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On Bell Numbers and Polynomials

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Dedication

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ABSTRACT

The Bell polynomials is widely used in many problems in physics and mathematics. The objective of this work is to study a class of generalizations of Bell numbers and polynomials, known as poly-Bell numbers and polynomials. In the first chapter, we introduce the basic concepts necessary for the development of the second chapter. The latter presents the class of Bell numbers and polynomials, along with their generalization. We define these concepts and discuss some fundamental properties such as generating functions, explicit expressions, recurrence relations, and certain identities involving these numbers and polynomials.

Keywords: Generating functions, explicit formula, recurrence relation, Stirling number, Bernoulli polynomials, Bell number, Bell polynomials

RÉSUMÉ

Les polynômes de Bell est largement utilisé dans de nombreux problèmes de physique et en mathématiques. L'objectif de ce mémoire est d'étudier une classe de généralisations des nombres et polynômes de Bell, appelés nombres et polynômes poly-Bell. Dans le premier chapitre, nous introduisons les notions de base nécessaires au développement du deuxième chapitre. Ce dernier présente la classe des nombres et des polynômes de Bell, ainsi que leur généralisation. Nous y définissons ces concepts et discutons de quelques propriétés fondamentales, telles que les fonctions génératrices, les expressions explicites, les relations de récurrence, ainsi que certaines identités impliquant ces nombres et polynômes.

Mots clé: Fonction génératrice, forme explicite, relation de recurrence, nombre de Stirling, polynôme de Bernoulli, nombre de Bell, polynôme de Bell .

Notations

$\mathbb{N}$	The set of natural numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$[n]$	The set $\{1, 2, \ldots, n\}$, for $n \in \mathbb{N}_*$.
$x^{\bar{n}}$	The product $x(x+1)\cdots(x+n-1)$ for $n \in \mathbb{N}$ and $x^{\bar{0}} = 1$.
$x^{\underline{n}}$	The product $x(x-1)\cdots(x-n+1)$ for $n \in \mathbb{N}$ and $x^{\underline{0}} = 1$.
$\binom{n}{k}$	The binomial coefficient.
${}_pF_q\left(\begin{matrix} a_1 \ldots a_p \\ b_1 \ldots b_q \end{matrix} ; x\right)$	Hypergeometric function.
$\Gamma(x)$	Gamma function.
$\beta(x, y)$	Beta function.
$s(n, k)$	Stirling numbers of the first kind.
$\mid s(n, k) \mid$	Unsigned Stirling numbers of the first kind.
$\left\{ {n \atop k} \right\}$	Stirling numbers of the second kind.
$\delta_{n,k}$	Kronecker symbol equal to 1 if $n = k$ and 0 otherwise.
$\left\{ {n \atop k} \right\}_r$	r-Stirling numbers of the second kind.
$:=$	Equality by definition (assignment).
$S_n^k(x)$	Weighted Stirling numbers of second kind.
$W_m(n, k)$	Withney Stirling numbers of the second kind.
$W_{m,r}(n, k)$	r-Withney Stirling numbers of the second kind.
B_n	Bernoulli numbers.

$B_n(x)$	Bernoulli polynomials.
$B_n^{(\alpha)}(x)$	Generalized Bernoulli polynomials.
ϕ_n	Bell numbers.
$\phi_n(x)$	Bell polynomials.
$B_{n,p}$	$p-$Bell numbers.
$\mathbb{B}_n^{(p)}$	Poly-Bell numbers.

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GENERAL INTRODUCTION

In this thesis entitled " On Bell Numbers and Polynomials", we gather several methods used to determine and study these class of numbers and polynomials, including explicit formulas, generating functions, and recurrence relations. The aim of this thesis is to study the generalization of Bell numbers. This is the Bell number, which represents the number of partitions of a set with n elements.

This study project consists of two chapters, in which:

Chapter 01:

In the first part, we will discuss the basic elements of combinatorial enumeration, namely the failling factorial, rising factorial, and binomial coefficients along with some of their properties. Following that, we will present the classical sequences of numbers and polynomials, including the Stirling numbers of the first and second kinds, generalized Stirling numbers of the second kind, as well as Bernoulli numbers, among others.

Chapter 02:

In the second part, we will focus on Bell polynomials, where we will study some properties of these numbers, this class were introduced by G. Guettai, D. Laissaoui, M. Rahmani and M. Sebaoui [17] .

CHAPTER 1

BASIC CONCEPTS

In this chapter, we give some introduce fundamental concepts that will be useful for what follows. We begin by recalling the rising and falling factorials, binomial coefficients, classic sequences (such as Stirling numbers, r-Stirling numbers, Whitney numbers, etc.), Bernoulli numbers, and Bernoulli polynomials.
This section is based on reference [8, 12, 15, 23, 24].

1.1 Binomial Coefficients

In this part we based on reference [15, 26]

Definition 1.1.1. *The binomial coefficient* $\binom{n}{k}$ *is the number of ways to choose a subset with* k *elements from a set with* n *elements.*

For $n \in \mathbb{N}$ *and* $k \in \mathbb{N}$*, the binomial coefficients is defined by*

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}. \tag{1.1}$$

Example 1. *From the set* $A = \{a, b, c, d, e\}$*, there are 10 subsets with two elements*

$$\{a, b\}\{a, c\}\{a, d\}\{a, e\}\{b, c\}\{b, d\}\{b, e\}\{c, d\}\{c, e\}\{d, e\}.$$

On the other hand

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = 10.$$

Properties 1.1.1. *For* $n \geq k \geq 0$

1. *Uper negation*

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k}. \tag{1.2}$$

2. *Symmetry*

$$\binom{n}{n-k} = \binom{n}{k}. \tag{1.3}$$

3. *Formula of Pascal*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{1.4}$$

4. *Newton binomial formula*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}. \tag{1.5}$$

Proof 1. *We begin by demonstrating the previous properties, for* $n \geq k \geq 0$

1. *We will demonstrate the formula (1.3)*

$$\begin{aligned}\binom{n}{n-k} &= \frac{n!}{(n-k)!(n-(n-k))!} \\ &= \frac{n!}{(n-k)!k!} \\ &= \binom{n}{k}.\end{aligned}$$

2. *We prove the formula of Pascal (1.4)*

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{k!(n-k-1)!(n-1)! + (n-1)!(k-1)!(n-k)!}{(k-1)!(n-k)!k!(n-k-1)!} \\ &= \frac{kk!(n-k-1)!(n-1)! + k(n-1)!(k-1)!(n-k)!}{k(k-1)!(n-k)!k!(n-k-1)!} \\ &= \frac{k(n-k-1)!(n-1)! + (n-1)!(n-k)!}{k!(n-k)!k(n-k-1)!} \\ &= \frac{k(n-k-1)!(n-1)! + (n-1)!(n-k-1)!(n-k)}{k!(n-k)!k(n-k-1)!} \\ &= \frac{k(n-1)! + (n-1)!(n-k)}{k!(n-k)!} \\ &= \frac{(n-1)!(k+n-k)}{k!(n-k)!} \\ &= \frac{(n-1)!n}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k}.\end{aligned}$$

3. *To prove Newton's binomial formula by induction on* n. *Suppose it is true for the first term* $n = 0$

$$(x+y)^0 = \sum_{k=0}^{0} x^0 y^{0-0} = 1.$$

Suppose $(x+y)^n$ *it is true, and we prove for* $(x+y)^{n+1}$, *we have*

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$$
$$(x+y)^{n+1} = (x+y)^n (x+y)^1.$$

On the assumption that

$$\begin{aligned}(x+y)^{n+1} &= (x+y)\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= x\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} + y\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}.\end{aligned}$$

We make an index change $k := k+1$ *in the first sum, we obtain*

$$\begin{aligned}(x+y)^{n+1} &= \sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k} \\ &= \binom{n}{n} x^{n+1} y^0 + \sum_{k=0}^{n} \binom{n}{k-1} x^k y^{n+1-k} + \binom{n}{0} x^0 y^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} \\ &= \binom{n}{n} x^{n+1} y^0 + \sum_{k=1}^{n} \left(\binom{n}{n-k} + \binom{n}{k} \right) x^k y^{n+1-k} + \binom{n}{0} x^0 y^{n+1}.\end{aligned}$$

According to Pascal's formula (1.4), we obtain

$$\begin{aligned}(x+y)^{n+1} &= x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} + y^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}.\end{aligned}$$

So $(x+y)^{n+1}$ *is true, from where* $(x+y)^n$ *is true .*

Pascal Triangl

From the formula of Pascal (1.4), we can construct Pascal's triangl

$n \diagdown k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Table 1.1: Triangl of Pascal

1.2 Generating Function

Definition 1.2.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, the ordinary generating function of this sequence is the function defined by*

$$A(x) = \sum_{n\geqslant 0} a_n x^n.$$

The exponential generating function is defined by

$$B(x) = \sum_{n\geqslant 0} a_n \frac{x^n}{n!}.$$

We can define the sum and the difference in the following

$$\sum_{n\geq 0} a_n x^n \pm \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0} (a_n \pm b_n) x^n,$$

and the product is defined by

$$\sum_{n\geq 0} a_n x^n \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0} \sum_{k=0}^{n} a_k b_{n-k} x^n.$$

The addition and multiplication of two exponential generating series are defined by

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} + \sum_{n\geq 0} b_n \frac{x^n}{n!} = \sum_{n\geq 0} (a_n + b_n) \frac{x^n}{n!},$$

and

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} \sum_{n\geq 0} b_n \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \frac{x^n}{n!}.$$

Definition 1.2.2. *Let n, m be natural numbers and $(a_{n,m})n, m \geq 0$ a sequence of complex numbers[16]. The double ordinary generating function is given by*

$$A(x, y) = \sum_{n\geq 0} \sum_{m\geq 0} a_{n,m} x^m y^n.$$

The double exponential generating function is given by

$$B(x,y)=\sum_{n\geq 0}\sum_{m\geq 0}a_{n,m}\frac{x^m}{m!}\frac{y^n}{n!},$$

and we define the mixed generating function by

$$C(x,y)=\sum_{n\geq 0}\sum_{m\geq 0}a_{n,m}\frac{x^m}{m!}y^n.$$

1.3 Gamma Function and Beta

See [1]. .

