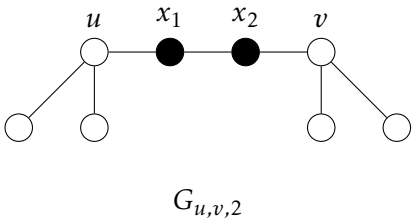


Figure 2.8: An illustration for (c), where $\gamma(G) = \gamma(G_{u,v,2}) = 2$.

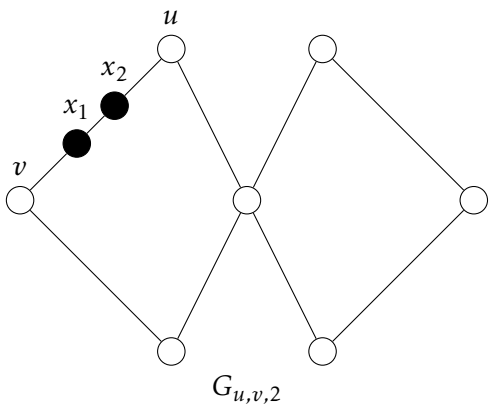


Figure 2.9: An illustration for (d), where $\gamma(G) = \gamma(G_{u,v,2}) = 3$.

Theorem 2.8 [7] *Let u and v be adjacent vertices of a graph G, then $\gamma(G_{u,v,3}) = \gamma(G) + 1$.*

Proof. *Let D be any $\gamma-$set of $G_{u,v,3}$, in all cases at least one of x_1, x_2 and x_3 is in D, if both u and v are γ-bad or γ-good vertices of G, then $x_2 \in D$ and $\gamma(G_{u,v,3}) = \gamma(G) + 1$, if $x_2 \notin D$ and $x_1, x_3 \in D$, then one of $(D - \{x_1, x_3\}) \cup \{u\}$ and $(D - \{x_1, x_3\}) \cup \{v\}$ is a $\gamma-$set of G, if $x_2, x_3 \notin D$ and $x_1 \in D$, then $v \in D$ and $D - \{x_1\}$ is a $\gamma-$set of G, then all of this leads to $\gamma(G_{u,v,3}) = \gamma(G) + 1$.*

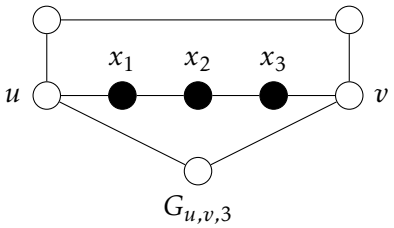


Figure 2.10: An illustration where $\gamma(G) = 2$ and $\gamma(G_{u,v,3}) = 3$.

Corollary 2.1 [7] *Let G be any graph with edges, then $epa_\gamma(G) = 1$ and $Epa_\gamma(G) \geq 2$ if and only if the set of all γ-bad vertices of G is neither empty nor independent.*

Proof. *This corollary is an immediate result from **Theorem 2.6**.*

Corollary 2.2 [7] *Let G be any graph with edges, then*
(a) $epa_\gamma(G) \geq 2$ if and only if the set of all γ-bad vertices of G is either empty or independent.
(b) $Epa_\gamma(G) = 2$ if and only if $i(G) \equiv \gamma(G)$ ($i(G)$ strongly $\gamma(G)$, that is, every $\gamma-$set of G is independent), and $V^-(G) = \emptyset$.

Proof.
*(a)$\Rightarrow$ Let $epa_\gamma(G) \geq 2$, suppose that the set of all γ-bad vertices of G is neither empty nor independent, but according to **Corollary 2.1**, in this case $epa_\gamma(G) = 1$, a contradiction.*
*(a)$\Leftarrow$ Let all the vertices of G be γ-good, which means that G is γ-excellent graph, then for $k \in \{1, 2\}$, $\gamma(G_{u,v,k}) = \gamma(G)$, but by **Theorem 2.8**, $\gamma(G_{u,v,3}) = \gamma(G) + 1$, hence $epa_\gamma(G) \geq 3$, and if the set of all γ-bad vertices of G is independent, that means every γ-bad vertex is adjacent to some γ-good vertices of G, but not adjacent to any γ-bad vertex, let D be a $\gamma-$set of $G_{u,v,1}$, then by (a) from **Theorem 2.6**, $\gamma(G_{u,v,1}) = \gamma(G)$, but by (b) from **Theorem 2.7**, $\gamma(G_{u,v,k}) = \gamma(G) + 1$ for $k \geq 2$, hence $epa_\gamma(G) \geq 2$.*

(b)$\Rightarrow$ Let $Epa_\gamma(G) = 2$ holds, if D is a $\gamma-$set of G and u, v are adjacent vertices, then D is a $\gamma-$set for $G_{u,v,2}$ with $\mid D \mid = \gamma(G)$, a contradiction.

(b)⇐ Let all the $\gamma-$sets of G be independent, if $u \in V^-(G)$ and U is a $\gamma-$set of $G-u$ with $|\ U\ | = \gamma(G)-1$, then for any adjacent vertex v of u, $U' = U \cup \{v\}$ is a $\gamma-$set of $G_{u,v,2}$ of cardinality $\gamma(G)$, hence $V^-(G)$ is empty, thus for any two adjacent vertices u and v, G is fulfilled either (a) or (b) of ***Theorem 2.7****, therefore, $Epa_\gamma(G) \leq 2$, and from* ***Corollary 2.1*** *we have $Epa_\gamma(G) \geq 2$, hence $Epa_\gamma(G) = 2$.*

Corollary 2.3 *[7] Let G be any graph with edges, and let u and v be adjacent vertices of G, then $1 \leq epa_\gamma(G) \leq Epa_\gamma(G) \leq 3$, moreover,*
(a) $Epa_\gamma(G) = 3$ if and only if G has a $\gamma-$set which is not independent.
(b) $epa_\gamma(G) = 3$ if and only if one of the following holds,
(i) There exists a $\gamma-$set of G which contains both u and v.
(ii) At least one of u and v is in $V^-(G)$.

Proof. *According to* ***Theorems 2.6, 2.7*** *and* ***2.8****, also from* ***Corollary 2.1****, we have the following inequality $1 \leq epa_\gamma(G) \leq Epa_\gamma(G) \leq 3$, and from* ***Corollary 2.2*** *we have $Epa_\gamma(G) \geq 2$.*

(a) Since $Epa_\gamma(G) = 2$ if and only if all the $\gamma-$sets of G are independent, then it is obvious that $Epa_\gamma(G) \geq 3$ if and only if G at least one $\gamma-$set which is not independent, and because $Epa_\gamma(G) \leq 3$, then $Epa_\gamma(G) = 3$.

(b) $epa_\gamma(G) = 3$ if and only if $\gamma(G_{u,v,2}) = \gamma(G)$, then according to ***(II)*** *from* ***Theorem 2.7****, $epa_\gamma(G) = 3$ if and only if one of (c) and (d) holds.*

Corollary 2.4 *[7] Let G be a graph with edges, if there is a set $S \subseteq V^-(G)$, such that S is a vertex covering set of G, then $epa_\gamma(G) = 3$, a graph which contains such a subset S is called a $vc-graph$.*

In what follows we will show the classes of graphs G with $\Delta(G) \geq 1$, such that $epa_\gamma(G) = 3$, let $A = \{G \mid epa_\gamma(G) = 3\}$ be the set of all graphs with $epa_\gamma(G) = 3$, and let A_i for $1 \leq i \leq 3$ be a subsets of A, such that:

- $A_1 = \{G \mid V^-(G)$ is a vertex cover of $G\}$.
- $A_2 = \{G \mid$ each two adjacent vertices belongs to some $\gamma-set$ of $G\}$.
- $A_3 = \{G \mid G$ is a $vc-graph\}$.

It is obvious that $A_3 \subseteq A_1$, and by **Corolaries 2.3** and **2.4**, $A_1 \cup A_2 \subseteq A$, now let be the feature sets R_i for $1 \leq i \leq 5$, such that, $R_0 = A/(A_1 \cup A_2)$, $R_1 = A_1/(A_2 \cap A_3)$, $R_2 = A_3/(A_2 \cap A_3)$, $R_3 = A_2 \cap A_3$, $R_4 = (A_1 \cap A_2)/(A_2 \cap A_3)$ and $R_5 = A_2/(A_1 \cup A_2)$, the illustration is in the figure below, as an examples, $G = H \circ K_1 \in R_0$ for any connecetd graph H of order $n \geq 2$, the graph $G(1,k) \in R_1$ which is obtained from the cycle C_{3k+1} for $k \geq 2$, by adding a vertex y and edges yu, yv to C_{3k+1}, also the graph $C_{3k+1} \in R_3$ for $k \geq 1$, the bipartite graphs $K_{2,n} \in R_4$ for $n \geq 3$ and the complete bipartite graph $K_{n,n} \in R_5$ for $n \geq 3$.

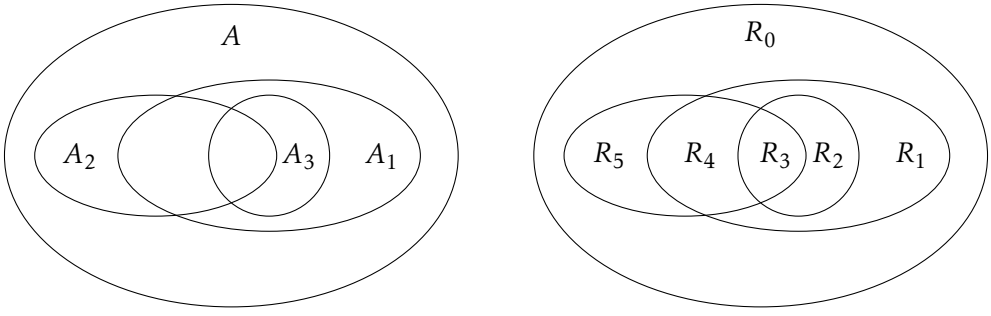
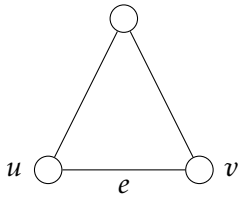


Figure 2.11: The classes of graphs with $epa_\gamma(G) = 3$.

2.3 The effect of path addition on the domination number

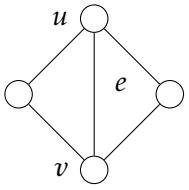
In this section we will prove that $pa_\gamma(G) \in \{1,2,3,4,5\}$ by providing necessary and sufficient conditions for $1 \leq \bar{e}pa_\gamma(G) \leq \overline{E}pa_\gamma(G) \leq 5$, where $uv \in \overline{E}(G)$ for any arbitrary graph G.

Theorem 2.9 [6] *Let G be a graph with $e \in \overline{E}(G)$, then $\gamma(G)-1 \leq \gamma(G+e) = \gamma(G_{u,v,0}) \leq \gamma(G)$, and also $\gamma(G_{u,v,k}) \leq \gamma(G_{u,v,k+1})$ for $k \geq 0$.*



$G+e = G_{u,v,0}$

Figure 2.12: $\gamma(G) = \gamma(G_{u,v,0}) = 1$.



$G+e = G_{u,v,0}$

Figure 2.13: $\gamma(G) = 2 \neq \gamma(G_{u,v,0}) = 1$.

Theorem 2.10 [7] *Let u and v be non adjacent vertices of a graph G, then*
$\gamma(G)-1 \leq \gamma(G_{u,v,1}) \leq \gamma(G)+1$, moreover,
(a) $\gamma(G_{u,v,1}) = \gamma(G)-1$ if and only if $\gamma(G-\{u,v\}) = \gamma(G)-2$, that is $u,v \in V^-(G)$.
(b) $\gamma(G_{u,v,1}) = \gamma(G)+1$ if and only if both u and v are γ-bad vertices of G, and $u \notin V^-(G-v)$, $v \notin V^-(G-u)$, and $x_1 \in V^-(G_{u,v,1})$ if $\gamma(G_{u,v,1}) = \gamma(G)+1$.
(c) $\gamma(G_{u,v,1}) = \gamma(G)$ if and only if at least one of u and v is γ-good vertex and exactly one of u and v is contained in each $\gamma-$set of G, or $\gamma(G-\{u,v\}) = \gamma(G)-1$, that is one of u in v is in $V^-(G)$.