Definition 1.3.1. *For all $x,y>0$, the gamma function is defined*

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt. \tag{1.6}$$

The integral of the function $\Gamma(x)$ is sometimes called the Eulerian integral of the second type.

Properties 1.3.1.

$$\Gamma(x+1)=x\Gamma(x).$$

Definition 1.3.2. *For all $x,y>0$, the beta function denoted $\beta(x,y)$, is a type of Euler integral defined by*

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{1.7}$$

The Beta function $\beta(x,y)$ is related to the Gamma function by the following relation.

Theorem 1.

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

1.4 Failling Factorial

Definition 1.4.1. *Let x be a real number and $n\in\mathbb{N}$, the failling factorial of x order n denoted $x^{\underline{n}}$, is defined by*

$$x^{\underline{n}}=\begin{cases}1 & \text{if } n=0\\ x(x-1)(x-2)\cdots(x-n+1) & \text{if } n\geq 1\end{cases} \tag{1.8}$$

If $x=n$ in (1.8), we get the factorial $n^{\underline{n}}=n!$.

Examples 1.

$$x^{\underline{0}}=1$$
$$x^{\underline{1}}=x$$
$$x^{\underline{2}}=x(x-1)=x^2-x$$
$$x^{\underline{3}}=x(x-1)(x-2)=x^3-3x^2+2x$$
$$x^{\underline{4}}=x(x-1)(x-2)(x-3)=x^4-6x^3+11x^2-6x$$
$$x^{\underline{5}}=x(x-1)(x-2)(x-3)(x-4)=x^5-10x^4+35x^3-50x^2+24x.$$

Properties 1.4.1. *Let's $x \in \mathbb{C}$ and $m, n \in \mathbb{Z}$, then*

$$x^{\underline{n+m}} = x^{\underline{n}}(x-n)^{\underline{m}}. \tag{1.9}$$

$$x^{\underline{-m}} = \frac{1}{(x+m)^{\underline{m}}}, x \neq -1, -2, -3... - m. \tag{1.10}$$

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{n-k}} y^{\underline{k}}. \tag{1.11}$$

$$n^{\underline{n-1}} = n!. \tag{1.12}$$

$$(x+n-1)^{\underline{n}} = (-1)^n (-x)^{\underline{n}}. \tag{1.13}$$

$$x^{\underline{n}} = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}. \tag{1.14}$$

Proof 2. *We're starting to demonstrate the previous properties .*

1. *We will proved that (1.9)*

$$\begin{aligned}
x^{\underline{n+m}} &= x(x-1)(x-2)\cdots(x-(n+m-1)) \\
&= \prod_{j=0}^{n+m-1} (x-j) \\
&= \prod_{j=0}^{n-1} (x-j) \prod_{j=n}^{n+m-1} (x-j) \\
&= \left[\prod_{j=0}^{n-1} (x-j)\right] [(x-n)(x-(n+1))(x-(n+2))\cdots(x-(n+m-1))] \\
&= \left[\prod_{j=0}^{n-1} (x-j)\right] [(x-n)(x-n-1)(x-n-2)\cdots(x-n-(m+1))] \\
&= \prod_{j=0}^{n-1} (x-j) \prod_{j=n}^{m-1} ((x-n)-j) \\
&= x^{\underline{n}}(x-n)^{\underline{m}}.
\end{aligned}$$

2. *To prove proprty (1.10), we've got (1.9),*
   We set : $n \coloneqq -m$

$$\begin{aligned}
x^{\underline{0}} &= x^{\underline{-m}}(x+m)^{\underline{m}} \\
1 &= x^{\underline{-m}}(x+m)^{\underline{m}} \\
x^{\underline{-m}} &= \frac{1}{(x+m)^{\underline{m}}}.
\end{aligned}$$

3. *For the demonstration of formula (1.11), we use the recurrence method.*

By the hypothesis of recurrence, we've got

$$\begin{aligned} z^{\underline{n}} &= z(z-1)(z-2)\cdots(z-n+1) \\ &= z(z-1-2...)(z-n)(z-n+1) \\ &= z^{\underline{n-1}}(z-n+1). \end{aligned}$$

For $n := 1$, we obtain

$$\begin{aligned} (x+y)^{\underline{1}} &= \sum_{k=0}^{1} \binom{1}{k} x^{\underline{1-0}} y^{\underline{0}} \\ &= \binom{1}{0} x^{\underline{1}} y^{0} + \binom{1}{1} x^{\underline{0}} y^{\underline{1}} \\ &= x + y. \end{aligned}$$

The formula is true.
Let $n \in \mathbb{N}$, suppose that $(x+y)^{\underline{n-1}}$ is true and let's show $(x+y)^{\underline{n}}$ is true,

$$\begin{aligned} (x+y)^{\underline{n-1}} &= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{\underline{n-1-k}} y^{\underline{k}}. \\ (x+y)^{\underline{n}} &= (x+y-n+1)(x+y)^{\underline{n-1}} \\ &= (x+y-n+1) \sum_{k=0}^{n-1} \binom{n-1}{k} x^{\underline{n-1}} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{\underline{n-1}} y^{\underline{k}} + \sum_{k=0}^{n-1} \binom{n-1}{k} x^{\underline{n-1-k}} y^{\underline{k+1}} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{\underline{n-k}} y^{\underline{k}} + \sum_{k=1}^{n-1} \binom{n-1}{k-1} x^{\underline{n-k}} y^{\underline{k}} \\ &= x^{\underline{n}} + \sum_{k=1}^{n-1} \left(\binom{n+1}{k} + \binom{n-1}{k-1} \right) x^{\underline{n-k}} y^{\underline{k}} + y^{\underline{n}}. \end{aligned}$$

According to Pascal's formula, we obtain

$$(x+y)^{\underline{n}} = \sum_{k=0}^{n} \binom{n}{k} x^{\underline{n-k}} y^{\underline{k}}.$$

So $(x+y)^{\underline{n}}$ is true.

4. *We will now demonstrate formula (1.12)*

$$\begin{aligned} n^{\underline{n-1}} &= n! \binom{n}{n-1} \\ &= (n-1)! \frac{n!}{(n-1)!(n-(n-1))} \\ &= n!. \end{aligned}$$

5. *We'll demonstrate formula (1.13)*

$$\begin{aligned}(x+n-1)^{\underline{n}} &= (x+n-1)(x+n-2)\cdots(x+1)x \\ &= x(x+1)(x+2)\cdots(x+n-1) \\ &= (-1)(-x)(-1)(-x-1)(-1)(-x-2)\cdots(-1)(-x-n+1) \\ &= (-1)^n(-x)^{\underline{n}}.\end{aligned}$$

6. *Demonstration of formula (1.14)*

$$\binom{x}{n} = \frac{\Gamma(x+1)}{n!\Gamma(x-n+1)} = \frac{x!}{n!(x-n)!}.$$

For a non-negative integer x, we have :

$$x\Gamma(x) = \Gamma(x+1). \tag{1.15}$$

$$\begin{aligned}x^{\underline{n}} &= \prod_{j=0}^{n-1}(x-j) \\ &= x(x-1)(x-2)\cdots(x-n+1) \\ &= \frac{x!}{(x-n)!} \\ &= \frac{\Gamma(x+1)}{\Gamma(x-n+1)}.\end{aligned}$$

1.5 Rising Factorial

Definition 1.5.1. *Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$, the rising factorial of x order n denoted by $x^{\overline{n}}$, is defined by*

$$x^{\overline{n}} = \begin{cases} 1 & \text{if } n = 0 \\ x(x+1)(x+2)\cdots(x+n-1) & \text{if } n \geq 1 \end{cases} \tag{1.16}$$

Examples 2.

$$\begin{aligned}x^{\overline{0}} &= 1 \\ x^{\overline{1}} &= x \\ x^{\overline{2}} &= x(x+1) = x^2 + x \\ x^{\overline{3}} &= x(x+1)(x+2) = x^3 + 3x^2 + 2x \\ x^{\overline{4}} &= x(x+1)(x+2)(x+3) = x^4 + 6x^3 + 11x^2 + 6x \\ x^{\overline{5}} &= x(x+1)(x+2)(x+3)(x+4) =^5 +10x^4 + 35x^3 + 50x^2 + 24x.\end{aligned}$$

Properties 1.5.1. *Let $x \in \mathbb{C}$ and $m, n \in \mathbb{Z}$, then*

$$x^{\overline{n+m}} = x^{\overline{n}}(x+n)^{\overline{m}}. \tag{1.17}$$

$$(x)^{\overline{n}} = (-1)^n(-x)^{\underline{n}}. \tag{1.18}$$

$$x^{\overline{n}} = (x+n-1)^{\underline{n}}. \tag{1.19}$$

$$x^{\overline{n}} = (-1)^{\bar{n}}(1-n-x)^{\overline{n}}. \tag{1.20}$$

$$x^{\overline{n}} = \frac{\Gamma(x+n)}{\Gamma(x)}. \tag{1.21}$$

$$(x+1)^{\overline{n}} = x^{\overline{n}}\frac{(x+n)}{x}. \tag{1.22}$$

$$x^{\overline{n+1}} = x(x+1)^{\overline{n}}. \tag{1.23}$$

For $x \in \mathbb{N}$, *we have* $$x^{\overline{n}} = \frac{(x+n-1)!}{(x-1)!}. \tag{1.24}$$

Proof 3. *We begin by demonstrating the previous properties*

1. *We'll demonstrate formula (1.17)*

$$\begin{aligned} x^{\overline{n+m}} &= \prod_{j=0}^{n+m-1}(x+j) \\ &= \prod_{j=0}^{n-1}(x+j)\prod_{j=n}^{n+m-1}(x+j) \\ &= \prod_{j=0}^{n-1}(x+j)\prod_{j=0}^{m-1}((x+n)+j) \\ &= x^{\overline{n}}(x+n)^{\overline{m}}. \end{aligned}$$

2. *We will now demonstrate formula (1.18)*

$$\begin{aligned} x^{\overline{n}} &= x(x+1)(x+2)\cdots(x+n-1) \\ &= (-1)(-x)(-1)(-x-1)(-1)(-x-2)\cdots(-1)(-x-n+1) \\ &= (-1)^n((-x)(-x-1)(-x-2)\cdots(-x-n+1)) \\ &= (-1)^n(-x)^{\underline{n}}. \end{aligned}$$