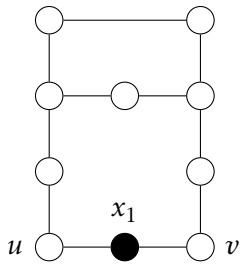
Proof. *From **Theorem 2.9** we know that $\gamma(G)-1 \leq \gamma(G_{u,v,0})$ and $\gamma(G_{u,v,0}) \leq \gamma(G_{u,v,1})$, hence we get $\gamma(G)-1 \leq \gamma(G_{u,v,1})$.*

(a)$\Rightarrow$ Let $\gamma(G_{u,v,1}) = \gamma(G)-1$ holds, and let D be any $\gamma-$set of $G_{u,v,1}$, if $x_1 \in D$ then $D' = (D-\{x_1\})\cup\{u,v\}$ is a dominating set of G, and $\gamma(G) \leq \mid D' \mid \leq \mid D \mid +1 \leq \gamma(G_{u,v,1})+1 = \gamma(G)$, thus $\gamma(G) \leq \mid D' \mid \leq \gamma(G)$, this implies that D' is a $\gamma-$set of G, hence $pn[x_1,D] = \{u,x_1,v\}$, since u and v are in some $\gamma-$sets of G, then $D'' = D'-\{u,v\}$ is a dominating set of $G-\{u,v\}$ with $\gamma(G)-2 \leq \gamma(G-\{u,v\}) \leq \mid D'' \mid = \gamma(G)-2$, then $\gamma(G-\{u,v\}) = \gamma(G)-2$.
(a)$\Leftarrow$ Let $\gamma(G-\{u,v\}) = \gamma(G)-2$ holds, then for any $\gamma-$set U of $G-\{u,v\}$, the set $U\cup\{x_1\}$ is a dominating set of $G_{u,v,1}$, this leads to $\gamma(G_{u,v,1}) \leq \mid U\cup\{x_1\} \mid = \gamma(G)-1 = \gamma(G_{u,v,1})$, hence, $\gamma(G_{u,v,1}) = \gamma(G)-1$.

(b)$\Rightarrow$ Let $\gamma(G_{u,v,1}) = \gamma(G)+1$ holds, then u and v are γ-bad vertices of G, and for any $\gamma-$set D of G, $D\cup\{x_1\}$ is a $\gamma-$set of $G_{u,v,1}$, hence $x_1 \in V^-(G_{u,v,1})$, now without loss of generality let $u \in V^-(G-v)$, and let U be a $\gamma-$set of $G-\{u,v\}$, since v is a γ-bad vertex of G and $u \in V^-(G-v)$, that means $\mid U \mid = \gamma(G-\{u,v\}) = \gamma(G)-1$, then $U' = U\cup\{x_1\}$ is a dominating set of $G_{u,v,1}$ and $\gamma(G)+1 = \gamma(G_{u,v,1}) \leq \mid U' \mid = \mid U \mid +1 = \gamma(G-\{u,v\})+1 \leq \gamma(G-v) = \gamma(G)$, a contradiction, thus $u \notin V^-(G-v)$, the same with $v \notin V^-(G-u)$.
(b)$\Leftarrow$ Let both u and v be γ-bad vertices of G, with $u \notin V^-(G-v)$ and $v \notin V^-(G-u)$, hence, $\gamma(G-\{u,v\}) \geq \gamma(G)$, let D be any $\gamma-$set of $G_{u,v,1}$, since u and v are γ-bad, if one of them is in D then $\gamma(G_{u,v,1}) = \gamma(G)+1$, so let x_1 be in every $\gamma-$set of $G_{u,v,1}$, then we have $pn[x_1,D] = \{u,x_1,v\}$, hence, $\mid D-\{x_1\} \mid = \gamma(G_{u,v,1})-1 = \gamma(G-\{u,v\}) \geq \gamma(G) = \gamma(G_{u,v,1})-1$, which leads to $\gamma(G_{u,v,1}) = \gamma(G)+1$.

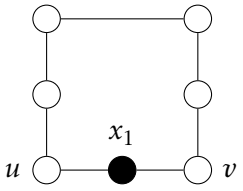
(c)$\Rightarrow$ Let $\gamma(G_{u,v,1}) = \gamma(G)$ holds, we know that from (a) if $u,v \in V^-(G)$ and $\gamma(G-\{u,v\}) = \gamma(G)-2$, then $\gamma(G_{u,v,1}) = \gamma(G)-1$, and from (b) if both u and v are γ-bad vertices of G and $u \notin V^-(G-v)$, $v \notin V^-(G-u)$, then $\gamma(G_{u,v,1}) = \gamma(G)+1$, then if $\gamma(G_{u,v,1}) = \gamma(G)$ then at least one of u and v is a γ-good vertex of G, and each $\gamma-$set of G contains at most one of u and u, or $\gamma(G-\{u,v\}) = \gamma(G)-1$.
(c)$\Leftarrow$ Let D be a $\gamma-$set of G, with at least one of u and v is a γ-good vertex of G, suppose without loss of

generality that $u \in D$ and $v \notin D$, hence in $G_{u,v,1}$, D dominates x_1 by the vertex u, that means D is a $\gamma-sets$ of $G_{u,v,1}$ of cardinality $\gamma(G)$, hence, $\gamma(G_{u,v,1}) = \gamma(G)$.



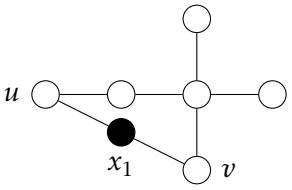
$G_{u,v,1}$

Figure 2.14: An illustration for (a), $\gamma(G) = 4 \neq \gamma(G_{u,v,1}) = 3$.



$G_{u,v,1}$

Figure 2.15: An illustration for (b), $\gamma(G) = 2 \neq \gamma(G_{u,v,1}) = 3$.



$G_{u,v,1}$

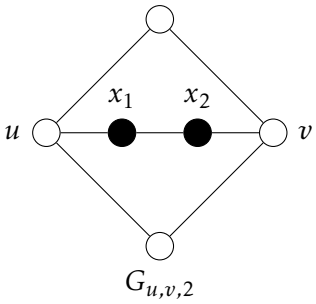
Figure 2.16: An illustration for (c), $\gamma(G) = \gamma(G_{u,v,1}) = 2$.

Theorem 2.11 [7] *Let u and v be non adjacent vertices of a graph G, then*
$\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$, moreover,
(I) *$\gamma(G_{u,v,2}) = \gamma(G)$ if and only if one of the following holds:*
(a) There is a $\gamma-set$ which contains both u and v.
(b) At least one of u and v is in $V^-(G)$.
(II) *$\gamma(G_{u,v,2}) = \gamma(G) + 1$ if and only if $u, v \notin V^-(G)$ and any $\gamma-set$ of G contains at most one of u and v.*

Proof. *First we will show that $\gamma(G_{u,v,2}) \in \{\gamma(G), \gamma(G)+1\}$, let D be any $\gamma-set$ of G, then $D \cup \{x_1\}$ or $D \cup \{x_2\}$ is a dominating set of $G_{u,v,2}$, hence, $\gamma(G_{u,v,2}) \leq \gamma(G)+1$, now let M be a $\gamma-set$ of $G_{u,v,2}$ and suppose that $\mid M \mid = \gamma(G_{u,v,2}) \leq \gamma(G) - 1$, then at least one of x_1 and x_2 is in M, if $x_1, x_2 \in M$ then $M' = (M - \{x_1, x_2\}) \cup \{u, v\}$ is a dominating set of G, and $\gamma(G) = \mid M' \mid \leq \gamma(G_{u,v,2})$, a contradiction, so without loss of generality let $x_1 \in M$ and $x_2 \notin M$, if $u \in M$ then $M'' = (M - \{x_1\}) \cup \{v\})$ or if $v \in M$ then $M'' = (M - \{x_1\}) \cup \{u\})$ is a dominating set of G and $\mid M'' \mid = \gamma(G) \leq \gamma(G_{u,v,2})$, a contradiction, also if $u, v \notin M$ and $x_1 \in M$, then $(M - \{x_1\}) \cup \{u\}$ or $(M - \{x_1\}) \cup \{v\}$ is a dominating set of G, and $\mid (M - \{x_1\}) \cup \{u\} \mid = \gamma(G) \leq \gamma(G_{u,v,2})$, a contradiction, thus $\gamma(G) \leq \gamma(G_{u,v,2}) \leq \gamma(G) + 1$.*

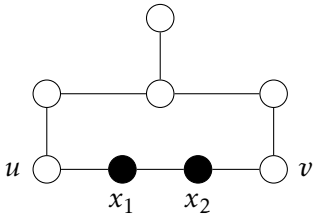
(I)$\Rightarrow$ *Let $\gamma(G_{u,v,2}) = \gamma(G)$ holds, suppose that there is no $\gamma-set$ of G which contains both u and v, and $u, v \notin V^-(G)$, let M be any $\gamma-set$ of $G_{u,v,2}$, if $x_1, x_2 \in M$, then $M' = (M - \{x_1, x_2\}) \cup \{u, v\}$ is a dominating set of G with $\mid M' \mid > \gamma(G)$ $(u, v \in M')$, a contradiction, without loss of generality let $x_1 \in M$ and $x_2 \notin M$, since $u \in pn[x_1, M]$, then $M'' = (M - \{x_1\}) \cup \{u\}$ is a $\gamma-set$ of G, and $\mid M'' \mid = \gamma(G)$ where $u \in V^-(G)$, a contradiction, thus there exists a $\gamma-set$ of G which contains both u and v, or at least one of u and v is in $V^-(G)$.*
(I)$\Leftarrow$ *Let D be a $\gamma-set$ of G and $u, v, \in D$, then D is a $\gamma-set$ of $G_{u,v,2}$, hence, $\gamma(G_{u,v,2}) = \mid D \mid = \gamma(G)$, now without loss of generality let $u \in V^-(G)$ and U is a $\gamma-set$ of $G-u$, then $U \cup \{x_1\}$ is a $\gamma-set$ of $G_{u,v,2}$ with $\mid U \cup \{x_1\} \mid = \gamma(G)$, thus $\gamma(G_{u,v,2}) = \gamma(G)$.*

(II)$\Rightarrow$ *Let* $\gamma(G_{u,v,2}) = \gamma(G)+1$ *holds, suppose that at least one of* u *and* v *is in* $V^-(G)$*, or there exists a* $\gamma-$*set of* G *which contains both* u *and* v*, but then according to* ***(I)****,* $\gamma(G_{u,v,2}) = \gamma(G)$*, a contradiction, hence,* $u,v \notin V^-(G)$*, and any* $\gamma-$*set of* G *contains at most one of* u *and* v*.*
(II)$\Leftarrow$ *Let* $u,v \notin V^-(G)$ *and any* $\gamma-$*set of* G *contains at most one of* u *and* v*, let* D *be any* $\gamma-$*set of* G*, it is obvious that* $D\cup\{x_1\}$ *or* $D\cup\{x_2\}$ *are* $\gamma-$*sets of* $G_{u,v,2}$ *of cardinality* $\gamma(G)+1$*, hence,* $\gamma(G_{u,v,2}) =\mid D\cup\{x_1\}\mid = \gamma(G)+1$*.*



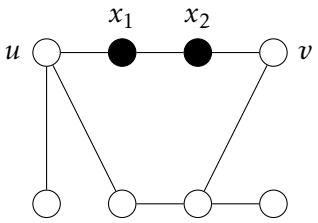
$G_{u,v,2}$

Figure 2.17: An illustration for (a), $\gamma(G) = \gamma(G_{u,v,2}) = 2$.



$G_{u,v,2}$

Figure 2.18: An illustration for (b), $\gamma(G) = \gamma(G_{u,v,2}) = 3$.



$G_{u,v,2}$

Figure 2.19: An illustration for **(II)**, $\gamma(G) = 2 \neq \gamma(G_{u,v,2}) = 3$.

Theorem 2.12 [7] *Let* u *and* v *be non adjacent vertices of a graph* G*, then* $\gamma(G) \leq \gamma(G_{u,v,3}) \leq \gamma(G)+1$*, moreover,*
$\gamma(G_{u,v,3}) = \gamma(G)$ *if and only if at least one of (i) and (ii) holds,*
(i) $u \in V^-(G)$ *and* v *is a* γ*-good of* $G-u$*.*
(ii) $v \in V^-(G)$ *and* u *is a* γ*-good of* $G-v$*.*

Proof. *Let* D *be any* $\gamma-$*set of* G*, then* $D\cup\{x_2\}$ *is a dominating set of* $G_{u,v,3}$ *with* $\mid D\cup\{x_2\}\mid = \gamma(G)+1$*, hence,* $\gamma(G_{u,v,3}) \leq \gamma(G)+1$*, and since we know that* $\gamma(G) \leq \gamma(G_{u,v,2})$ *from* ***Theorem 2.11****, and* $\gamma(G_{u,v,2}) \leq \gamma(G_{u,v,3})$ *from* ***Theorem 2.9****, then* $\gamma(G) \leq \gamma(G_{u,v,3}) \leq \gamma(G)+1$*.*

$\Rightarrow$ *Let* $\gamma(G_{u,v,3}) = \gamma(G)$ *holds, and let* D *be a* $\gamma-$*set of* $G_{u,v,3}$*, it is obvious that at most only one vertex of* x_1,x_2,x_3 *is in* D*, if* $x_2 \in D$*, then* $D-\{x_2\}$ *is a* $\gamma-$*set of* G *with* $\mid D-\{x_2\}\mid = \gamma(G)-1$*, a contradiction, now without loss of generality let* $x_1 \in D$*, then* $v \in D$*, since* $pn[x_1,D] = \{u,x_1,x_2\}$ *and* $x_3 \in pn[v,D]$*, this implies that* $D' = (D-\{x_1\})\cup\{u\}$ *is a* $\gamma-$*set of* G*, with* $pn[u,D'] = \{u\}$ *and* $v \in D'$*, then* $u \in V^-(G)$ *and* v *is a* γ*-good vertex of* $G-u$*.*
$\Leftarrow$ *Without loss of generality, let* $u \in V^-(G)$ *and* v *is a* γ*-good vertex of* $G-u$*, then there is a* $\gamma-$*set* D *of* G *such that* $u,v \in D$ *and* $D-\{u\}$ *is a* $\gamma-$*set of* $G-u$*, then* $(D-\{u\})\cup\{x_1\}$ *is a dominating set of* $G_{u,v,3}$ *of cardinality* $\gamma(G)$*, then* $\mid (D-\{u\})\cup\{x_1\}\mid = \mid (D-\{v\})\cup\{x_3\}\mid = \gamma(G)$*, hence* $\gamma(G_{u,v,3}) = \gamma(G)$*.*

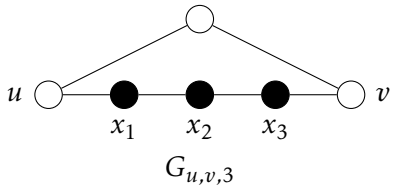


Figure 2.20: $\gamma(G) = 1 \neq \gamma(G_{u,v,3}) = 2$.

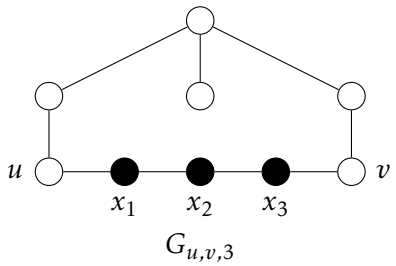


Figure 2.21: $\gamma(G) = \gamma(G_{u,v,2}) = 3$.

Theorem 2.13 [7] *Let u and v be non adjacent vertices of a graph G, then $\gamma(G) \leq \gamma(G_{u,v,4}) \leq \gamma(G) + 2$, moreover,*
(I) *If $\gamma(G_{u,v,1}) \leq \gamma(G)$ then $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$.*
(II) *$\gamma(G_{u,v,4}) = \gamma(G) + 2$ if and only if $\gamma(G_{u,v,1}) = \gamma(G) + 1$.*
(III) *If $\gamma(G_{u,v,1}) = \gamma(G)$ and $\gamma(G_{u,v,i}) = \gamma(G) + 1$ for some $i \in \{2, 3\}$, then $\gamma(G_{u,v,4}) = \gamma(G) + 1$.*
(IV) *Let $\gamma(G_{u,v,3}) = \gamma(G)$, then $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$, it holds if and only if $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$.*
(V) *$\gamma(G_{u,v,4}) = \gamma(G)$ if and only if $\gamma(G - \{u, v\}) = \gamma(G) - 2$.*

Proof. *Since $\gamma(G_{u,v,3}) \geq \gamma(G)$ from **Theorem 2.12**, and $\gamma(G_{u,v,4}) \geq \gamma(G_{u,v,3})$ from **Theorem 2.9**, then $\gamma(G_{u,v,4}) \geq \gamma(G)$, now let D be a $\gamma-$set of G, then $D \cup \{x_1, x_4\}$ is a dominating set of $G_{u,v,4}$ with $\gamma(G_{u,v,4}) \leq \gamma(G) + 2$, hence, $\gamma(G) \leq \gamma(G_{u,v,4}) \leq \gamma(G) + 2$.*

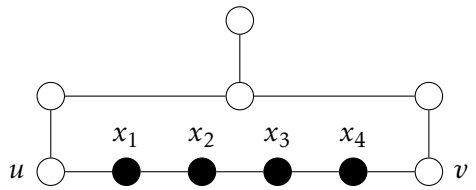
(I) *From (c) of **Theorem 2.10**, $\gamma(G_{u,v,1}) = \gamma(G)$ if and only if at least one of u and v is a γ-good vertex of G, and each $\gamma-$set contains exactly one of u and v, let D be a $\gamma-$set of G, without loss of generality let $u \in D$, then $D \cup \{x_3\}$ is a $\gamma-$set of $G_{u,v,4}$, hence $\gamma(G_{u,v,4}) = \gamma(G) + 1$, now from (a) of **Theorem 2.10**, $\gamma(G_{u,v,1}) = \gamma(G) - 1$ if and only if $\gamma(G - \{u, v\}) = \gamma(G) - 2$, let M be a $\gamma-$set of G, then $(D - \{u, v\}) \cup \{x_1, x_4\}$ is a $\gamma-$set of $G_{u,v,4}$ and $\gamma(G_{u,v,4}) = \gamma(G)$, hence, $\gamma(G) \leq \gamma(G_{u,v,4}) \leq \gamma(G) + 1$.*

(II)$\Rightarrow$ *Let $\gamma(G_{u,v,4}) = \gamma(G) + 2$ holds, by **(I)**, $\gamma(G_{u,v,1}) > \gamma(G)$, then according to **Theorem 2.10**, $\gamma(G_{u,v,1}) = \gamma(G) + 1$.*
(II)$\Leftarrow$ *Let $\gamma(G_{u,v,1}) = \gamma(G) + 1$, then by (b) from **Theorem 2.10**, both u and v are γ-bad vertices of G, $u \notin V^-(G - v)$ and $v \notin V^-(G - u)$, let D be a $\gamma-$set of $G_{u,v,4}$, clearly, at most two vertices of x_1, x_2, x_3 and x_4 are in D, assume that there is only one of them is in D, so without loss of generality let $x_2 \in D$, then $D - \{x_2\}$ is a dominating set of G and $v \in D - \{x_2\}$ then $| D - \{x_2\} | > \gamma(G)$, which leads to $\gamma(G_{u,v,4}) = | D | > \gamma(G) + 1$, now assume that there are two vertices of them in D and wihtout loss of generality, let $x_1, x_4 \in D$, since $| D |$ is of minimum cardinality, then $u, v \notin D$ and $D - \{x_1, x_4\}$ is a dominating set of $G - \{u, v\}$, but then $\gamma(G_{u,v,4}) = | D - \{x_1, x_4\} | + 2 \geq \gamma(G - \{u, v\}) + 2 \geq \gamma(G - u) + 2 = \gamma(G - v) + 2 = \gamma(G) + 2$, hence $\gamma(G_{u,v,4}) = \gamma(G) + 2$.*

(III) *Let $\gamma(G_{u,v,1}) = \gamma(G)$ holds, then by **(I)**, $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$, if at least one of $\gamma(G_{u,v,i}) = \gamma(G) + 1$ for $i \in \{1, 2\}$, then since $\gamma(G_{u,v,4}) \geq \gamma(G_{u,v,i})$, we obtain $\gamma(G_{u,v,4}) = \gamma(G) + 1$.*

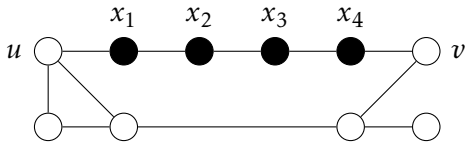
(IV) *Let $\gamma(G_{u,v,3}) = \gamma(G)$ holds, hence, at least one of (i) and (ii) of **Theorem 2.12** holds, then $\gamma(G_{u,v,4}) \leq \gamma(G) + 1$, suppose that $\gamma(G_{u,v,4}) = \gamma(G) + 1$, if $\gamma(G - \{u, v\}) = \gamma(G) - 2$, then for any $\gamma-$set W of $G - \{u, v\}$, $W \cup \{x_1, x_2\}$ is a dominating set of $G_{u,v,4}$, hence $\gamma(G_{u,v,4}) = \gamma(G)$, a contradiction, let now $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$, without loss of generality let (i) of **Theorem 2.12** be fulfilled, suppose that $\gamma(G_{u,v,4}) = \gamma(G)$, hence, for each $\gamma-$set M of $G_{u,v,4}$, $x_1, x_4 \in M$, $x_2, x_3, u, v \notin M$ with $pn[x_1, M] = \{u, x_1, x_2\}$ and $pn[x_4, M] = \{x_3, x_4, v\}$, but in this case $\gamma(G - \{u, v\}) = \gamma(G) - 2$, a contradiction, hence, $\gamma(G_{u,v,4}) = \gamma(G) + 1$.*

(V)$\Rightarrow$ *If $\gamma(G_{u,v,4}) = \gamma(G)$ then $\gamma(G_{u,v,3}) = \gamma(G)$ and by **(IV)**, $\gamma(G - \{u, v\}) = \gamma(G) - 2$.*
(V)$\Leftarrow$ *Now let $\gamma(G - \{u, v\}) = \gamma(G) - 2$, then for each $\gamma-$set D of $G - \{u, v\}$, the set $D \cup \{x_1, x_4\}$ is a $\gamma-$set of $G_{u,v,4}$, thus $\gamma(G_{u,v,4}) = \gamma(G)$.*



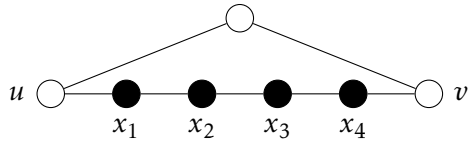
$G_{u,v,4}$

Figure 2.22: An illustration for **(I)** and **(V)**, $\gamma(G) = 3$, $\gamma(G_{u,v,1}) = 2$ and $\gamma(G_{u,v,4}) = 3$.



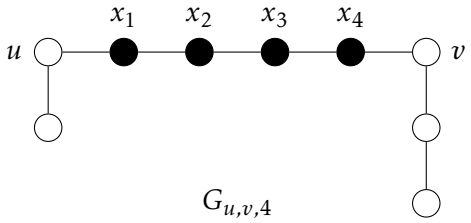
$G_{u,v,4}$

Figure 2.23: An illustration for **(I)**, $\gamma(G) = 2$, $\gamma(G_{u,v,1}) = 2$ and $\gamma(G_{u,v,4}) = 3$.



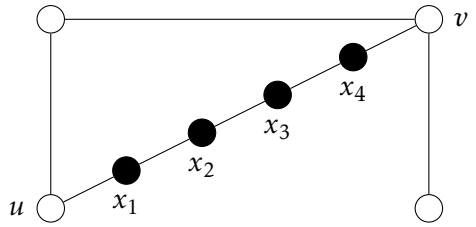
$G_{u,v,4}$

Figure 2.24: An illustration for **(II)**, $\gamma(G) = 1$, $\gamma(G_{u,v,1}) = 2$ and $\gamma(G_{u,v,4}) = 3$.



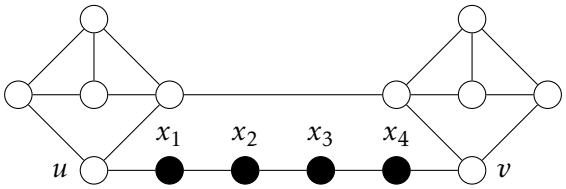
$G_{u,v,4}$

Figure 2.25: An illustration for **(III)**, $\gamma(G) = \gamma(G_{u,v,1}) = 2$ and $\gamma(G_{u,v,i}) = 3$ for $i \in \{2,3,4\}$.



$G_{u,v,4}$

Figure 2.26: An illustration for **(IV)**, $\gamma(G) = \gamma(G_{u,v,3}) = 2$ and $\gamma(G_{u,v,4}) = 3$.



$G_{u,v,4}$

Figure 2.27: An illustration for **(IV)**, $\gamma(G) = \gamma(G_{u,v,3}) = 4$ and $\gamma(G_{u,v,4}) = 4$.

Theorem 2.14 [7] *Let u and v be non adjacent vertices of a graph G,*
- *If $\gamma(G_{u,v,k}) = \gamma(G)$, then $k \leq 4$.*
- *If $k \geq 5$, then $\gamma(G_{u,v,k}) \geq \gamma(G) + 1$.*
- *If $\gamma(G_{u,v,4}) = \gamma(G)$, then $\gamma(G_{u,v,5}) = \gamma(G) + 1$.*

Proof. *We know that $\gamma(G) \leq \gamma(G_{u,v,4}) \leq \gamma(G) + 2$ from **Theorem 2.13**, if $\gamma(G_{u,v,4}) \geq \gamma(G) + 1$, then according to **Theorem 2.9**, $\gamma(G_{u,v,k}) \geq \gamma(G) + 1$ for $k \geq 5$, let $\gamma(G_{u,v,4}) = \gamma(G)$, then by **(III)** of **Theorem 2.13**, $\gamma(G - \{u,v\}) = \gamma(G) - 2$, but then since $u, v \in V^-(G)$, for each $\gamma - set$ D of $G - \{u,v\}$, the set $D \cup \{x_1, x_3, x_5\}$ is a dominating set of $G_{u,v,5}$, hence $\gamma(G_{u,v,5}) \geq \gamma(G) + 1$, now let M be a $\gamma - set$ of $G_{u,v,5}$ it is obvious that at least one of x_2, x_3, x_4 is in M, hence $\gamma(G_{u,v,5}) = \mid M \mid \leq \gamma(G) + 1$, thus $\gamma(G_{u,v,5}) = \gamma(G) + 1$, now from **Theorem 2.9**, $\gamma(G_{u,v,i}) \geq \gamma(G) + 1$ for $i \geq 5$.*

Corollary 2.5 [7] *Let G be a non complete graph, then $1 \leq \overline{e}pa_\gamma(G) \leq \overline{E}pa_\gamma(G)$, moreover,*
(a) $\overline{e}pa_\gamma(G) = 1$ if and only if there are no two non adjacent γ-bad vertices u and v of G, such that $u \in V^-(G - v)$ and $v \in V^-(G - u)$.
(b) $\overline{E}pa_\gamma(G) = 1$ if and only if $\gamma(G) = 1$.