3. *Now let us demonstrate formula (1.19)*

$$\begin{aligned} x^{\overline{n}} &= x(x+1)\cdots(x+n-2)(x+n-1) \\ &= (x+n-1)(x+n-2)\cdots(x+1)x \\ &= \prod_{j=0}^{n-1}((x+n-1)-j) \\ &= (x+n-1)^{\underline{n}}. \end{aligned}$$

4. *We use (1.19) to demonstrate property (1.20)*

$$\begin{aligned}(x+n-1)^{\underline{n}} &= (x+n-1)(x+n-2)(x+n-3)\cdots(x-1)x \\ &= (-1)(-x_n+1)(-1)(-x-n+2)(-1)(-x-n+3)\cdots(-1)(-x-1)(-1)(-x) \\ &= (-1)^n \prod_{j=0}^{n-1}((-x-n+1)+j) \\ &= (-1)^n(-x-n+1)^{\overline{n}}.\end{aligned}$$

5. *To demonstrate property (1.21), we turn to (1.15)*

$$\begin{aligned}\Gamma(x+1) &= x\Gamma(x) \\ \Gamma(x+2) &= (x+1)\Gamma(x+1) = x(x+1)\Gamma(x) \\ \Gamma(x+3) &= x(x+1)(x+2)\Gamma(x) \\ &\vdots \\ \Gamma(x+n) &= x(x+1)(x+2)\cdots(x+n-1)\Gamma(x) \\ \Gamma(x+n) &= x^{\overline{n}}\Gamma(x).\end{aligned}$$

Then : $x^{\overline{n}} = \frac{\Gamma(x+1)}{\Gamma(x)}$.

6. *Proof of formula (1.22)*

$$\begin{aligned}(x+1)^{\overline{n}} &= (x+1)(x+2)\cdots(x+1+n-1) \\ &= \frac{x}{x}(x+1)(x+2)\cdots(x+n-1)(x+n) \\ &= x^{\overline{n}}\frac{(x+n)}{x}.\end{aligned}$$

7. *We proved the property(1.23)*

$$\begin{aligned}x^{\overline{n+1}} &= x(x+1)(x+2)\cdots(x+n) \\ &= x(x+1)(x+2)\cdots((x+1)+n-1) \\ &= x(x+1)^{\overline{n}}.\end{aligned}$$

8. *We will demonstrate formula (1.24)*

$$\begin{aligned}\frac{(x+n-1)!}{(x-1)!} &= \frac{(1)(2)(3)\cdots(x-1)(x)(x+1)\cdots(x+n-1)}{(1)(2)(3)\cdots(x-1)} \\ &= x(x+1)(x+2)\cdots(x+n-1) \\ &= x^{\overline{n}}.\end{aligned}$$

1.6 Hypergeometric Function

Definition 1.6.1. *For $x \in \mathbb{C}$, we defined a generalized hypergeometric function by*

$$ {}_pF_q\left(\begin{array}{c} a_1 \dots a_p \\ b_1 \dots b_q \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{a_1^{\overline{n}} \cdots a_p^{\overline{n}}}{b_1^{\overline{n}} \cdots b_q^{\overline{n}}} \frac{x^n}{n!}. \tag{1.25} $$

If $p = q = 0$, we get the expononcial function

$$ {}_0F_0\left(\begin{array}{c} , \\ \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x. $$

If $p = 1, q = 0$, we obtain the binomial function

$$ {}_1F_0\left(\begin{array}{c} a, \\ \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{a^{\overline{n}} x^n}{n!} = (1-x)^a. $$

If $p = 2, q = 1$, we have

$$ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} ; x \right) = \sum_{n=0}^{\infty} \frac{a^{\overline{n}} b^{\overline{n}} x^n}{c^{\overline{n}} n!}; \qquad |x| \leq 1. $$

Theorem 2. *(**Differencial representation**)[14] For $x > 0$ and $c > b > a > 0$*

$$ \frac{d}{dx} {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} ; x \right) = \frac{ab}{c} {}_2F_1\left(\begin{array}{c} a+1, b+1 \\ c+1 \end{array} ; x \right). \tag{1.26} $$

Proof 4.

$$ \begin{aligned} \frac{d}{dx} {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} ; x \right) &= \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{a^{\overline{n}} b^{\overline{n}}}{c^{\overline{n}}} \frac{x^n}{n!} \right) \\ &= \sum_{n=1}^{\infty} \frac{a^{\overline{n}} b^{\overline{n}}}{c^{\overline{n}}} \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{a^{\overline{n+1}} b^{\overline{n+1}} x^n}{c^{\overline{n+1}} n!}. \end{aligned} $$

We have (1.23)

$$ \begin{aligned} \sum_{n=0}^{\infty} \frac{a^{\overline{n+1}} b^{\overline{n+1}} x^n}{c^{\overline{n+1}} n!} &= \sum_{n=0}^{\infty} \frac{\left(a(a+1)^{\overline{n}}\right)\left(b(b+1)^{\overline{n}}\right)}{(c(c+1)^{\overline{n}})} \frac{x^n}{n!} \\ &= \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)^{\overline{n}} (b+1)^{\overline{n}} x^n}{(c+1)^{\overline{n}} n!} \\ &= \frac{ab}{c} {}_2F_1\left(\begin{array}{c} a+1, b+1 \\ c+1 \end{array} ; x \right). \end{aligned} $$

Theorem 3. ***(Euler's integral representation)**[23]*
For $x > 0$ and $c > b > 0$

$$ {}_2F_1\left(\begin{array}{c} a, b \\ c \end{array} ; x \right) = \frac{\Gamma(c)}{\Gamma(a)(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} (1-xt)^{-b} dt. \tag{1.27} $$

Many important functions employed in analysis can be expressed by means of hypergeomertic functions

$$\frac{1}{(1-x)^a} = \sum_{n\geq 0} a^{\overline{n}} \frac{x^n}{n!} = {}_2F_1\left(\begin{matrix} a, 1 \\ 1 \end{matrix} ; x \right). \tag{1.28}$$

$$\ln(1+x) = x {}_2F_1\left(\begin{matrix} 1, 1 \\ 2 \end{matrix} ; -x \right). \tag{1.29}$$

$$e^x = \lim_{b\to\infty} {}_2F_1\left(\begin{matrix} 1, b \\ 1 \end{matrix} ; \frac{x}{b} \right). \tag{1.30}$$

1.7 Confluent Hypergeometric Function

Definition 1.7.1. *For $c \neq 0$, the Kummer confluent hypergeometric function [14] is defined by*

$$ {}_1F_1\left(\begin{matrix} a \\ c \end{matrix} ; x \right) = \sum_{n=0}^{\infty} \frac{a^{\overline{n}} x^n}{c^{\overline{n}} n!}, \qquad -\infty < x < +\infty. \tag{1.31}$$

We have

$$\lim_{b\to\infty} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; \frac{x}{b} \right) = {}_1F_1\left(\begin{matrix} a \\ c \end{matrix} ; x \right). \tag{1.32}$$

Where ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; \frac{x}{b} \right)$ is the Gauss formula.

Proof 5.

$$\begin{aligned}
\lim_{b\to\infty} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; \frac{x}{b} \right) &= \lim_{b\to\infty} \sum_{n=0}^{\infty} \frac{a^{\overline{n}} b^{\overline{n}} \frac{x^n}{b}}{c^{\overline{n}} n!} \\
&= \lim_{b\to\infty} \sum_{n=0}^{\infty} \frac{a^{\overline{n}} (b[1(1+\frac{1}{b})(1+\frac{2}{b})\cdots(1+\frac{n-1}{b})])\frac{x^n}{b}}{c^{\overline{n}} n!}. \\
&= \lim_{b\to\infty} \sum_{n=0}^{\infty} \frac{a^{\overline{n}} (1+\frac{1}{b})(1+\frac{2}{b})\cdots(1+\frac{n-1}{b}) x^n}{c^{\overline{n}} n!} \\
&= \sum_{n=0}^{\infty} \frac{a^{\overline{n}} x^n}{c^{\overline{n}} n!} \\
&= {}_1F_1\left(\begin{matrix} a \\ c \end{matrix} ; x \right).
\end{aligned}$$

Representation of function

$$ {}_1F_1\left(\begin{matrix} a \\ a \end{matrix} ; x \right) = e^x.$$

According to the Kummer transformation [1], we can write

$$ {}_1F_1\left(\begin{matrix} 1 \\ p+1 \end{matrix} ; e^z - 1 \right) = \exp(e^z - 1) {}_1F_1\left(\begin{matrix} p \\ p+1 \end{matrix} ; 1 - e^z \right). \tag{1.33}$$

Theorem 4. ***(Differencial representation)***

$$\frac{d}{dx}{}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right) = \frac{a}{c}{}_1F_1\left(\begin{array}{c} a+1 \\ c+1 \end{array};x\right). \tag{1.34}$$

Proof 6.

$$\begin{aligned}\frac{d}{dx}{}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right) &= \frac{d}{dx}\sum_{n\geqslant 0}\frac{a^{\overline{n}}x^n}{c^{\overline{n}}n!} \\ &= \sum_{n\geq 1}\frac{a^{\overline{n}}x^{n-1}}{c^{\overline{n}}(n-1)!} \\ &= \sum_{n\geq 0}\frac{a^{\overline{n+1}}x^n}{c^{\overline{n+1}}(n)!}.\end{aligned}$$

We have (1.23), we get

$$\begin{aligned}\sum_{n\geq 0}\frac{a^{\overline{n+1}}x^n}{c^{\overline{n+1}}(n)!} &= \sum_{n\geq 0}\frac{a(a+1)^{\overline{n}}x^n}{c(c+1)^{\overline{n}}n!} \\ &= \frac{a}{c}{}_1F_1\left(\begin{array}{c} a+1 \\ c+1 \end{array};x\right).\end{aligned}$$

In general

$$\begin{aligned}\frac{d^k}{dx^k}{}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right) &= \frac{a(a+1)\cdots(a+k-1)}{c(c+1)\cdots(c+k-1)}{}_1F_1\left(\begin{array}{c} a+k \\ c+k \end{array};x\right) \\ &= \frac{a^{\overline{k}}}{c^{\overline{k}}}{}_1F_1\left(\begin{array}{c} a+k \\ c+k \end{array};x\right).\end{aligned}$$

Theorem 5. ***(Integral representation)****[1] Based on Euler's integral representation for the ${}_1F_1$ hypergeometric function, on might expect that the confluent hypergeometric function satisfies, For $c > a > 0$*

$$\lim_{b\to\infty}{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};\frac{x}{b}\right) = {}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right). \tag{1.35}$$

$${}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{c-a-1}dt. \tag{1.36}$$

Theorem 6. *For $c > a > 0$, we have*

$${}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right) = e^x{}_1F_1\left(\begin{array}{c} c-a \\ c \end{array};-x\right).$$

Proof 7. *For $c > a > 0$, we have*

$$
\begin{aligned}
{}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right) &= \frac{\Gamma(c)e^x}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{-x+xt}t^{a-1}(1-t)^{c-a-1}dt \qquad (1.37)\\
&= e^x\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1\sum_{n\geqslant 0}\frac{(-x(1-t))^n}{n!}t^{a-1}(1-t)^{c-a-1}dt\\
&= e^x\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\sum_{n\geqslant 0}\frac{(-x)^n}{n!}\int_0^1(1-t)^n t^{a-1}(1-t)^{c-a-1}dt\\
&= e^x\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\sum_{n\geqslant 0}\frac{(-x)^n}{n!}\int_0^1 t^{a-1}(1-t)^{n+c-a-1}dt\\
&= e^x\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\sum_{n\geqslant 0}\frac{(-x)^n}{n!}\beta(a,n+c-a)\\
&= e^x\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\sum_{n\geqslant 0}\frac{(-x)^n}{n!}\frac{\Gamma(a).\Gamma(n+c-a)}{\Gamma(c+n)}\\
&= e^x\sum_{n\geqslant 0}\frac{(-x)^n}{n!}\frac{\Gamma(c).\Gamma(c-a+n)}{\Gamma(c+n)\Gamma(c-a)}\\
&= e^x\sum_{n\geqslant 0}\frac{(-x)^n}{n!}\frac{(c-a)^{\overline{n}}}{c^{\overline{n}}}\\
&= e^x{}_1F_1\left(\begin{array}{c} c-a \\ c \end{array};-x\right). \qquad (1.38)
\end{aligned}
$$

$e^{-x}{}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{-x(1-t)}t^{a-1}(1-t)^{c-a-1}dt,$
where $s=1-t\Rightarrow dt=-ds$

$$\frac{\Gamma(c)}{\Gamma(c-a)}\int_0^1 e^{s(-x)}(1-s)^{a-1}s^{c-a-1}ds={}_1F_1\left(\begin{array}{c} c-a \\ c \end{array};-x\right),$$

$c-(c-a)-1=a-1$

$$
\begin{aligned}
{}_1F_1\left(\begin{array}{c} c-a \\ c \end{array};-x\right) &= {}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right)\\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{c-a-1}dt.
\end{aligned}
$$

Confluent Hypergeometric Equation

The confluent hypergeometric equation established by "Kummer" [6] defines the hypergeometric function $U={}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};x\right)$ as a solution of Kummer equation

$$x(1-x)U^{''}(x)+(c-(a+b+1)x)U^{'}(x)-abU(x)=0. \qquad (1.39)$$

We take the change of variable $x:=\frac{t}{b}$,

$$(1-\frac{t}{b})U^{''}(x)+(c-t-\frac{a+1}{b}t)U^{'}(x)-aU(x)=0, \qquad (1.40)$$

Whene $b \to \infty$ we have

$$xU'' + (c-t)U' - aU = 0,$$

For $c \notin \mathbb{Z}$, can be written

$$U(x) = A\,{}_1F_1\left(\begin{array}{c} a \\ c \end{array}; x\right) + Bx^{1-c}\,{}_1F_1\left(\begin{array}{c} a+1-c \\ 2-c \end{array}; x\right),$$

where A and B arbitrary constants.

1.7.1 Recurrence Relation

1.

$$(c-a-1)\,{}_1F_1\left(\begin{array}{c} a \\ c \end{array}; x\right) + a\,{}_1F_1\left(\begin{array}{c} a+1 \\ c \end{array}; x\right) = (c-1)\,{}_1F_1\left(\begin{array}{c} a \\ c-1 \end{array}; x\right). \tag{1.41}$$

2.

$$c\,{}_1F_1\left(\begin{array}{c} a \\ c \end{array}; x\right) - c\,{}_1F_1\left(\begin{array}{c} a-1 \\ c \end{array}; x\right) = x\,{}_1F_1\left(\begin{array}{c} a \\ c+1 \end{array}; x\right). \tag{1.42}$$

Proof 8. *We will explain the recurrence relation*

1. *We proved (1.41)*

$$(c-a-1)\,{}_1F_1\left(\begin{array}{c} a \\ c \end{array}; x\right) + a\,{}_1F_1\left(\begin{array}{c} a+1 \\ c \end{array}; x\right) = (c-a-1)\sum_{n\geqslant 0}\frac{a^{\overline{n}}x^n}{c^{\overline{n}}n!} + a\sum_{n\geq 0}\frac{(a+1)^{\overline{n}}x^n}{c^{\overline{n}}n!}.$$

We have (1.22), we obtain

$$\begin{aligned}(c-a-1)\sum_{n\geq 0}\frac{a^{\overline{n}}}{(c-1)^n}\frac{c-1}{c-1+n}\frac{x^n}{n!} + a\sum_{n\geq 0}\frac{a^{\overline{n}}(a+n)}{(c-1)^{\overline{n}}a}\frac{(c-1)x^n}{(c-1+n)n!} &\\ = \sum_{n\geq 0}\frac{a^{\overline{n}}}{(c-1)^{\overline{n}}}\frac{c-1}{c-1+n}(c-a-1+a+n)\frac{x^n}{n!} &\\ = (c-1)\sum_{n\geq 0}\frac{a^{\overline{n}}(c+n-1)}{(c-1)^{\overline{n}}(c-1+n)}\frac{x^n}{n!} &\\ = (c-1)\sum_{n\geq 0}\frac{a^{\overline{n}}x^n}{(c-1)^{\overline{n}}n!} &\\ = (c-1)\,{}_1F_1\left(\begin{array}{c} a \\ c-1 \end{array}; x\right). &\end{aligned}$$

2. *We proved (1.42)*

$$c\,{}_1F_1\left(\begin{array}{c} a \\ c \end{array}; x\right) - c\,{}_1F_1\left(\begin{array}{c} a-1 \\ c \end{array}; x\right) = c\sum_{n\geq 0}\frac{a^{\overline{n}}x^n}{c^{\overline{n}}n!} - c\sum_{n\geq 0}\frac{(a-1)^{\overline{n}}x^n}{c^{\overline{n}}n!}.$$

We have (1.22)

$$
\begin{aligned}
c\sum_{n\geq 0}\frac{a^{\overline{n}}x^n}{c^{\overline{n}}n!}-c\sum_{n\geq 0}\frac{(a-1)^{\overline{n}}x^n}{c^{\overline{n}}n!} &= c\sum_{n\geq 0}\frac{a^{\overline{n}}}{(c+1)^{\overline{n}}c}(c+n)\frac{x^n}{n!}-c\sum_{n\geq 0}\frac{a^{\overline{n}}(a-1)(c+n)}{(a+n-1)(c+1)^{\overline{n}}c}\frac{x^n}{n!}\\
&= \sum_{n\geq 0}\left(\frac{a^{\overline{n}}(c+n)}{(c+1)^{\overline{n}}}-\frac{a^{\overline{n}}(a-1)}{(a+n-1)}\frac{(c+n)}{(c+1)^{\overline{n}}}\right)\frac{x^n}{n!}\\
&= \sum_{n\geq 0}\left[\frac{a^{\overline{n}}(c+n)}{(c+1)^{\overline{n}}}\left(1-\frac{a-1}{(a+n-1)}\right)\right]\frac{x^n}{n!}\\
&= \sum_{n\geq 0}\left[\frac{a^{\overline{n}}(c+n)}{(c+1)^{\overline{n}}}\frac{n}{(a+n-1)}\right]\frac{x^n}{n!}\\
&= \sum_{n\geq 1}\frac{a^{\overline{n}}}{(c+1)^{\overline{n}}}\frac{c+n}{(a+n-1)}\frac{x^n}{(n-1)!}\\
&= x\sum_{n\geq 1}\frac{a^{\overline{n}}}{(c+1)^{\overline{n}}}\frac{c+n}{(a+n-1)}\frac{x^{n-1}}{(n-1)!}\\
&= x\sum_{n\geq 0}\frac{a^{\overline{n+1}}}{(c+1)^{\overline{n+1}}}\frac{c+n+1}{(a+n)}\frac{x^n}{n!}\\
&= x\sum_{n\geq 0}\frac{a^{\overline{n}}(a+n)}{(c+1)^{\overline{n}}(c+n+1)}\frac{c+n+1}{(a+n)}\frac{x^n}{n!}\\
&= x\,{}_1F_1\left(\begin{matrix}a\\ c+1\end{matrix};x\right).
\end{aligned}
$$

Example 2. *For* $\Re(a) > 0$

$$\begin{aligned}
y(a,x) &= \int_0^x e^{-t} t^{a-1} dt \\
&= \int_0^x \sum_{n\geq 0} \frac{(-t)^n}{n!} t^{a-1} dt \\
&= \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_0^x (t)^n t^{a-1} dt \\
&= \sum_{n\geq 0} \frac{(-1)^n}{n!} \int_0^x (t)^{n+a-1} dt \\
&= \sum_{n\geq 0} \frac{(-1)^n}{n!} \left[\frac{t^{n+a}}{n+a} \right]_0^x \\
&= \sum_{n\geq 0} \frac{(-1)^n}{n!} \frac{x^{n+a}}{n+a} \\
&= x^a \sum_{n\geq 0} \frac{1}{n+a} \frac{(-x)^n}{n!} \\
&= x^a \sum_{n\geq 0} \frac{a^{\overline{n}}}{(a+n)^{\overline{n}} a} \frac{(-x)^n}{n!} \\
&= a^{-1} x^a \sum_{n\geq 0} \frac{a^{\overline{n}}}{(a+n)^{\overline{n}}} \frac{(-x)^n}{n!} \\
&= a^{-1} x^a {}_1F_1 \left(\begin{array}{c} a \\ a+1 \end{array} ; -x \right).
\end{aligned}$$