Proof. *From* ***Theorem 2.9*** *we have* $1 \leq \bar{e}pa_\gamma(G)$, *and it is obvious that* $\bar{e}pa_\gamma(G) \leq \overline{E}pa_\gamma(G)$.

(a) It is an immediate result from (b) of ***Theorem 2.10****.*

(b) It is obvious that if $\gamma(G) = 1$, *then* $\overline{E}pa_\gamma(G) = 1$, *and if* $\gamma(G) \geq 2$, *then G has two non adjacent vertices at least one of them is* γ*-good, then by (a) of* ***Theorem 2.10****,* $\overline{E}pa_\gamma(G) \geq 2$.

Corollary 2.6 [7] *Let G be a non complete graph, then at least one of the following holds,*
(a) $\bar{e}pa_\gamma(G) \leq 2$ *if and only if there are two non adjacent vertices* $u, v \in V(G) - V^-(G)$, *such that any* $\gamma -$*set of G contains at most one of them.*
(b) $\overline{E}pa_\gamma(G) = 2$ *if and only if* $\gamma(G) \geq 2$ *and each* $\gamma -$*set is a clique.*

Proof.
(a) Is an immediate result from ***(II)*** *of* ***Theorem 2.11****.*

(b)$\Rightarrow$ *Let* $\overline{E}pa_\gamma(G) = 2$ *holds, by* ***Corollary 2.5****,* $\gamma(G) \geq 2$, *suppose that G has a* $\gamma -$*set D wich is not a clique, then* $u, v \in D$ *are not adjacent, by* ***(I)*** *of* ***Theorem 2.11****,* $\gamma(G_{u,v,2}) = \gamma(G)$, *a contradiction, thus each* $\gamma -$*set of G is a clique.*
(b)$\Leftarrow$ *Let* $\gamma(G) \geq 2$ *and each* $\gamma -$*set of G is a clique, if there exists a vertex* $u \in V^-(G)$, *and let U be a* $\gamma -$*set of* $G - u$, *then* $U' = U \cup \{u\}$ *is a* $\gamma -$*set of G of cardinality* $\gamma(G)$, *and by* ***Theorem 2.2****, u is an isolate in the subgraph induced by U', which means that U' is not a clique, a contradiction, thus* $V^-(G)$ *is empty, now by* ***(II)*** *from* ***Theorem 2.11****,* $\overline{E}pa_\gamma(G) = 2$.

Corollary 2.7 [7] *Let G be a non complete graph, then the following is true,*
(a) $\bar{e}pa_\gamma(G) \leq 3$ *if and only if there is a pair of non adjacent vertices u and v such that neither (i) nor (ii) of* ***Theorem 2.12*** *is valid.*
(b) $\bar{e}pa_\gamma(G) = \overline{E}pa_\gamma(G) = 3$ *if and only if all vertices of G are* γ*-good,* $V^-(G) = \emptyset$, *and for every two non adjacent vertices u and v of G there is a* $\gamma -$*set of G which contains both u and v.*

Proof.
(a) Is an immediate result of the previous theorems.

(b)$\Rightarrow$ *Let* $\bar{e}pa_\gamma(G) = \overline{E}pa_\gamma(G) = 3$ *holds, without loss of generality if* $u \in V^-(G)$ *and U is a* $\gamma -$*set of* $G - u$, *then for every* $u, v \in V^-(G)$, *(i) or (ii) of* ***Theorem 2.12*** *if fulfilled, but then* $\bar{e}pa_\gamma(G) \neq 3$, *a contradiction, so* $V^-(G) = \emptyset$, *now suppose that there are some* γ*-bad vertices in G, which means that there is a* γ*-bad vertex not adjacent to another* γ*-good vertex of G, but in this case* $\bar{e}pa_\gamma(G) = 2 < 3$, *since* ***(II)*** *of* ***Theorem 2.11*** *is fulfilled, also a contradiction, thus all the vertices of G are* γ*-good, if there is no* $\gamma -$*set of G which contains both u and v, then again by* ***(II)*** *of* ***Theorem 2.11****,* $\gamma(G_{u,v,2}) = \gamma(G) + 1$, *hence* $\overline{E}pa_\gamma(G) = 2 \neq 3$, *a contradiction.*
(b)$\Leftarrow$ *Let* $V^-(G) = \emptyset$, *and let D be a* $\gamma -$*set of G with* $u, v \in D$, *then by* ***(I)*** *of* ***Theorem 2.11****,* $\gamma(G_{u,v,2}) = \gamma(G)$, *and by* ***Theorem 2.12****,* $\gamma(G_{u,v,3}) = \gamma(G) + 1$, *hence* $\bar{e}pa_\gamma(G) = \overline{E}pa_\gamma(G) = 3$.

Corollary 2.8 [7] *Let G be a non complete graph, then* $\bar{e}pa_\gamma(G) \leq \overline{E}pa_\gamma(G) \leq 5$, *moreover,*
(a) $\overline{E}pa_\gamma(G) = 5$ *if and only if there are non adjacent vertices* u, v *of G with* $\gamma(G - \{u, v\}) = \gamma(G) - 2$.
(b) $\bar{e}pa_\gamma(G) = 5$ *if and only if G is edgeless* $(E(G) = \emptyset)$.
(c) $\bar{e}pa_\gamma(G) = \overline{E}pa_\gamma(G) = 4$ *if and only if for each pair* u, v *of non adjacent vertices of G,* $\gamma(G - \{u, v\}) \geq \gamma(G) - 1$, *and at least one of (i) and (ii) of* ***Theorem 2.12*** *holds.*

Proof.
(a)$\Rightarrow$ *Let* $\overline{E}pa_\gamma(G) = 5$, *then* $\gamma(G_{u,v,4}) = \gamma(G)$, *which means that* $\gamma(G - \{u, v\}) = \gamma(G) - 2$.
(a)$\Leftarrow$ *Let* $\gamma(G - \{u, v\}) = \gamma(G) - 2$ *and D a* $\gamma -$*set of* $G - \{u, v\}$, *hence* $D' = D \cup \{x_1, x_4\}$ *is a dominating set of* $G_{u,v,4}$ *and* $| D' | = \gamma(G)$, *this implies* $\gamma(G_{u,v,4}) = \gamma(G)$, *then by* ***Theorem 2.13****,* $\gamma(G_{u,v,5}) = \gamma(G) + 1 > \gamma(G)$.

(b)⇒ Let G has no edges, then the result is obvious, since it is a γ-critical graph.
*(b)⇐ Let $\overline{e}pa_\gamma(G) = 5$, now suppose that G has an edge $(u,w) \in E(G)$ then one of u and w is in $V^-(G)$ and $\gamma(G-\{u,v\}) \geq \gamma(G)-1$, then by **(IV)** of **Theorem 2.13**, $\gamma(G_{u,v,4}) = \gamma(G)+1$, hence $\overline{e}pa_\gamma(G) = 4$, a contradiction, then G is edgless.*

*(c)⇒ Let $\overline{E}pa_\gamma(G) = \overline{e}pa_\gamma(G) = 4$, then $\forall u,v \in V(G)$, such that u and v are not adjacent, $\gamma(G_{u,v,3}) = \gamma(G) < \gamma(G_{u,v,4})$, and by **(IV)** of **Theorem 2.13**, $\gamma(G-\{u,v\}) \geq \gamma(G)-1$, and by **Theorem 2.12**, at least one of (i) and (ii) is valid.*
*(c)⇐ Let u,v be any two non adjacent vertices of G, with $\gamma(G-\{u,v\}) \geq \gamma(G)-1$ and at least one of (i) and (ii) of **Theorem 2.12** is valid, then $\gamma(G_{u,v,3}) = \gamma(G)$ and by **Theorem 2.13**, $\overline{E}pa_\gamma(G) = \overline{e}pa_\gamma(G) = 4$.*

Corollary 2.9 [7] *Let G be a non complete graph with edges, then*
(i) $2 \leq epa_\gamma(G) + \overline{E}pa_\gamma(G) \leq 8$.
(ii) $2 \leq epa_\gamma(G) + \overline{e}pa_\gamma(G) \leq 7$.
(iii) $3 \leq Epa_\gamma(G) + \overline{E}pa_\gamma(G) \leq 8$.
(iv) $3 \leq Epa_\gamma(G) + \overline{e}pa_\gamma(G) \leq 7$.

Proof. *The left side of inequalities (i)-(iv) are an immedate results of **Corollaries 2.1** and **2.5**, the right side of inequalities (i)-(iv) are an immedate results of **Corollaries 2.2** and **2.7**.*

Chapter 3

Changing and unchanging connected domination

In this chapter we will apply the same proceedure of the previous chapter on the connected domination number, to study how $\gamma_c(G)$ is affected when a change is made up in the graph, for instance edge subdivision and path addition operations, the behavior of $\gamma_c(G)$ will be demonstrated in the following sections by providing necessary and sufficient conditions.

3.1 Definitions and notations

We are interested in the minimum and the maximum values of number k with respect to γ_c, which denoted by $pa_{\gamma_c}(u,v)$, such that $\gamma_c(G) \neq \gamma_c(G_{u,v,k})$, for any connected graph G of order $n \geq 2$, furthermore, we will study the behavior of the connected domination number of an arbitrary connected graph G for two different cases, when a subdivision is made up in one of its edges, and when we add a path P_k between two of its distinct non adjacent vertices.

Let $epa_{\gamma_c}(G)$ and $\overline{e}pa_{\gamma_c}(G)$ denotes the minimum values of $pa_{\gamma_c}(u,v)$ for a subdivision and a path addition respectively, such that:

- $epa_{\gamma_c}(G) = min\{pa_{\gamma_c}(u,v) \mid u,v \in V(G), uv \in E(G)\}$.
- $\overline{e}pa_{\gamma_c}(G) = min\{pa_{\gamma_c}(u,v) \mid u,v \in V(G), uv \notin E(G)\}$.

Let $Epa_{\gamma_c}(G)$ and $\overline{E}pa_{\gamma_c}(G)$ denotes the maximum values (also called the upper values) of $pa_{\gamma_c}(u,v)$ for a subdivision and a path addition respectively, such that:

- $Epa_{\gamma_c}(G) = max\{pa_{\gamma_c}(u,v) \mid u,v \in V(G), uv \in E(G)\}$.
- $\overline{E}pa_{\gamma_c}(G) = max\{pa_{\gamma_c}(u,v) \mid u,v \in V(G), uv \notin E(G)\}$.

If G is a complete graph, then $\overline{e}pa_{\gamma_c}(G) = \overline{E}pa_{\gamma_c}(G) = \infty$, all the graphs in the first section are of order $n \geq 2$, and of order $n \geq 3$ in the seconde section.

Theorem 3.1 *If a vertex u belongs to a γ_c – set D of a connected graph $G = (V,E)$, then u has at least one vertex as a private neighbor in $V - D$, or u is a support vertex in D and $pn[u,D] = \{u\}$, (in other words $N(u) \cap (V - D) = \emptyset$).*

Proof. *Let D be any γ_c – set of G with $\mid D \mid = \gamma_c(G)$, let $u \in D$, suppose that u has no private neighbor in $V - D$ and it is not a support vertex in $G[D]$, that is, all the vertices of $N(u)$ are either adjacent to some other vertices in D or $N(u) \cap (V - D) = \emptyset$, now if $N(u) \cap (V - D) \neq \emptyset$, then all the vertices that are adjacent to u are dominated by $D - \{u\}$, and since u is adjacent to some vertices in D, then $D - \{u\}$ is a γ_c – set of G and $\gamma_c(G) = \mid D - \{u\} \mid$, a contradiction, and if $N(u) \cap (V - D) = \emptyset$, then $D - \{u\}$ also is a γ_c – set of G, a contradiction.*

3.2 The effect of subdivision on the connected domination number

In this section we will prove that $pa_{\gamma_c}(G) \in \{1,2,3\}$ by providing necessary and sufficient conditions for $1 \leq epa_{\gamma_c}(G) \leq Epa_{\gamma_c}(G) \leq 3$, where $uv \in E(G)$ for any connected graph G of order $n \geq 2$.