So that

$$y(a,x) = a^{-1} x^a {}_1F_1 \left(\begin{array}{c} a \\ a+1 \end{array} ; -x \right).$$

Example 3.

$$\begin{aligned}
erF(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \\
&= \frac{2x}{\sqrt{\pi}}{}_1F_1\left(\begin{array}{c}\frac{1}{2}\\ \frac{3}{2}\end{array};-x^2\right) \\
\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt &= \frac{2}{\sqrt{\pi}}\int_0^x \sum_{n\geq 0}\frac{(-t^2)^n}{n!}dt \\
&= \frac{2}{\sqrt{\pi}}\int_0^x \sum_{n\geq 0}(-1)^n\frac{(t^2)^n}{n!}dt \\
&= \frac{2}{\sqrt{\pi}}\sum_{n\geq 0}\frac{(-1)^n}{n!}\int_0^x t^{2n}dt \\
&= \frac{2}{\sqrt{\pi}}\sum_{n\geq 0}\frac{(-1)^n}{n!}\left[\frac{1}{2n+1}t^{2n+1}\right]_0^x \\
&= \frac{2}{\sqrt{\pi}}\sum_{n\geq 0}\frac{(-1)^n}{n!}\frac{1}{2n+1}x^{2n+1} \\
&= \frac{2x}{\sqrt{\pi}}\sum_{n\geq 0}\frac{1}{2n+1}\frac{(-x^2)^n}{n!}.
\end{aligned}$$

So

$$\begin{aligned}
erF(x) &= \frac{2x}{\sqrt{\pi}}\sum_{n\geq 0}\frac{(\frac{1}{2})^{\overline{n}}}{(\frac{3}{2})^{\overline{n}}}\frac{(-x^2)^n}{n!} \\
&= \frac{2x}{\sqrt{\pi}}{}_1F_1\left(\begin{array}{c}\frac{1}{2}\\ \frac{3}{2}\end{array};-x^2\right).
\end{aligned}$$

1.8 Stirling numbers

All the results discussed in this chapter are taken from the following references [12, 15, 22, 27]

1.8.1 The Stirling Number of the First Kind

Definition 1.8.1. *Let $s(n,k)$ denoted the usigned Stirling numbers of the first kind, are the coefficients of the developement of the failling factorial $x^{\underline{n}}$, are defined by*

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)x^k. \tag{1.43}$$

x^k is the canonical basis, and we defined by rising factorial

$$\begin{aligned} x^{\overline{n}} &= \sum_{k=0}^{n} |s(n,k)|x^k \\ &= \sum_{k=0}^{n} (-1)^{n-k} s(n,k)x^k. \end{aligned}$$

Explicit Formula

The explicit formula for the Stirling numbers of the first kind is given by

$$s(n,k) = \frac{n!}{k!} \sum_{i_1+i_2+\cdots+i_k=n} \frac{1}{i_1 \cdots i_k}. \tag{1.44}$$

Generating Function

The generating function of the Stirling numbers of the first kind [22] is given by

1. The $s(n,k)$ we can written by double generating function

$$\sum_{n,k\geq 0} s(n,k)\frac{t^n}{n!}x^k = (1+t)^x. \tag{1.45}$$

2. The exponentiel generating function is

$$\sum_{n\geq 0} s(n,k)\frac{t^n}{n!} = \frac{(\ln(1+t))^k}{k!}. \tag{1.46}$$

Proof 9. *We will demonstrate the generating functions of $s(n,k)$*

1. *We proved (1.45), we using Newton's binomial to prove this formula*

$$\begin{aligned} (1+t)^x &= \sum_{n\geq 0} \binom{x}{n} t^n \\ &= \sum_{n\geq 0} x^{\underline{n}} \frac{t^n}{n!}. \end{aligned}$$

whene $x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)x^k$.
So we written

$$(1+t)^x = \sum_{n,k\geq 0} S(n,k)\frac{t^n}{n!}x^k.$$

2. *As a demonstration of formula (1.46), we use the formula (1.45)*

$$\begin{aligned}\sum_{n,k\geq 0} s(n,k)\frac{t^n}{n!}x^k &= (1+t)^x \\ &= e^{x\ln(1+t)} \\ &= \sum_{k\geq 0}\frac{x^k(\ln 1+t)^k}{k!} \\ &= \sum_{k\geq 0}\frac{x^k(\ln 1+t)^k}{k!}.\end{aligned}$$

So

$$\begin{aligned}\sum_{n\geq 0} s(n,k)\frac{t^n}{n!}\sum_{k\geq 0}x^k &= \sum_{k\geq 0}\frac{x^k(\ln 1+t)^k}{k!} \\ \sum_{n\geq 0} s(n,k)\frac{t^n}{n!} &= \frac{(\ln 1+t)^k}{k!}.\end{aligned}$$

Equating the coeffitient x^k.

Recurrence Relation

The Stirling number of the first kind is given by the recurrence relation for $n,k\geq 1$

$$\begin{aligned}s(n,k) &= s(n-1,k-1)-(n-1)s(n-1,k); \\ s(n,0) &= s(0,k)=0, \text{except} \quad \text{s(0,0)=1 .}\end{aligned}$$

The First Values For Stirling Numbers of The First kind

$n \diagdown k$	1	2	3	4	5	6	7	8	9
1	1								
2	- 1	1							
3	2	- 3	1						
4	-6	11	-6	1					
5	24	-50	35	-10	1				
6	-120	274	-225	85	-15	1			
7	720	- 1764	1624	- 735	175	-21	1		
8	-5040	13068	-13132	6769	-1960	322	- 28	1	
9	40320	-109584	118124	-67284	22449	-4536	546	-36	1

Table 1.2: The first values for Stirling numbers of the first kind .

1.8.2 Stirling Number of the Second kind

Definition 1.8.2. *The Stirling number of the second kind* $\left\{ {n \atop k} \right\}$ *is the number of partitions of* $[n]$*:* $[n] = \{1, 2, 3 \cdots n\}$ *into exactly* k *parts.*

Explicit Formula

For $n \geqslant k$, the explicit formula of Stirling number of the second kind is

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{0 \leq i \leq k} (-1)^i \binom{k}{i} (k-i)^n. \tag{1.47}$$

Generating Fnction

The generating functions of the Stirling number of the second kind are given by

1. The exponential generating function of the Stirling numbers of second kind

$$\sum_{n \geq k} \left\{ {n \atop k} \right\} \frac{t^n}{n!} = \frac{(e^t - 1)^k}{k!}, \qquad k \geq 0. \tag{1.48}$$

Proof 10. *For* $n \geq 0$

$$\begin{aligned}
\sum_{n\geq k} \left\{ n \atop k \right\} \frac{t^n}{n!} &= \sum_{n\geq 0} \left\{ n \atop k \right\} \frac{t^n}{n!} \\
&= \sum_{n\geq 0} \left(\frac{1}{k!} \sum_{0\leq i\leq k} (-1)^i \binom{k}{i} (k-i)^n \right) \frac{t^n}{n!} \\
&= \frac{1}{k!} \sum_{n\geqslant 0} \sum_{0\leqslant i\leqslant k} (-1)^i \binom{k}{i} (k-i)^n \frac{t^n}{n!} \\
&= \frac{1}{k!} \sum_{0\leqslant i\leqslant k} (-1)^i \binom{k}{i} \sum_{n\geqslant 0} \frac{t^n (k-i)^n}{n!} \\
&= \frac{1}{k!} \sum_{0\leqslant i\leqslant k} (-1)^i \binom{k}{i} e^{(t)^{k-i}} \\
&= \frac{1}{k!} (e^t - 1)^k.
\end{aligned}$$

2. The ordinary generating function for Stirling number is given by

$$\sum_{n\geq k} \left\{ n \atop k \right\} t^n = \frac{x^k}{(1-x)(1-2x)...(1-kx)}. \tag{1.49}$$

3. Double generating function

$$\sum_{k\geqslant 0} \sum_{n\geq k} \left\{ n \atop k \right\} \frac{x^n}{n!} U^k = e^{U(e^x - 1)}. \tag{1.50}$$

Recurrence Relation

The Stirling numbers of the second kind satisfy the following recurrence relation

1. For $n, k \geq 1$

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\}. \tag{1.51}$$

2. The vertical recurrence relation

$$\left\{ n \atop k \right\} = \sum_{m=k}^{n} \left\{ m-1 \atop k-1 \right\} k^{n-m}. \tag{1.52}$$

3. The horizontal recurrence relations

$$\left\{ n \atop k \right\} = \sum_{k\leq i\leq n} \left\{ n+1 \atop i+1 \right\} n^{i-k}. \tag{1.53}$$

The First Values Stirling Numbers of The Second Kind

$n \diagdown k$	1	2	3	4	5	6	7	8	9
1	1								
2	1	1							
3	1	3	1						
4	1	7	6	1					
5	1	15	25	10	1				
6	1	31	90	65	15	1			
7	1	63	301	350	140	21	1		
8	1	127	966	1701	1050	266	28	1	
9	1	255	3025	7770	6951	2646	462	36	1

Table 1.3: The first values for Stirling numbers of the second kind .