Theorem 3.2 [7] *Let u and v be adjacent vertices of an arbitrary connected graph G, then $\gamma(G) \leq \gamma(G_{u,v,k}) \leq \gamma(G_{u,v,k+1})$, for $k \geq 1$.*

Proof. *Let D be any γ_c-set of G, suppose that at least one of u and v is in D, then $D' = D \cup \{x_1\}$ is a connected dominating set of the graph $G_{u,v,1}$, with $\gamma_c(G) \leq \mid D' \mid = \mid D \cup \{x_1\} \mid = \gamma_c(G)+1$, same thing if both u and v are γ_c-bad vertices of G, and from **Theorem 2.4**, $\gamma(G_{u,v,k}) \leq \gamma(G_{u,v,k+1})$ for $k \geq 1$.*

Theorem 3.3 *Let u and v be adjacent vertices of a connected graph G, then $\gamma_c(G) \leq \gamma_c(G_{u,v,1}) \leq \gamma_c(G)+1$, moreover,*
(I) *$\gamma_c(G_{u,v,1}) = \gamma_c(G)$ if and only if one of the following holds,*
(a) Both u and v are in the same γ_c-set of G, and the edge uv belongs to a cylce in the subgraph induced by this γ_c-set.
(b) u is a γ_c-good vertex and v is a γ_c-bad vertex of G, where v is dominated by at least another vertex besides u.
(c) Both u and v are γ_c-good vertices, and every γ_c-set contains at most one of u and v, where without loss of generality, v is dominated by at least another vertex besides u.
(II) *$\gamma_c(G_{u,v,1}) = \gamma_c(G)+1$ if and only if one of the following holds,*
(d) Both u and v are in the same γ_c-set of G, and the edge uv belongs to no cycle in the subgraph induced by this γ_c-set.
(e) At least one of u and v is a γ_c-good vertex and every γ_c-set of G contains one of u and v, where witout loss of generality v is a private neighbor of u.
(f) Both u and v are γ_c-bad vertices of G.

Proof.
(a)$\Rightarrow$ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G)$ holds, let D be any γ_c-set of G with $u,v \in D$, suppose that the edge uv belongs to no cycle in $G[D]$, now since the graph $G_{u,v,1}$ is obtained from G by the subdivision of the edge uv, then because the edges ux_1) and x_1v belongs to the same path of uv in $G[M]$ where $M = D \cup \{x_1\}$, then D is not a connected dominating set of $G_{u,v,1}$, thus M is a γ_c-set of $G_{u,v,1}$, and $\gamma_c(G_{u,v,1}) = \mid M \mid = \mid D \cup \{x_1\} \mid = \gamma_c(G)+1$, a contradiction, then the edge uv belongs to a cycle in $G[D]$.
(a)$\Leftarrow$ Let U be a γ_c-set of G with $u,v \in U$, and let the edge uv belongs to a cycle in $G[U]$, since x_1 is adjacent only to u and v in $G_{u,v,1}$ with $pn[x_1,U'] = \{x_1\}$ where U' is a dominating set of $G_{u,v,1}$, then without lose of generality, if $x_1 \in U'$ and $u \notin U'$, then every vertex $w \in pn[u,U]$ is not dominated by $U' - \{u\}$, thus u and v must be included in U', if $u,v \in U'$ and $x \in U'$, then $\mid U' \mid > \gamma_c(G)$, all of this implies that $x_1 \in V - U$ where $x_1 \in N(u)$ and $x_1 \in N(v)$, hence $\gamma_c(G_{u,v,1}) = \mid U' \mid = \mid U \mid = \gamma_c(G)$.

(b)$\Rightarrow$ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G)$ holds, let D be a γ_c-set of G with $u \in D$ and v is a γ_c-bad vertex of G, now suppose that v is adjacent only to u in D ($\mid N(v) \cap D \mid = \{u\}$), then in the graph $G_{u,v,1}$, $pn[x_1,D'] = \{v\}$ and $D' = D \cup \{x_1\}$, and x_1 is adjacent to u in D', which means that D' is a γ_c-set of $G_{u,v,1}$ with $\gamma_c(G_{u,v,1}) = \mid D' \mid = \gamma_c(G)+1$, a contradiction, thus $\mid N(v) \cap D \mid \geq 2$.
(b)$\Leftarrow$ Let M be a γ_c-set of G with $u \in D$ and let v be a γ_c-bad vertex of G, where v is adjacent to u and w in D, in the graph $G_{u,v,1}$, $x_1 \in pn[u,D]$ and $v \in pn[w,D]$, which means that x_1 is dominated by u and v is dominated by w, more generaly, the vertices u,v,w belongs to a cycle C_n in G, hence, u,x_1,v,w also belongs to the cycle C_{n+1} in $G_{u,v,1}$, thus D is a γ_c-set of $G_{u,v,1}$ and $\gamma_c(G_{u,v,1}) = \gamma_c(G)$.

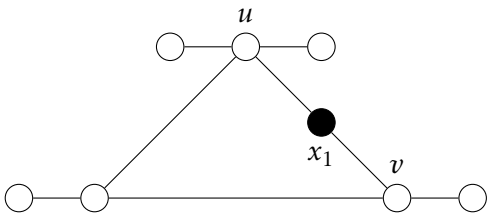
(c)$\Rightarrow$ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G)$ holds, let D be a γ_c-set of G with D contains one of u and v where both u and v are γ_c-good vertices, now without loss of generality suppose that $u \in D$, and $v \in pn[u,D]$, in this case v can not be a γ_c-good vertex of G, then v is γ_c-bad vertex, then according to (b), $\gamma_c(G_{u,v,1}) = \gamma_c(G)+1$, a contradiction, thus v is a γ_c-good vertex and $\mid N(v) \cap D \mid \geq 2$.
(c)$\Leftarrow$ Let M be a γ_c-set of G, let both u and v be γ_c-good vertices of G, with M contains one of u and v, without loss of generality, let $u \in D$ where $\mid N(v) \cap M \mid \geq 2$, in the graph $G_{u,v,1}$, x_1 is dominated by u and v

by some vertices in $D-\{u\}$, hence, $x_1 \in pn[u,D]$, thus D is a γ_c-set of $G_{u,v,1}$ and $\gamma_c(G_{u,v,1}) = \gamma_c(G)$.

(d)⇒ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G)+1$ holds, let D be a γ_c-set of G with $u,v \in D$, suppose that the edge uv belongs to a cycle in $G[D]$, hence (a) is fulfilled which leads to $\gamma_c(G_{u,v,1}) = \gamma_c(G)$, a contradiction, then the edge uv belongs to no cycle in $G[D]$.
(d)⇐ Let M be a γ_c-set of G with $u,v \in M$ and let the edge uv belongs to a path in $G[D]$, in the graph $G_{u,v,1}$, the set D is not connected, hence, the connected set of the minimum cardinality of $G_{u,v,1}$ is $M = D \cup \{x_1\}$, then $\gamma_c(G_{u,v,1}) = \mid M \mid = \mid D \cup \{x_1\} \mid = \gamma_c(G)+1$.

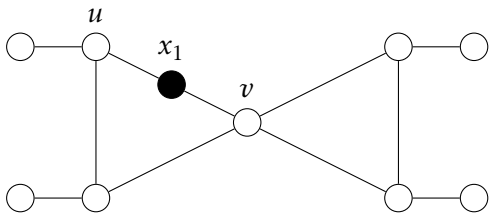
(e)⇒ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G)+1$ holds, and let D be any γ_c-set of $G_{u,v,1}$, the set $D-\{x_1\}$ is a γ_c-set of G where $\mid D-\{x_1\} \mid = \gamma_c(G)$, hence, since $v \in pn[x_1,D]$, then $\gamma_c(G_{u,v,1}) = \mid D \mid = \gamma_c(G)+1$.
(e)⇐ Let M be a γ_c-set of G with $u \in M$ and $v \notin M$ (both cases are possible, v is a γ_c-bad vertex of G, or is a γ_c-good vertex of G), and $v \in pn[u,M]$, in the graph $G_{u,v,1}$, the set M dominates the vertices in $V(G_{u,v,1})-\{v\}$, hence $M \cup \{x_1\}$ is a γ_c-set of $G_{u,v,1}$ and $\gamma_c(G_{u,v,1}) = \mid M \cup \{x_1\} \mid = \gamma_c(G)+1$.

(f)⇒ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G)+1$ holds, and let D be any γ_c-set of $G_{u,v,1}$, now without loss of generality, the set $D-\{u\}$ is a γ_c-set of G since $x_1 \in pn[u,D]$, (in the cases (e) and (d) the set $D-\{x_1\}$ is a γ_c-set of G), where $\mid D-\{u\} \mid = \gamma_c(G)$, thus the set D is of cardinality $\gamma_c(G)+1$, hence, $\gamma_c(G_{u,v,1}) = \gamma_c(G)+1$.
(f)⇐ Let both u and v be γ_c-bad vertices of G, let M be a γ_c-set of G and U a γ_c-set of $G_{u,v,1}$, the set M dominates $V(G_{u,v,1})-\{x_1\}$, hence, one of u and v must be in U, then $U = M \cup \{u\}$ or $U = M \cup \{v\}$ are both γ_c-sets of $G_{u,v,1}$ and $\gamma_c(G_{u,v,1}) = \mid U \mid = \gamma_c(G)+1$.



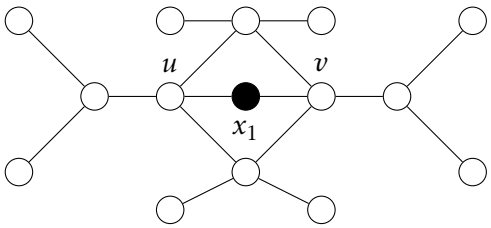
$G_{u,v,1}$

Figure 3.1: An illustration for (a), $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 3$, in the case of $\mid pn[u,D] \mid \geq 2$ and $\mid pn[v,D] \mid \geq 2$.



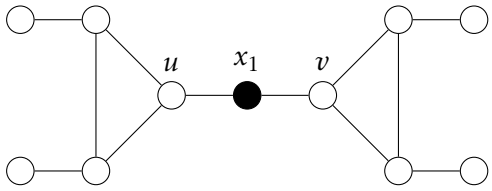
$G_{u,v,1}$

Figure 3.2: An illustration for (a), $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 5$, in the case of $\mid pn[u,D] \mid \geq 2$ and $pn[v,D] = \{v\}$, (v is a support vertex $G[D]$).



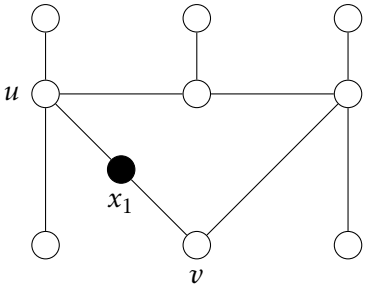
$G_{u,v,1}$

Figure 3.3: An illustration for (a), $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 6$ in the case of $pn[u,D] = \{u\}$ and $pn[v,D] = \{v\}$, (u and v are support vertices $G[D]$).



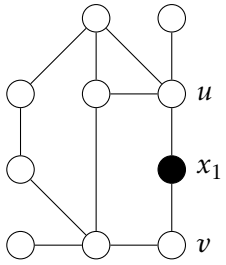
$G_{u,v,1}$

Figure 3.4: An illustration for (a), $\gamma_c(G) = 6 \neq \gamma_c(G_{u,v,1}) = 7$, both u and v belongs to a cycle in $G[D]$ but the edge uv not.



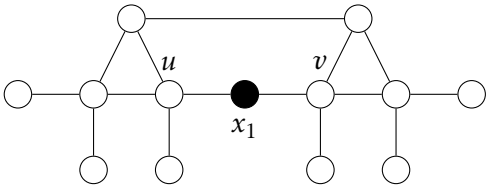
$G_{u,v,1}$

Figure 3.5: An illustration for (b), $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 3, \mid N(v) \cap D \mid = 2$.



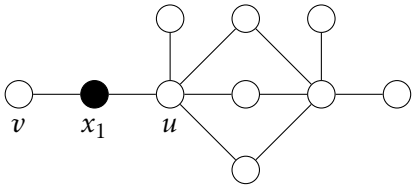
$G_{u,v,1}$

Figure 3.6: An illustration for (c), $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 4, \mid N(v) \cap D \mid = 2$.



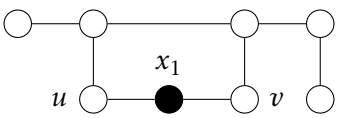
$G_{u,v,1}$

Figure 3.7: An illustration for (d), $\gamma_c(G) = 4 \neq \gamma_c(G_{u,v,1}) = 5$.



$G_{u,v,1}$

Figure 3.8: An illustration for (e), $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,1}) = 4$, u and v are respectively γ_c-good and γ_c-bad vertices of G.



$G_{u,v,1}$

Figure 3.9: An illustration for (f), $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,1}) = 4$.

Theorem 3.4 *Let u and v be adjacent vertices of a connected graph G, then $\gamma_c(G) \leq \gamma_c(G_{u,v,2}) \leq \gamma_c(G)+2$, moreover,*
(I) *$\gamma_c(G_{u,v,2}) = \gamma_c(G)$ if and only if both u and v are in the same γ_c-set of G, and the edge uv belongs to a cylce in the subgraph induced by this γ_c-set.*
(II) *$\gamma_c(G_{u,v,2}) = \gamma_c(G)+1$ if and only if at least one of u and v is a γ_c-good vertex of G, and every γ_c-set contains at most one of u and v, where without loss of generality v is dominated by at least another vertex.*
(III) *$\gamma_c(G_{u,v,2}) = \gamma_c(G)+2$ if and only if one of the following holds,*
(a) At least one of u and v is a γ_c-good vertex of G, and every γ_c-set contains at most one of u and v, where without loss of generality, v is a private neighbor of u.
(b) Both u and v are in the same γ_c-set of G, and the edge uv belongs to no cyle in the subgraph induced by this γ_c-set.
(c) Both u and v are γ_c-bad vertices of G.

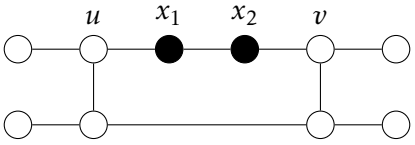
Proof.
(I)$\Rightarrow$ *Let $\gamma_c(G_{u,v,2}) = \gamma_c(G)$ holds, and let D be any γ_c-set of G where $u,v \in D$, in the graph $G_{u,v,2}$, $x_1 \in pn[u,D]$ and $x_2 \in pn[v,D]$, in this case the set D dominates all the vertices in $V(G_{u,v,2})$, and according to (a) of* ***Theorem 3.3*** *the edge uv belongs to a cycle in $G[D]$, hence, D is a γ_c-set of $G_{u,v,2}$ and $\gamma_c(G_{u,v,2}) = \mid D \mid = \gamma_c(G)$.*
(I)$\Leftarrow$ *Let M be a γ_c-set of G with $u,v \in M$ and the edge uv belongs to a cycle in $G[M]$, since $N(x_1) = \{u,x_2\}$ and $N(x_2) = \{x_1,v\}$, it is obvious that if at least one of x_1 and x_2 is in M, then $\mid M \mid > \gamma_c(G_{u,v,2})$, thus M dominates all the vertices of $G_{u,v,2}$ with $x_1,x_2 \notin M$, which leads to M is a γ_c-set of $G_{u,v,2}$, hence $\gamma_c(G_{u,v,2}) = \mid M \mid = \gamma_c(G)$.*

(II)$\Rightarrow$ *Let $\gamma_c(G_{u,v,2}) = \gamma_c(G)+1$ holds, let D be any γ_c-set of G, according to (b) and (c) of* ***Theorem 3.3****, $\gamma_c(G) = \gamma_c(G_{u,v,1})$, now in the graph $G_{u,v,2}$, let $u \in D$ and $v \notin D$, then $D \cup \{x_1\}$ or $D \cup \{v\}$ are both γ_c-sets of $G_{u,v,2}$ and $\gamma_c(G_{u,v,2}) = \mid D \cup \{x_1\} \mid = \mid D \cup \{v\} \mid = \gamma_c(G)+1$.*
(II)$\Leftarrow$ *Let M be any γ_c-set of G with $u \in M$, $v \notin M$, and $\mid N(v) \cap M \mid \geq 2$, then in the graph $G_{u,v,2}$, the vertex u dominates x_1 and v is dominated by $M-\{u\}$, but M dominates only the vertices of $V(G_{u,v,2})-\{x_2\}$, hence, $M \cup \{x_1\}$ and $M \cup \{v\}$ are both γ_c-sets of $G_{u,v,2}$ and $\gamma_c(G_{u,v,2}) = \mid M \cup \{x_1\} \mid = \gamma_c(G)+1$.*

(a)$\Rightarrow$ *Let $\gamma_c(G_{u,v,2}) = \gamma_c(G)+2$ holds, let D be a γ_c-set of $G_{u,v,1}$, according to (e) of* ***Theorem 3.3****, $\gamma_c(G_{u,v,1}) = \gamma_c(G)+1$, which means that $u,x_1 \in D$ and $D-\{x_1\}$ is a γ_c-set of G, and $v \in pn[u,D-\{x_1\}]$, then every γ_c-set of G contains one of u and v, now in the graph $G_{u,v,2}$, D dominates all the vertices in $V(G_{u,v,2})-\{v\}$, hence, $D \cup \{x_2\}$ is a γ_c-set of $G_{u,v,2}$ and $\gamma_c(G_{u,v,2}) = \mid D \cup \{x_2\} \mid = \gamma_c(G)+2$.*
(a)$\Leftarrow$ *Let M be a γ_c-set of G, with $u \in M$, $v \notin M$ and $v \in pn[u,M]$, then since v is dominated only by u in G, $M \cup \{x_1,x_2\}$ is a γ_c-set of $G_{u,v,2}$ and $\gamma_c(G_{u,v,2}) = \mid M \cup \{x_1,x_2\} \mid = \gamma_c(G)+2$.*

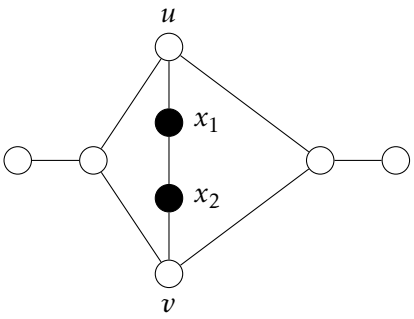
(b)$\Rightarrow$ *Let $\gamma_c(G_{u,v,2}) = \gamma_c(G)+2$ holds, let D be a γ_c-set of G with $u,v \in D$, suppose that the edge uv belongs to a cycle in $G[D]$, then* ***(I)*** *is fulfilled and $\gamma_c(G_{u,v,2}) = \gamma_c(G)$, a contradiction, thus the edge uv belongs to no cycle in $G[D]$, which implies that $D \cup \{x_1,x_2\}$ is a γ_c-set of $G_{u,v,2}$, hence, $\gamma_c(G_{u,v,2}) = \mid D \cup \{x_1,x_2\} \mid = \gamma_c(G)+2$.*
(b)$\Leftarrow$ *Let M be a γ_c-set of G with $u,v \in M$, and let the edge uv belongs to no cylce in $G[M]$, since $N(x_1) = \{u,x_2\}$ and $N(x_2) = \{x_1,v\}$, and from (d) of* ***Theorem 3.3****, $M \cup \{x_1,x_2\}$ is a γ_c-set of $G_{u,v,2}$ and $\gamma_c(G_{u,v,2}) = \mid M \cup \{x_1,x_2\} \mid = \gamma_c(G)+2$.*

(c)$\Rightarrow$ *Let $\gamma_c(G_{u,v,2}) = \gamma_c(G)+2$ holds, let D be a γ_c-set of $G_{u,v,1}$, according to (f) of* ***Theorem 3.3****, $\gamma_c(G_{u,v,1}) = \gamma_c(G)+1$, hence, $\gamma_c(G) \in \{\mid D-\{u\} \mid, \mid D-\{v\} \mid\}$, then both of u and v are γ_c-bad vertices of G, and only one of them is in D, now without loss of generality, let $u \in D$ and $v \notin D$, then D dominates all the vertices of $V(G_{u,v,2})-\{x_2\}$, which means that since $N(x_1) = \{u,x_2\}$ and $N(x_2) = \{x_1,v\}$ one of x_1, x_2 and v is in any γ_c-set M of $G_{u,v,2}$, where $\gamma_c(G_{u,v,2}) = \mid M \mid = \mid D \cup \{x_1\} \mid = \mid D \cup \{v\} \mid = \gamma_c(G)+2$.*
(c)$\Leftarrow$ *Let both u and v be γ_c-bad vertices of G, since both u and v are dominated by any γ_c-set U of G, then in $G_{u,v,2}$, U dominates $V(G_{u,v,2})-\{x_1,x_2\}$, this implies that $U \cup \{u,x_1\}$ and $U \cup \{v,x_2\}$ are both γ_c-sets of $G_{u,v,2}$ and $\gamma_c(G_{u,v,2}) = \mid U \cup \{u,x_1\} \mid = \gamma_c(G)+2$.*



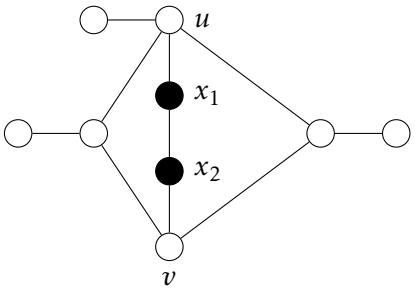
$G_{u,v,2}$

Figure 3.10: An illustration for **(I)**, $\gamma_c(G) = \gamma_c(G_{u,v,2}) = 4$.



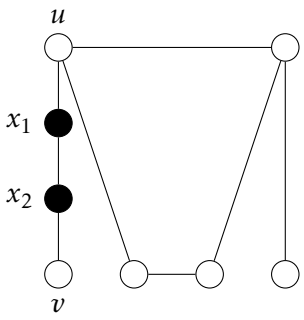
$G_{u,v,2}$

Figure 3.11: An illustration for **(II)**, $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,2}) = 4$, where both u and v are γ_c-good vertices of G.



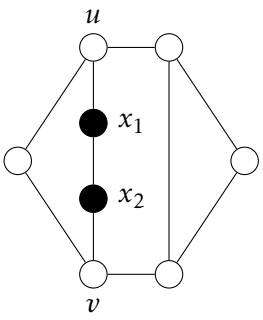
$G_{u,v,2}$

Figure 3.12: An illustration for **(II)**, $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,2}) = 4$, where u and v are respectively γ_c-good and γ_c-bad vertices of G.



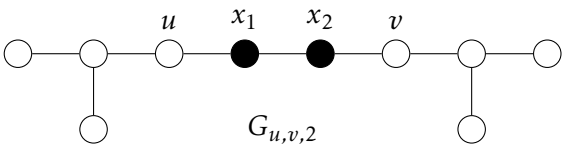
$G_{u,v,2}$

Figure 3.13: An illustration for (a), $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,2}) = 4$, where u and v are respectively γ_c-good and γ_c-bad vertices of G.



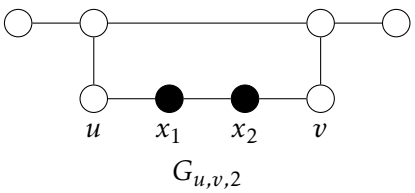
$G_{u,v,2}$

Figure 3.14: An illustration for (a), $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,2}) = 3$, where both u and v are γ_c-good vertices of G.



$G_{u,v,2}$

Figure 3.15: An illustration for (b), $\gamma_c(G) = 4 \neq \gamma_c(G_{u,v,2}) = 6$.



$G_{u,v,2}$

Figure 3.16: An illustration for (c), $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,2}) = 4$.

Theorem 3.5 *Let u and v be adjacent vertices of a connected graph G, then $\gamma_c(G_{u,v,3}) \geq \gamma_c(G)+1$.*

Proof. *Let D be a γ_c-set of G and M a γ_c-set of $G_{u,v,3}$, in all cases at least three vertices of u, v, x_1, x_2 and x_3 are in M, if $u,v \in D$ and the edge uv belongs to a cycle in $G[D]$, then one of x_1 and x_3 is in M besides u and v, thus $M = D \cup \{x_1\}$ or $M = D \cup \{x_3\}$, hence $\gamma_c(G_{u,v,3}) = \gamma_c(G)+1$, now in the other cases, if $u \in D$, $v \notin D$ and $\mid N(v) \cap D \mid \geq 2$, then $D \cup \{v,x_1\}$, $D \cup \{x_1,x_2\}$ and $D \cup \{v,x_1\}$ are all γ_c-sets of $G_{u,v,3}$ with $\gamma_c(G_{u,v,3}) =\mid D \cup \{v,x_1\} \mid = \gamma_c(G)+2 > \gamma_c(G)$, if $u \in D$, $v \notin D$ and $N(v) \cap D = \{u\}$, then $D \cup \{x_1,x_2,x_3\}$ is a γ_c-set of $G_{u,v,3}$ with $\gamma_c(G_{u,v,3}) =\mid D \cup \{x_1,x_2,x_3\} \mid = \gamma_c(G)+3 > \gamma_c(G)$, if $u,v \in D$ and the edge uv belongs to no cycle in $G[D]$, then $D \cup \{x_1,x_2,x_3\}$ is a γ_c-set of $G_{u,v,3}$ and $\gamma_c(G_{u,v,3}) =\mid D \cup \{x_1,x_2,x_3\} \mid = \gamma_c(G)+3 > \gamma_c(G)$, at last if both u and v are γ_c-bad vertices of G, then $D \cup \{u,v,x_1\}$ and $D \cup \{u,v,x_3\}$ are γ_c-sets of $G_{u,v,3}$ with $\gamma_c(G_{u,v,3}) =\mid D \cup \{u,v,x_3\} \mid = \gamma_c(G)+3 > \gamma_c(G)$, all of this leads to $\gamma_c(G_{u,v,3}) \geq \gamma_c(G)+1$.*

Corollary 3.1 [29] *Let u and v be adjacent vertices of a connected graph G of order $n \geq 3$, then*
(i) $Epa_{\gamma_c}(G) = 1$ if and only if $\gamma_c(G) = 1$.
(ii) If $\gamma_c(G) = 2$ and G contains a γ_c-set such that $N(u) \cap N(v) = \emptyset$, then $Epa_{\gamma_c}(G) = 1$.
(iii) If the graph G contains two ajdacent vertices of degree 2, then $Epa_{\gamma_c}(G) = 1$.