1.9 $r-$Stirling Numbers of the Second Kind

Definition 1.9.1. *The $r-$Stirling numbers of the second kind are defined for all positives r, denoted $\left\{ {n \atop k} \right\}_r$ count the number of partitions of set n into k non-empty subset [5].*
The $r-$Stirling numbers correspond to the coefficients in the devloppement ($r \leq n$)

$$(x+r)^n = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r x^{\underline{n}}. \tag{1.54}$$

1.9.1 Generating Function

The exponential generating function of the second Kind $r-$Stirling number is given by

$$\sum_{n \geq k} \left\{ {n+r \atop k+r} \right\}_r \frac{x^n}{n!} = \frac{1}{k!} e^{rx} (e^x - 1)^k.$$

The ordinary generating functions of the second space $r-$Stirling number is given by

$$\sum_{n \geq k} \left\{ {n+r \atop k+r} \right\}_r x^n = \frac{x^k}{(1-rx)(1-(r+1)x)\cdots(1-(k+r)x)}; \qquad k \geq r \geq 0.$$

The double generating function is given by

$$\sum_{n,k \geq 0} \left\{ {n+r \atop k+r} \right\}_r \frac{x^n}{n!} y^k = e^{y(e^x-1)+rx}.$$

1.9.2 Explicit Formula

The explicit formula for $r-$Stirling numbers is given by

$$\left\{ {n+r \atop k+r} \right\}_r = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (r+j)^n. \tag{1.55}$$

1.9.3 Recurrence Relation

These numbers verify the following recurrence relation for $k \geq 0$ and $n \geq r$,

$$\left\{ n \atop k \right\}_r = \left\{ n-1 \atop k-1 \right\}_r + k \left\{ n-1 \atop k \right\}.$$

For $n \leq r$

$$\left\{ n \atop k \right\}_r = 0.$$

For $n = r$

$$\left\{ n \atop k \right\}_r = \delta_{k,r};$$

$$\delta_{k,r} = \begin{cases} 1 & \text{if } k = r \\ 0 & \text{if } k \neq r \end{cases}$$

Another forme of recurrence can also be found in the liturature, namly this relation
For $n \geq k \geq r \geq 0$

$$\left\{ n \atop k \right\}_r = \left\{ n \atop k \right\}_{r-1} - (r-1) \left\{ n-1 \atop k \right\}_{r-1}.$$

For $n \geq r$

$$\left\{ n \atop k \right\}_r = r^{n-r}.$$

The first values

Recurrence allows us to illustrate the first values of these numbers for $r = 2, r = 3$ given by the tables below

$n \diagdown k$	2	3	4	5	6	7	8	9
2	1							
3	2	1						
4	4	5	1					
5	8	19	9	1				
6	16	65	55	14	1			
7	32	211	285	215	20	1		
8	64	655	1351	910	245	27	1	
9	128	2059	6069	5901	2380	434	35	1

Table 1.4: The first values for 2-Stirling numbers of the second kind .

$n \diagdown k$	3	4	5	6	7	8	9	10
3	1							
4	3	1						
5	9	7	1					
6	27	37	12	1				
7	81	175	97	18	1			
8	243	781	660	205	25	1		
9	729	3367	4081	1890	380	33	1	
10	2181	14197	23772	15421	4550	644	42	1

Table 1.5: The first values for 3-Stirling numbers of the second kind

1.10 Generalized Stirling Transform

This part is based on reference [20]
The sequence b_n is the Stirling transformation of the sequence a_n

$$b_n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} a_n. \tag{1.56}$$

The inverse of the transformation is

$$a_n = \sum_{k=0}^{n} s(n,k) b_k. \tag{1.57}$$

We study a general identity that generalises the Stirling transform and considers the corresponding generating function as will. We generate an infinite matrix $\mathbf{S} := a_{n,m}$, for $a_m := a_{0,m}(m \geqslant 0)$. The initial sequence of the matrix is the first row $a_{0,m}$, and $b_n = a_{n,0}(n \geqslant 0)$ is the first column and is called the terminal sequence each element of the matrix $a_{0,m}$. Given recursively

$$a_{n+1,m} = a_{n,m+1} + m a_{n,m}, \tag{1.58}$$

and we can recovred the matrix $\mathbf{S}$ by the recurrence relation

$$a_{n,m+1} = a_{n+1,m} - m a_{n,m}. \tag{1.59}$$

Theorem 7. *Using (1.58), and the initial sequence $a_{0,m}(m \geqslant 0)$, we determine the matrix $\mathbf{S}$. The element of the initial matrix $\mathbf{S}$are given by*

$$a_{n,m} = \sum_{k=0}^{n} \begin{Bmatrix} n+m \\ k+m \end{Bmatrix}_m a_{0,m+k}. \tag{1.60}$$

Theorem 8. *Given a final sequence $a_{n,0}(n \geqslant 0)$, define the matrix $\mathbf{S}$ by (1.59).Then, the entries of the infinite matrix $\mathbf{S}$ are given by*

$$a_{n,m} = \sum_{k=0}^{n} s(n,k) a_{n+k,0}. \tag{1.61}$$

Corollary 1.

$$\sum_{k=0}^{n} s(n,k)b_{n+k} = \sum_{k=0}^{n} \left\{ {n+m \atop k+m} \right\}_{m} a_{m+k}. \tag{1.62}$$

Example 4. *Put* $a_{0,m} = 1$ *in (1.62). wee get the result*

$$\sum_{k=0}^{m} s(m,k)B_{n+k} = \sum_{k=0}^{n} \left\{ {n+m \atop k+m} \right\}_{m}.$$

1.11 The Weighted Numbers of the Second Kind

1.11.1 Generating Function

The generating function of the weighted numbers of the second kind [7, 16], denoted $\mathcal{S}_n^k(x)$ is defined by

$$\sum_{n \geqslant k} \mathcal{S}_n^k(x) \frac{z^n}{n!} = \frac{1}{k!} e^{xz} (e^z - 1)^k. \tag{1.63}$$

1.11.2 Explicit Formula

The explicit formula is given by

$$\mathcal{S}_n^k(x) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j)^n.$$

1.12 The Whitney Numbers of the Second Kind

For a variety of results concerning the Whitney numbers, see [21]

1.12.1 Generating Function

The Whitney numbers of the second kind denoted $W_m(n,k)$ are defined by the generating function

$$\sum_{n \geq k} W_m(n,k) \frac{z^n}{n!} = \frac{e^z}{m^k k!} (e^{mz} - 1)^k.$$

1.12.2 Explicit Formula

The explicit formula is given by

$$\begin{aligned} W_m(n,k) &= \sum_{j=k}^{n} \binom{n}{j} m^{j-k} \left\{ {j \atop k} \right\} \\ &= \frac{1}{m^k k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (mj+1)^n. \end{aligned}$$

1.12.3 Recurrence Relation

Given with the formula

$$W_m(n,k) = (1+mk)W_m(n-1,k) + W_m(n-k,k-1).$$

For all integer $m \geq 0$ we will define $r-$Whitney numbers of second kind

$$W_{m,r}(n,k) = m^{n-k}\mathcal{S}_n^k(\frac{r}{m}). \tag{1.64}$$

We use (1.54), we get

$$(mx+r)^r = \sum_{k=0}^{n} m^k W_{m,r}(n,k)x^{\underline{n}}.$$

1.13 Bernoulli Numbers

Reference based [3, 15]

Definition 1.13.1. *For all $n \in \mathbb{N}$, the Bernoulli numbers denoted B_n, are defined by*

$$B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k, \tag{1.65}$$

with $B_0 = 1$.

1.13.1 Generating Function

The generating function of the Bernoulli numbers is given by

$$\sum_{n\geq 0} B_n\frac{t^n}{n!} = \frac{t}{e^t-1}; \qquad |t| < 2\pi.$$

1.13.2 Explicit Formula

The explicit formula for Bernoulli numbers is given by

$$B_n = \sum_{k=0}^{n}\left\{{n \atop k}\right\}(-1)^k\frac{k!}{k+1}.$$

1.13.3 Recurrence Relation

The recurrence relation of Bernoulli numbers is given by

$$\begin{cases} B_0 = 1 \\ \sum_{k=0}^{n}\binom{n+1}{k}B_n = 0. \end{cases}$$

The first values of the Bernoulli numbers are given in the following table

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

Table 1.6: The first values of Bernouli numbers

1.14 Bernoulli Polynomials

Definition 1.14.1. *For $n \in \mathbb{N}$, the Bernoulli polynomials denoted $B_n(x)$, are given by*

$$\begin{cases} B_n(0) = B_n \\ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \end{cases} .$$

1.14.1 Generating Function

The exponontial generating function of Bernoulli polynomials is given by

$$\sum_{n\geq 0} B_n(x)\frac{t^n}{n!} = \frac{te^{tx}}{e^t - 1}; \qquad |t| < 2\pi. \tag{1.66}$$

The first values of these polynomials

$$\begin{aligned} B_0(x) &= 1 \\ B_1(x) &= x - \frac{1}{2} \\ B_2(x) &= x^2 - x + \frac{1}{6} \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\ B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30} \\ B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x. \end{aligned}$$

1.15 Generalized Bernoulli

For a series of results on the general Bernoulli, see [4, 16, 25]

Definition 1.15.1. *For $\alpha \in \mathbb{C}$, we define generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ by the generating function [16]*

$$\frac{t^\alpha e^{xt}}{(e^t - 1)^\alpha} = \sum_{k=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \qquad |t| < 2\pi. \tag{1.67}$$

If $\alpha = 1$ we say classical Bernoulli polynomials, and $B_n^{(\alpha)}(0) = B_n^{(\alpha)}$.

Explicit Formula

We can write the generalized Bernoulli polynomials by the explicit formula

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(\alpha)} x^{n-k}.$$

CHAPTER 2

BELL NUMBERS AND POLYNOMIALS

We present in this chapter some properties related to the p-Bell numbers, the poly-Bell numbers, the p-Bell polynomials by generating functions and recurrence relation.
All of the results that are discussed in this chapter have been taken from the following references [16, 17, 23].

2.1 Bell Numbers

Definition 2.1.1. *We call the Bell numbers [22] the number of all partitions in the set* $[n]$, *and denoted that* ϕ_n
They are defined as the sum of the Stirling numbers of the second kind.
Let $k \in \mathbb{N}$

$$\phi_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}; \qquad n \geq 1 \tag{2.1}$$

If $n = 0$, *we get* $\phi_0 = 1$.

The first set of ϕ_n values is shown in the table below

n	0	1	2	3	4	5	6	7	8	9	10
ϕ_n	1	1	2	5	15	52	203	877	4140	21147	115975

Table 2.1: The first values set of ϕ_n

Example 5. *The set of partitins for the* $X = \{a, b, c\}$, *consisting of three elements is* $\{p1, p2, p3, p4, p5\}$ *with*

$$\begin{aligned} p1 &= \{\{a\}, \{b\}, \{c\}\} \\ p2 &= \{\{a, b\}, \{c\}\} \\ p3 &= \{\{a, c\}, \{b\}\} \\ p4 &= \{\{b, c\}, \{a\}\} \\ p5 &= \{a, b, c\}. \end{aligned}$$

Example 6. *The following example illustrates the meaning of Bell numbers, we have*

$$\phi_4 = \left\{ {4 \atop 0} \right\} + \left\{ {4 \atop 1} \right\} + \left\{ {4 \atop 2} \right\} + \left\{ {4 \atop 3} \right\} + \left\{ {4 \atop 4} \right\}$$

$\left\{ {4 \atop 0} \right\} = 0$,

$\left\{ {4 \atop 1} \right\} = 1$, *counts the number of partitions of a set of 4 elements in a subset* $\{1, 2, 3, 4\}$.