Proof.
(i)$\Rightarrow$ Let $Epa_{\gamma_c}(G) = 1$ holds, since $\gamma_c(G) = 1$, then $\exists u \in V(G)$ such that u is a universel vertex in G, that is, $d_G(u) = n-1$, now in the graph $G_{u,v,1}$, $\forall v \in V(G_{u,v,1})$, $d_{G_{u,v,1}}(v) \leq n-2$, which means that u is no longer a universel vertex in $G_{u,v,1}$, then any γ_c-set of $G_{u,v,1}$ contains at least two vertices, thus, $\gamma_c(G_{u,v,1}) \geq 2$, hence, $\gamma_c(G_{u,v,1}) > \gamma_c(G) = 1$.
(i)$\Leftarrow$ Let $\gamma_c(G) = 1$, it is obvious that in any case $\gamma_c(G_{u,v,1}) > \gamma_c(G)$.

(ii) Let $\gamma_c(G) = 2$ and let D be any γ_c-set of G with $u,v \in D$, where $N(u) \cap N(v) = \emptyset$, now every γ_c-set of $G_{u,v,1}$ contains at least one of u and v if $u,v \notin D$, or at least two vertices of u,v,x_1 and $N(u) \cup N(v)$, hence, $\gamma_c(G_{u,v,1}) \geq 3 > \gamma_c(G) = 2$.

(iii) Let u and v be adjacent vertices of degree 2 in a connected graph G, in $G_{u,v,1}$, an induced path a,u,x_1,v,b is obtained (it is possible to obtain a cycle if a and b are adjacent or if $a = b$), let M be a γ_c-set of $G_{u,v,1}$, without loss of generality, either $\{a,u,x_1\}$ or $\{a,v,b\}$ is included in M, if $a \neq b$, and $\{a,v\}$ is included in M if $a = b$, then the sets $M' = M-\{x_1\}$, $M'' = M-\{u\}$ when $a \neq b$ and, $M''' = M-\{v\}$ when $a = b$ are γ_c-sets of G, therefore, $\gamma_c(G_{u,v,1}) > \gamma_c(G)$.

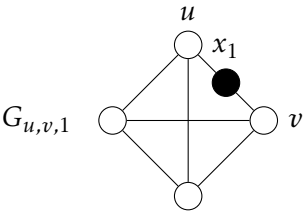


Figure 3.17: An illustration for (i), $\gamma_c(G) = 1 \neq \gamma_c(G_{u,v,1}) = 2$.

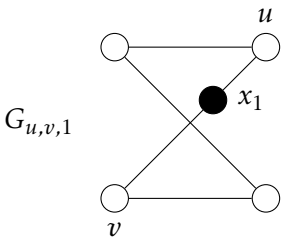


Figure 3.18: An illustration for (ii), $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,1}) = 3$.

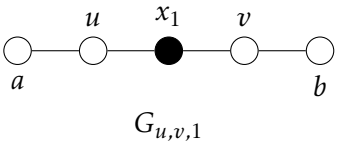


Figure 3.19: An illustration for (iii), $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,1}) = 3$ and $a \neq b$ with $(a,b) \notin E(G)$.

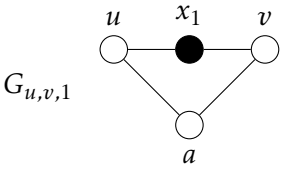


Figure 3.20: An illustration for (iii), $\gamma_c(G) = 1 \neq \gamma_c(G_{u,v,1}) = 2$ and $a = b$.

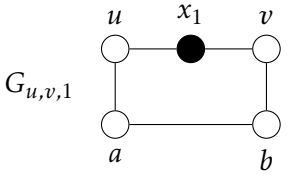


Figure 3.21: An illustration for (iii), $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,1}) = 3$ and $a \neq b$ with $(a,b) \in E(G)$.

Corollary 3.2 *Let u and v be adjacent vertices of a connected graph G of order $n \geq 3$, then*
(iv) $Epa_{\gamma_c}(G) = 1$ for any complete graph K_n, a star $K_{1,n-1}$, wheel W_n, or any connected graph G with a universel vertex, for $n \geq 3$.
(v) $epa_{\gamma_c}(G) = 2$ for any complete bipartite graph $K_{p,q}$ with $p,q \geq 2$.
(vi) $Epa_{\gamma_c}(G) = 1$ for every cycle C_n and a path P_n, with $n \geq 3$.

Proof.
*(iv) Is an immediate result of (i) from **Crollary 3.1**.*
*(v) Is an immediate result of (ii) from **Crollary 3.1**.*
*(vi) Is an immediate result of (iii) from **Crollary 3.1**.*

Corollary 3.3 *Let T be a tree of order $n \geq 3$, then $Epa_{\gamma_c}(T) = 1$.*

Proof. *Since any tree T is an acyclic graph, then every edge $uv \in E(T)$ belongs to a path in T, let D be a γ_c-set of T and M a γ_c-set of $T_{u,v,1}$ with $\mid D \mid = \gamma_c(T)$, now in $T_{u,v,1}$, there is at least one leav vertex which is not dominated by any set of cardinality $\mid D \mid$, since $x_1 \in M$ (otherwise M is not connected), hence, for dominating the leavs, their support vertices must be in M, thus $M = D \cup \{x_1\}$ is a γ_c-set of $T_{u,v,1}$ and $\gamma_c(T_{u,v,1}) = \mid D \cup \{x_1\} \mid > \gamma_c(T)$.*

Corollary 3.4 *Let u and v be adjacent vertices of a connected graph G of order $n \geq 3$, and let D be a γ_c-set of G, then there are seven cases related to the subdivision effect,*
♦ *$u,v \in D$ and the edge (u,v) belongs to a cycle in $G[D]$.*
♦ *$u \in D$, v is a γ_c-bad vertex and $\mid N(v) \cap D \mid \geq 2$.*
♦ *Both u and v are γ_c-good vertices of G, every set D contains one of u and v, where without loss of generality, $u \in D$ and $\mid N(v) \cap D \mid \geq 2$.*
♦ *$u \in D$ and v is a γ_c-bad vertex, with $\mid N(v) \cap D \mid = \{u\}$.*
♦ *Both u and v are γ_c-good vertices of G, every set D contains one of u and v, where without loss of generality, $u \in D$ and $N(v) \cap D = \{u\}$.*
♦ *$u,v \in D$ and the edge (u,v) belongs to no cycle in $G[D]$.*
♦ *Both u and v are γ_c-bad vertices of G.*

3.3 The effect of path addition on the connected domination number

In this section we will prove that $pa_{\gamma_c}(G) \in \{0,1,2,3\}$ by providing necessary and sufficient conditions for $0 \leq \bar{e}pa_{\gamma_c}(G) \leq \overline{E}pa_{\gamma_c}(G) \leq 3$, where $uv \in \overline{E}(G)$ for any connected graph G of order $n \geq 3$.

Theorem 3.6 *Let u and v be non adjacent vertices of a connected graph G, and let D be a γ_c-set of G, if $u,w \in D$ with $pn[w,D] = \{v\}$ and the removal of w do not disconnect $G[D-\{w\}]$, then $\gamma_c(G_{u,v,0}) = \gamma_c(G)-1$.*

Proof. *Let D be a γ_c-set of G and let $u,w \in D$ with $pn[w,D] = \{v\}$ and the removal of w do not disconnect $G[D-\{w\}]$, then in the graph $G_{u,v,0}$, by adding an edge uv to G, v will be adjacent to u and w, and it is obvious that w is adjacent to at least one vertex in D, since its removal do not disconnect $G[D-\{w\}]$, then by **Theorem 3.1**, $pn[w,D] = \{w\}$ and it is not a support vertex in $G_{u,v,0}[D]$, hence $D-\{w\}$ is a γ_c-set of $G_{u,v,0}$, thus, $\gamma_c(G_{u,v,0}) = \mid D-\{w\} \mid = \gamma_c(G)-1$.*

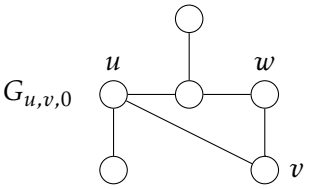


Figure 3.22: $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,0}) = 2$.

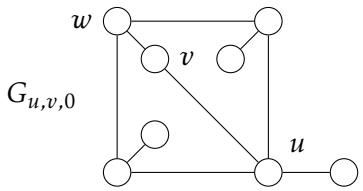
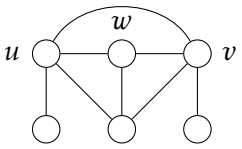


Figure 3.23: $\gamma_c(G) = 4 \neq \gamma_c(G_{u,v,0}) = 3$.

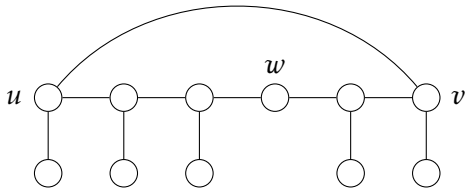
Theorem 3.7 *Let u and v be non adjacent vertices of a connected graph G, and let D be a γ_c-set of G, if $u,v,w \in D$ with $pn[w,D] = \{w\}$ (w is a support vertiex in $G[D]$), where w belongs to a path $P_{u,v}(G)$ which starts with u and ends with v in $G[D]$, then $\gamma_c(G_{u,v,0}) = \gamma_c(G) - 1$.*

Proof. *Let D be a γ_c-set of G and let $u,v,w \in D$ with $pn[w,D] = \{w\}$, where w belongs to a path $P_{u,v}(G)$ which starts with u and ends with v in $G[D]$, then in the graph $G_{u,v,0}$, by adding an edge uv to G, a cycle in $G[D]$ is obtained which contains u,v,w, again by **Theorem 3.1**, the vertex w is not a support vertex and $pn[w,D] = \{w\}$, then $D - \{w\}$ is a γ_c-set of $G_{u,v,0}$, thus, $\gamma_c(G_{u,v,0}) = \mid D - \{w\} \mid = \gamma_c(G) - 1$.*



$G_{u,v,0}$

Figure 3.24: $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,0}) = 2$.



$G_{u,v,0}$

Figure 3.25: $\gamma_c(G) = 6 \neq \gamma_c(G_{u,v,0}) = 5$.

Theorem 3.8 *Let u and v be non adjacent vertices of a connected graph G of order $n \geq 4$, let D be a γ_c-set of G, if there is a path $P_{u,v}(G)$ in $G[D]$, which starts with u and ends with v, such that $P_{u,v}(G)$ contains two adjacent vertices $t,w \in D$ where $pn[t,D] = \{t\}$ and $pn[w,D] = \{w\}$ (w and t are support vertices), then $\gamma_c(G_{u,v,0}) = \gamma_c(G) - 2$.*

Proof. *Let D be a γ_c-set of G with $u,v,t,w \in D$, and let $G[D]$ contains the path $P_{u,v}(G)$ of the endpoints u and v, which contains the adjacent vertices t and w, now in the graph $G_{u,v,0}$, since it is obtained by adding an edge uv between u and v in G, then a cycle is obtained in $G_{u,v,0}[D]$, which also contains u,w,t and v, if both t and w are support vertices but not adjacent, then this implies that the removal of either vertices would disconnect $G_{u,v,0}[D]$, now if at least one of them has a private neighbor in $V - D$, then **Theorem 3.7** is fulfilled, a contradiction, thus both t and w are adjacent support vertices (ie: $pn[w,D]) = \{w\}$ and $pn[t,D]) = \{t\}$), which implies that since both of them are adjacent to at least one vertex in D (besides each other) and belongs to a cycle in $G_{u,v,0}$, then the removal of w and t will not disconnect the γ_c-set $M = D - \{w,t\}$ of $G_{u,v,0}$ such that $\gamma_c(G_{u,v,0}) = \mid M \mid = \mid D - \{w,t\} \mid = \gamma_c(G) - 2$.*

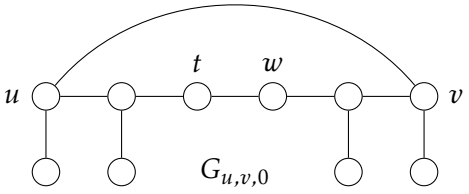
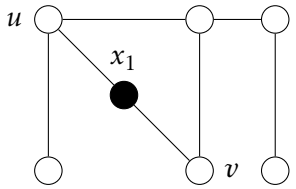


Figure 3.26: $\gamma_c(G) = 6 \neq \gamma_c(G_{u,v,0}) = 4$.

Theorem 3.9 *Let u and v be non adjacent vertices of a connected graph G, of order $n \geq 3$, then $\gamma_c(G) \leq \gamma_c(G_{u,v,1}) \leq \gamma_c(G) + 1$, moreover,*
(a) $\gamma_c(G_{u,v,1}) = \gamma_c(G)$ if and only if at least one of u and v is a γ_c-good vertex of G.
(b) $\gamma_c(G_{u,v,1}) = \gamma_c(G) + 1$ if and only if both u and v are γ_c-bad vertices of G.

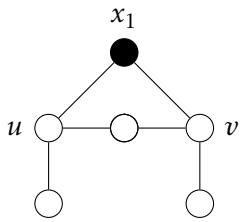
Proof.
(a)$\Rightarrow$ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G)$ holds, let D be a $\gamma_c -$set of G, suppose that u and v are both γ_c-bad vertices of G, then one of u and v must be in any $\gamma_c -$set M of $G_{u,v,1}$, hence, that leads to $\gamma_c(G_{u,v,1}) =\mid M \mid=\mid D \cup \{u\} \mid=\mid D \cup \{v\} \mid= \gamma_c(G) + 1$, a contradiction, thus, at least one of u and v is a γ_c-good vertex of G.
(a)$\Leftarrow$ Let M be a $\gamma_c -$ set of G, first let be $u \in M$ and v is a γ_c-bad vertex of G, then in $G_{u,v,1}$ the vertex v is dominated by some vertices in $M - \{u\}$ (since u and v are not adjacent in G), and $x_1 \in pn[u, M]$, then M is a $\gamma_c -$ set of $G_{u,v,1}$ with $\gamma_c(G_{u,v,1}) =\mid M \mid= \gamma_c(G)$, now if both $u, v \in M$, then it is obvious that $\gamma_c(G_{u,v,1}) = \gamma_c(G)$.

(b)$\Rightarrow$ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G) + 1$ holds, let D be a $\gamma_c -$ set of $G_{u,v,1}$, since $N(x_1) = \{u, v\}$, then one of $D - \{u\}$ or $D - \{v\}$ is a $\gamma_c -$ sets of G, and $\gamma_c(G) =\mid D - \{u\} \mid= \gamma_c(G_{u,v,1}) - 1$, thus both u and v are γ_c-bad vertices of G.
(b)$\Leftarrow$ Let M be a $\gamma_c -$ set of G, and let both u and v be γ_c-bad vertices of G, in the graph $G_{u,v,1}$, $N(x_1) = \{u, v\}$, thus both $M \cup \{u\}$ and $M \cup \{v\}$ are $\gamma_c -$ sets of $G_{u,v,1}$ with $\gamma_c(G_{u,v,1}) =\mid M \cup \{u\} \mid= \gamma_c(G) + 1$.



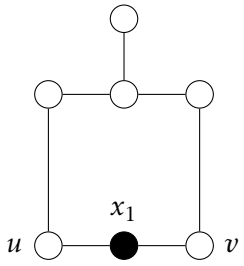
$G_{u,v,1}$

Figure 3.27: An illustration for (a), $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 3$, u and v are respectevely γ_c-good and γ_c-bad vertices of G.



$G_{u,v,1}$

Figure 3.28: An illustration for (a), $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 3$, u and v are both γ_c-good vertices of G.



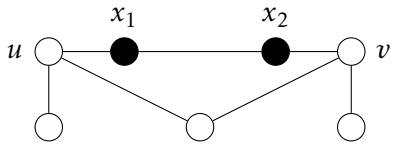
$G_{u,v,1}$

Figure 3.29: An illustration for (b), $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,1}) = 4$.

Theorem 3.10 *Let u and v be non adjacent vertices of a connected graph G, of order $n \geq 3$, then $\gamma_c(G) \leq \gamma_c(G_{u,v,2}) \leq \gamma_c(G) + 2$, moreover,*
(a) $\gamma_c(G_{u,v,2}) = \gamma_c(G)$ if and only if both u and v are in the same $\gamma_c -$ set of G.
(b) $\gamma_c(G) + 2 \geq \gamma_c(G_{u,v,2}) \geq \gamma_c(G) + 1$ if and only if every $\gamma_c -$ set of G contains at most one of u and v.

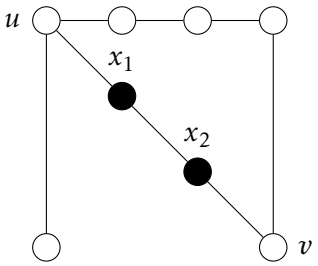
Proof.
(a)⇒ Let $\gamma_c(G_{u,v,2}) = \gamma_c(G)$ holds, let M be a γ_c-set of G, suppose that at most one of u and v is in M, then in $G_{u,v,2}$, at least one of x_1 and x_2 is not dominated by M, which leads to that any γ_c-set M' of $G_{u,v,2}$ is of cardinality $\mid M' \mid > \gamma_c(G)$, a contradiction, then $u, v \in M$.
(a)⇐ Let D be a γ_c-set of G and $u, v \in D$, now in the graph $G_{u,v,2}$, $x_1 \in pn[u, D]$ and $x_2 \in pn[v, D]$, hence, D is a γ_c-set of $G_{u,v,2}$ and $\gamma_c(G_{u,v,2}) = \gamma_c(G)$.

(b)⇒ Let $\gamma_c(G_{u,v,2}) \geq \gamma_c(G) + 1$ holds, let D be a γ_c-set of G, suppose that $u, v \in D$, then (a) is fulfilled and $\gamma_c(G_{u,v,2}) = \gamma_c(G)$, a contradiction, thus, D contains at most one of u and v.
(b)⇐ Let M be a γ_c-set of G, if $u \in M$ and $v \notin M$, then in $G_{u,v,2}$, x_2 is not dominated by the set M, thus $M \cup \{x_1\}$ or $M \cup \{v\}$ are both γ_c-sets of $G_{u,v,2}$, and $\gamma_c(G_{u,v,2}) = \mid M \cup \{v\} \mid = \gamma_c(G) + 1$, now if $u, v \notin M$, then both x_1 and x_2 are not dominated by M, thus $M \cup \{u, v\}$ is a γ_c-set of $G_{u,v,2}$ and $\gamma_c(G_{u,v,2}) = \mid M \cup \{u, v\} \mid = \gamma_c(G) + 2$, hence, $\gamma_c(G) + 2 \geq \gamma_c(G_{u,v,2}) \geq \gamma_c(G) + 1$.



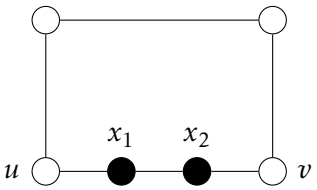
$G_{u,v,2}$

Figure 3.30: An illustration for (a), $\gamma_c(G) = \gamma_c(G_{u,v,2}) = 3$.



$G_{u,v,2}$

Figure 3.31: An illustration for (b), $\gamma_c(G) = 4 \neq \gamma_c(G_{u,v,2}) = 5$, u and v are respectevely γ_c-good and γ_c-bad vertices of G.



$G_{u,v,2}$

Figure 3.32: An illustration for (b), $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,2}) = 4$, u and v are both γ_c-bad vertices of G.

Theorem 3.11 *Let u and v be non adjacent vertices of a connected graph G, of order $n \geq 3$, then $\gamma_c(G_{u,v,3}) \geq \gamma_c(G) + 1$.*

Proof. *Let D be a γ_c-set of G, if $u, v \in D$, then in the graph $G_{u,v,3}$, the set D dominates all the vertices in $V(G_{u,v,3}) - \{x_2\}$, thus $D \cup \{x_1\}$ or $D \cup \{x_3\}$ is a γ_c-set of $G_{u,v,3}$ and $\gamma_c(G_{u,v,3}) = \mid D \cup \{x_1\} \mid = \gamma_c(G) + 1$, now if $u \in D$ and $v \notin D$, then in $G_{u,v,3}$, the set D dominates all the vertices in $V(G_{u,v,3}) - \{x_2, x_3\}$, thus, $D \cup \{x_1, v\}$, $D \cup \{x_1, x_2\}$ and $D \cup \{x_3, v\}$ are γ_c-sets of $G_{u,v,3}$ and $\gamma_c(G_{u,v,3}) = \mid D \cup \{x_1, v\} \mid = \gamma_c(G) + 2$, at last, if $u, v \notin D$, then D dominates all the vertices in $V(G_{u,v,3}) - \{x_1, x_2, x_3\}$, thus, $D \cup \{u, x_1, v\}$ is a γ_c-set of $G_{u,v,3}$ and $\gamma_c(G) = \mid D \cup \{u, x_1, v\} \mid = \gamma_c(G) + 3$, hence, $\gamma_c(G_{u,v,3}) \geq \gamma_c(G) + 1$.*

Corollary 3.5 *Let T be a tree of order $n \geq 4$ and D its γ_c-set, let u and v be non adjacent vertices of T,*
(i) Without loss of generality, if u is either a root or a support vertex of T and v is a leav, such that $pn[w, D] = \{v\}$, then $\gamma_c(T_{u,v,0}) = \gamma_c(T) - 1$.

(ii) If both u and v are not leavs of T, such that the path $P_{u,v}(T)$ which starts by u and ends by v contains one or two adjacent support vertices with no neighbors in $V-D$, then $\gamma_c(T_{u,v,0}) \in \{\gamma_c(T)-1, \gamma_c(T)-2\}$.
(iii) If both u and v are leavs, then $\gamma_c(T_{u,v,1}) = \gamma_c(T)+1$.

Proof.
*(i) Let D be a γ_c-set of T and let $u,w \in D$, with $pn[w,D]=\{v\}$, since the support vertex w is adjacent to some vertices in D, then in $T_{u,v,0}$, v is dominated by u, and by **Theorem 3.1**, w is no longer a support vertex in $T_{u,v,0}$ and $D-\{w\}$ is a γ_c-set of $T_{u,v,0}$ with $\gamma_c(T_{u,v,0}) =\mid D-\{w\} \mid= \gamma_c(T)-1$.*

*(ii) Is an immediate resulte from **Theorem 3.7** and **Theorem 3.8**.*

*(iii) Is an immediate resulte from (b) of **Theorem 3.9**.*

Corollary 3.6 *Let u and v be non adjacent vertices of a connected graph G with a universal vertex, then $\overline{E}pa_{\gamma_c}(G)=1$.*

Proof. *Let $w \in V(G)$ be a universal vertex, such that $d_G(w)=n-1$, with $\gamma_c(G)=1$, now in the graph $G_{u,v,1}$, $\forall t \in V(G_{u,v,1})$, $d_G(t) \leq n-2$, hence, there are no universal vertices in $G_{u,v,1}$, which implies that for any γ_c-set M of $G_{u,v,1}$, $\gamma_c(G_{u,v,1}) =\mid M \mid \geq 2 > \gamma_c(G)=1$. thus, $\overline{E}pa_{\gamma_c}(G)=1$.*

Corollary 3.7 *For any cycle C_n with $n \geq 4$, $\overline{E}pa_{\gamma_c}(C_n)=0$, and $\gamma_c(C'_n) \in \{\gamma_c(C_n)-1, \gamma_c(C_n)-2\}$ (ie: $C'_n=(V(C_n), E(C_n) \cup (u,v))$).*

Proof. *Is an immediate resulte from **Theorems 3.7** and **Theorems 3.8**.*

Corollary 3.8 *For any Path P_n with $n \geq 3$, $0 \leq \overline{e}pa_{\gamma_c}(P_n) \leq \overline{E}pa_{\gamma_c}(P_n) \leq 1$.*

Proof. *Is an immediate resulte from **Theorems 3.7** and **3.8**, and (b) of **Theorem 3.9**.*

Corollary 3.9 *Let G be a simple connected graph of order $n \geq 3$, then*
(i) $1 \leq epa_{\gamma_c}(G)+\overline{E}pa_{\gamma_c}(G) \leq 6$.
(ii) $1 \leq epa_{\gamma_c}(G)+\overline{e}pa_{\gamma_c}(G) \leq 5$.
(iii) $1 \leq Epa_{\gamma_c}(G)+\overline{E}pa_{\gamma_c}(G) \leq 6$.
(iv) $1 \leq Epa_{\gamma_c}(G)+\overline{e}pa_{\gamma_c}(G) \leq 5$.

Proof. *The inequalities are an immediat results from the previous theorems.*

Conclusions and perspectives

Conclusions on the changing and unchanging connected domination

The behavior of the parameter $\gamma_c(G)$ in the edge subdivision is subject to the relations between the edge's endpoints and the neighboring vertices, also to the type of the subgraph induced by $\gamma_c - sets$ which the endpoints of the edge belongs to, and in the path addition operation, $\gamma_c(G)$ is subject to the type of the endpoints of the added path, if whether they are γ_c-good or γ_c-bad vertices of the graph.

Open problems in the changing and unchanging domination

Problem 1 *Characterize all acyclic graphs G with* $Epa_{\gamma_c}(G) = 2$.

Problem 2 *Characterize all the graphs graphs G with* $Epa_{\gamma_c}(G) = \overline{E}pa_{\gamma_c}(G)$ *or* $epa_{\gamma_c}(G) = \overline{e}pa_{\gamma_c}(G)$.

Problem 3 *Characterize all the graphs G of* $\gamma_c(G) \geq 2$ *with* $\overline{E}pa_{\gamma_c}(G) = 1$.

Problem 4 *Characterize all the graphs G with* $\overline{e}pa_{\gamma_c}(G) = 3$.

Problem 5 *Find results on the connected* γ_c*-excellent graphs with* $\overline{e}pa_{\gamma_c}(G) = 0$.

Problem 6 *Characterize all the graphs that attain the bounds in* ***Corollary 3.9***.

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