$\left\{ {4 \atop 2} \right\} = 7$, *counts the number of partitions of 4 elements into two subsets :* $\{\{1\}, \{2, 3, 4\}\}, \{\{2\}, \{1, 3, 4\}\}, \{\{3\}, \{1, 2, 4\}\}, \{\{4\}, \{1, 2, 3\}\}, \{\{1, 2\}, \{3, 4\}\}, \{\{1, 3\}, \{2, 4\}\}$, $\{\{1, 4\}, \{2, 3\}\}$,

$\left\{ {4 \atop 3} \right\} = 6$, *counts the number of partitions of 4 elements into three subsets :* $\{\{1\}\{2\}\{3, 4\}\}, \{\{1\}\{3\}\{2, 4\}\}, \{\{1\}\{4\}\{2, 3\}\}, \{\{3\}\{2\}\{1, 4\}\}, \{\{3\}\{4\}\{1, 2\}\}, \{\{4\}\{2\}\{, 3\}\}$

$\left\{ {4 \atop 4} \right\} = 1$, *counts the number of partitions of 4 elements into four subsets :* $\{\{1\}, \{2\}, \{3\}, \{4\}\}$.

So : $\phi_4 = 0 + 1 + 7 + 6 + 1 = 15$.

Proposition 2.1.1. (Dobinski formula) *For* $n \geq 0$, *we have*

$$\phi_n = \frac{1}{e} \sum_{k \geq 0} \frac{k^n}{k!}. \tag{2.2}$$

Proof 11. *Let* f *a function be defined on* $\mathbb{R}$ *by :* $f(z) = e^{e^z - 1}$.
Accroding to formula (2.1) since the function f *is derivable on this interval.*
We have

$$\begin{aligned}
f'(z) &= e^z e^{e^z - 1} \\
&= \left(\sum_{n \geq 0} \frac{z^n}{n!} \right) \left(\sum_{n \geq 0} \phi_n \frac{z^n}{n!} \right) \\
&= \left(\sum_{n \geq 0} \frac{z^n}{n!} \right) \left(\sum_{k=0}^{n} \binom{n}{k} \phi_k \right) \\
&= \sum_{n \geq 0} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \phi_k \frac{z^n}{n!} \\
&= \sum_{n \geq 0} \sum_{k=0}^{n} \frac{\phi_k}{k!(n-k)!} z^n \\
&= \sum_{n \geq 0} \sum_{k=0}^{n} \frac{z^{n-k}}{(n-k)!} \frac{z^k}{k!} \phi_k \\
&= \sum_{k=0}^{n} \phi_k \frac{z^k}{k!} \sum_{n \geq k} \frac{z^{n-k}}{(n-k)!} \\
&= f(z) e^z.
\end{aligned}$$

So there exists $c \in \mathbb{R}$, such that $f(z) = ce^{e^z}$, then

$$f(z) = e^{-1}e^{e^z},$$

For $z \in \mathbb{R}$

$$\begin{aligned} e^{e^z} &= \sum_{n\geq 0} \frac{(e^z)^n}{n!} \\ &= \sum_{n\geq 0} \frac{e^{nz}}{n!} \\ &= \sum_{n\geq 0}\sum_{k\geq 0} \frac{(nz)^k}{k!n!} \\ &= \sum_{n\geq 0} \frac{z^n}{n!} \sum_{k\geq 0} \frac{k^n}{k!}. \end{aligned}$$

So

$$e^{e^z-1} = \sum_{n\geq 0} \frac{z^n}{n!} \sum_{k\geq 0} \frac{k^n}{k!} \frac{1}{e}.$$

And we have

$$\sum_{n\geq 0} \phi_n \frac{z^n}{n!} = e^{e^z-1}.$$

So

$$\begin{aligned} \sum_{n\geq 0} \phi_n \frac{z^n}{n!} &= \frac{1}{e} \sum_{n\geq 0} \frac{z^n}{n!} \sum_{k\geq 0} \frac{k^n}{k!} \\ \phi_n &= \frac{1}{e} \sum_{k\geq 0} \frac{k^n}{k!}. \end{aligned}$$

Equating the coefficients of $\frac{z^n}{n!}$, we get the result.

2.1.1 Generating Function

The Bell numbers have the following generating functions

1. The exponential generation function

$$\sum_{n\geq 0} \phi_n \frac{z^n}{n!} = e^{e^z-1}. \tag{2.3}$$

2. The ordinary generating function

$$\sum_{n\geq 0} \phi_n z^n = \sum_{n\geq 0} \frac{z^n}{(1-z)(1-2z)\cdots(1-nz)}. \tag{2.4}$$

2.1.2 Recurrence Relation

We defined Bell numbers in termes of the following recurrence relation, for all integer $n \geq 0$, $\phi_0 = 1$

$$\phi_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \phi_k. \tag{2.5}$$

Proof 12. *For $k \in [0, n]$, consider the set E_k of partitions of $[1, n+1]$for which the part of $[1, n+1]$ containing $n+1$ is of cardinal $k+1$. card$E_k = \binom{n}{k}\phi_{n-k}$.In effet, we choose the k which will be with $n+1$it then suffit to count the partitions of the $n-k$ elements. Since $E_0, E_1, \cdots, E_n$ form a partition of the set $[1, n+1]$, we obtain*

$$\begin{aligned}
\phi_{n+1} &= \sum_{k=0}^{n} \binom{n}{k} \phi_{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{n-k} \phi_k \\
&= \sum_{k=0}^{n} \binom{n}{k} \phi_k.
\end{aligned}$$

2.2 Bell Polynomials

Definition 2.2.1. *For $n \in \mathbb{N}$, the Bell polynomials $\phi_n(x)$ are definis by*

$$\phi_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k. \tag{2.6}$$

These polynomials are called Bell polynomials with a single variable or exponential polynomials.

2.2.1 Generating Function

The exponential generating function of the Bell polynomial is

$$\sum_{n \geq 0} \phi_n(x) \frac{t^n}{n!} = e^{x(e^t - 1)}. \tag{2.7}$$

The ordinary generating function of the Bell polynomial is

$$\sum_{n \geq 0} \phi_n x^n = \sum_{n \geq 0} x^n \prod_{i=0}^{n} (1 - ix)^{-1}. \tag{2.8}$$

2.2.2 Explicit Formula

The explicit formula of Bell polynomials [11] is given by

$$\phi_n(x) = e^{-x} \sum_{k \geq 0} \frac{k^n}{k!} x^k \textbf{(Dobinski formula)}. \tag{2.9}$$

2.2.3 Recurrence Relation

The Bell polynomials are realized by the following recurrence relation

$$\phi_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \phi_n(x). \tag{2.10}$$

The First Few Bell Polynomials

n	$\phi_n(x)$
0	1
1	x
2	$x^2 + x$
3	$x^3 + 3x^2 + x$
4	$x^4 + 6x^3 + 7x^2 + x$
5	$x^5 + 10x^4 + 25x^3 + 15x^2 + x$
6	$x^6 + 15x^5 + 65x^4 + 90x^3 + 31x^2 + x$

Table 2.2: The first few values of the Bell polynomials.

2.3 $p-$Bell Numbers

Definition 2.3.1. *For every integer $p \geq 0$ and $n \geq 0$, the sequence of rational numbers $B_{n,p}$ is defined by the following generating function*

$$f_p(z) = \sum_{n\geq 0} B_{n,p} \frac{z^n}{n!} = {}_1F_1\left(\begin{array}{c} 1 \\ p+1 \end{array} ; e^z - 1 \right) = \sum_{n\geq 0} \binom{n+p}{p}^{-1} \frac{(e^z-1)^n}{n!}. \tag{2.11}$$

If $p = 0$, the p-Bell numbers reduce to the classical Bell numbers : $B_{n,0} = \phi_n = {}_1F_1\left(\begin{array}{c} a \\ c \end{array} ; z \right)$ the kummer confluent hypergeometric function .

The first exponential generating are

$$\sum_{n\geq 0} B_{n,1} \frac{z^n}{n!} = \frac{\exp(e^z - 1) - 1}{e^z - 1}. \tag{2.12}$$

$$\sum_{n\geq 0} B_{n,2} \frac{z^n}{n!} = \frac{2(\exp(e^z - 1) - e^z)}{(e^z - 1)^2}. \tag{2.13}$$

$$\sum_{n\geq 0} B_{n,3} \frac{z^n}{n!} = \frac{3(2\exp(e^z - 1) - e^{2z} - 1)}{(e^z - 1)^3}. \tag{2.14}$$

Proof 13. *Will demonstrate (2.12)*

$$\begin{aligned}
\sum_{n\geq 0} B_{n,1}\frac{x^n}{n!} &= \sum_{n\geq 0}\binom{n+1}{1}^{-1}\frac{(e^z-1)^n}{n!}\\
&= \sum_{n\geq 0}\frac{1!n!}{(n+1)!}\frac{(e^z-1)^n}{n!}\\
&= \sum_{n\geq 0}\frac{(e^z-1)^n}{(n+1)!}\\
&= \sum_{n\geq 0}\frac{(e^z-1)^{n+1-1}}{(n+1)!}\\
&= (e^z-1)^{-1}\sum_{n\geq 1}\frac{(e^z-1)^{n+1}}{(n+1)!}\\
&= (e^z-1)^{-1}\left(\sum_{n\geq 1}(e^z-1)^n+1-1\right)\\
&= \frac{exp(e^z-1)-1}{e^z-1}.
\end{aligned}$$

Will demonstrate (2.13)

$$\begin{aligned}
\sum_{n\geq 0} B_{n,2}\frac{z^n}{n!} &= \sum_{n\geq 0}\binom{n+2}{2}^{-1}\frac{(e^z-1)^n}{n!}\\
&= \sum_{n\geq 0}\frac{2!n!}{(n+2)!}\frac{(e^z-1)^n}{n!}\\
&= 2\sum_{n\geq 0}\frac{(e^z-1)^{n+2-2}}{(n+2)!}\\
&= \frac{2}{(e^z-1)^2}\sum_{n\geq 0}\frac{(e^z-1)^{n+2}}{(n+2)!}\\
&= \frac{2}{(e^z-1)^2}\sum_{n\geq 2}\frac{(e^z-1)^n}{n!}\\
&= \frac{2}{(e^z-1)^2}\sum_{n\geq 2}\frac{(e^z-1)^n}{n!}+(e^z-1)-(e^z-1)\\
&= \frac{2}{(e^z-1)^2}\sum_{n\geq 1}\frac{(e^z-1)^n}{n!}-(e^z-1)+1-1\\
&= \frac{2}{(e^z-1)^2}\sum_{n\geq 0}\frac{(e^z-1)^n}{n!}-e^z\\
&= \frac{2(exp(e^z-1)-e^z)}{(e^z-1)^2}.
\end{aligned}$$

2.3.1 Generating Functions and Explicit Formulas

We begin with an explicit formula for $p-$Bell numbers $B_{n,p}$ with Stirling numbers of the second kind.

Theorem 9. *For $p \geq 0$, we have*

$$B_{n,p} = \sum_{k=0}^{n} \binom{k+p}{k}^{-1} \left\{ n \atop k \right\}. \tag{2.15}$$

Proof 14. *By using (1.48), we obtain*

$$\begin{aligned} B_{n,p} &= \sum_{n\geq 0} \binom{k+p}{k}^{-1} \left\{ n \atop k \right\} \\ \sum_{n\geq 0} B_{n,p} \frac{z^n}{n!} &= \sum_{n\geq 0} \sum_{k=0}^{n} \binom{k+p}{k}^{-1} \left\{ n \atop k \right\} \frac{z^n}{n!} \\ &= \sum_{k\geq 0} \frac{k!p!}{(k+p)!} \frac{(e^z-1)^k}{k!}. \end{aligned}$$

We have : $1^{\overline{k}} = 1.2\cdots(1+k-1) = k!$, *and*

$$\begin{aligned} \frac{p!}{(k+p)!} &= \frac{1.2\cdots(p-1)p}{1.2\cdots p(p+1)\cdots(p+k-1)(p+k)} \\ &= \frac{1}{(p+1)(p+2)\cdots(p+k-1)(p+k)} \\ &= \frac{1}{(p+1)^{\overline{k}}}. \end{aligned}$$

So

$$\begin{aligned} \sum_{k\geq 0} \frac{k!p!}{(k+p)!} \frac{(e^z-1)^k}{k!} &= \sum_{k\geq 0} \frac{(1)^{\overline{k}}}{(p+1)^{\overline{k}}} \frac{(e^z-1)}{k!} \\ &= {}_1F_1\left(\begin{matrix} 1 \\ p+1 \end{matrix} ; e^z-1 \right) \\ &= \sum_{n\geq 0} B_{n,p} \frac{z^n}{n!}. \end{aligned}$$

Proposition 2.3.1. *For $n \geq 0$, we have*

$$B_{n,1} = \sum_{k=0}^{n} \binom{n}{k} \frac{\phi_{k+1} B_{n-k}}{k+1}. \tag{2.16}$$

Proof 15.

$$\begin{aligned} B_{n,1} &= \sum_{k=0}^{n} \frac{1}{k+1} \left\{ n \atop k \right\} \\ &= \int_0^1 \phi_n(t)dt. \end{aligned}$$

We have

$$\begin{aligned}\phi_n(z) &= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} z^k \\ \int_0^1 \phi_n(z)dz &= \int_0^1 \sum_{k=0}^{n} \left\{ {n \atop k} \right\} z^k dz \\ &= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \int_0^1 z^k dz \\ &= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \left[\frac{1}{k+1} z^{k+1} \right]_0^1 \\ &= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \frac{1}{k+1}.\end{aligned}$$

$$B_{n,1} = \sum_{k=0}^{n} \binom{n}{k} \frac{\phi_{k+1} B_{n-k}}{k+1}.$$

We have (2.7),

$$\begin{aligned}\frac{d}{dz}\left(\sum_{n\geq 0} \phi_n(t) \frac{z^n}{n!} \right) &= \frac{d}{dz}\left(exp(e^z - 1) \right) \\ \sum_{n\geq 0} \frac{\phi_n(t)}{n!} \frac{d}{dz} z^n &= e^z e^{e^z - 1} \\ \sum_{n\geq 1} \frac{\phi_n(t)}{(n-1)!} z^{n-1} &= e^z e^{e^z - 1} \\ \sum_{n\geq 0} \frac{\phi_{n+1}(t)}{n!} z^n &= e^z e^{e^z - 1}.\end{aligned}$$

We have (2.12),

$$\sum_{n\geq 0} B_{n,1} \frac{z^n}{n!} = \frac{(exp(e^z - 1) - 1)z}{(e^z - 1)z}.$$

Theorem 10. *The double exponential generating function of $B_{n,p}$ is given by*

$$\sum_{p\geq 0} \sum_{p\geq 0} B_{n,p} \frac{z^n}{n!} \frac{y^n}{p!} = \frac{(e^z - 1)\exp(e^z - 1)}{e^z - 1 - y}. \tag{2.17}$$

Proof 16.

$$\begin{aligned}
\sum_{p\geq 0}\sum_{n\geq p} B_{n,p}\frac{z^n}{n!}\frac{y^p}{p!} &= \sum_{p\geq 0}\sum_{n\geq 0}\binom{n+p}{p}^{-1}\frac{(e^z-1)^n}{n!}\frac{y^p}{p!}\\
&= \sum_{p\geq 0}\sum_{n\geq 0}\frac{p!n!}{(n+p)!}\frac{(e^z-1)^n}{n!}\frac{y^p}{p!}\\
&= \sum_{p\geq 0}\sum_{n\geq 0}\frac{(e^x-1)^{n+p-p}}{(n+p)!}y^p\\
&= \sum_{n\geq p}\frac{(e^z-1)^{n+p}}{(n+p)!}\sum_{p\geq 0}\left(\frac{y}{e^z-1}\right)^p\\
&= exp(e^z-1)\frac{1}{1-\frac{y}{(e^z-1)}}\\
&= \frac{exp(e^z-1)(e^z-1)}{e^z-1-y}.
\end{aligned}$$

To establish some properties of $B_{n,p}$, recall that the $r-$Stirling numbers of the second kind [5] $\left\{{n \atop k}\right\}_r$ counts the number of partitions of a set of n objects into exactly k non-empty disjoint subsets, such that the first r elements are in different subsets. Now, by means of the generalized Stirling transform [20], we obtain the following result.

Theorem 11. *For $n, m \geq 0$, we have*

$$\sum_{k=0}^{m} s(m,k)B_{n+k,p} = \sum_{k=0}^{n}\left\{{n+m \atop k+m}\right\}_m\binom{m+k+p}{p}^{-1}. \tag{2.18}$$

Theorem 12. *For $p \geq 1$, we have*

$$\sum_{n\geq 0} B_{n,p}\frac{z^n}{n!} = p\int_0^1 \exp((e^z-1)t)(1-t)^{p-1}dt. \tag{2.19}$$

Proof 17. *We have (1.36), by setting $a \coloneqq 1, c \coloneqq 1+p$ and $z \coloneqq 1$*

$$\begin{aligned}
{}_1F_1\left(\begin{matrix}1\\ p+1\end{matrix};e^z-1\right) &= \frac{\Gamma(1+p)}{\Gamma(1+p-1)\Gamma(1)}\int_0^1 \exp((e^z-1)t)t^{1-1}(1-t)^{p+1-1-1}dt\\
&= \frac{\Gamma(1+p)}{\Gamma(p)\Gamma(1)}\int_0^1 \exp((e^z-1)t)(1-t)^{p-1}dt.
\end{aligned}$$

We have (1.15), we get the result

$$\begin{aligned}
\frac{\Gamma(1+p)}{\Gamma(p)\Gamma(1)}\int_0^1 \exp((e^z-1)t)(1-t)^{p-1}dt &= \frac{p\Gamma(p)}{\Gamma(p}\int_0^1 \exp((e^z-1)t)(1-t)^{p-1}dt\\
&= p\int_0^1 \exp((e^z-1)t)(1-t)^{p-1}dt.
\end{aligned}$$

Corollary 2. *For $p \geq 1$, we have*

$$B_{n,p} = p\int_0^1 (1-t)^{p-1}\phi_n(t)dt. \tag{2.20}$$

Proof 18. *The integral representation can also be demonstrated using Theorem 9, along with the well-known result by Sury [26].*

$$\binom{n}{k}^{-1} = (n+1)\int_0^1 x^k(1-x)^{n-k}dx.$$

By using (2.15), we have

$$\begin{aligned}
B_{n,p} &= \sum_{k=0}^{n}\binom{k+p}{k}^{-1}\left\{{n \atop k}\right\} \\
&= \int_0^1 (1-x)^p \sum_{k=0}^{n}(k+p+1)\left\{{n \atop k}\right\}x^k dx \\
&= \int_0^1 (1-x)^p \left(\sum_{k=0}^{n}(p+1)\left\{{n \atop k}\right\}x^k + \sum_{k=0}^{n}k\left\{{n \atop k}\right\}x^{k+1-1}\right)dx \\
&= \int_0^1 (1-x)^p \left((p+1)\phi_n(x) + x\frac{d}{dx}\phi_n(x)\right)dx \\
&= (p+1)\int_0^1 (1-x)^p\phi_n(x)dx + \int_0^1 (1-x)^p x\frac{d}{dx}\phi_n(x)dx \\
&= p\int_0^1 (1-x)^p\phi_n(x)dx + \int_0^1 (1-x)^p\phi_n(x)dx + \int_0^1 (1-x)^p x\frac{d}{dx}\phi_n(x)dx \\
&= p\int_0^1 (1-x)^{p+1-1}\phi_n(x)dx + p\int_0^1 x(1-x)^{p-1}\phi_n(x)dx \\
&= p\int_0^1 (1-x)^{p-1}\phi_n(x)dx.
\end{aligned}$$

So

$$\begin{aligned}
\sum_{n\geq 0}B_{n,p}\frac{x^n}{n!} &= \sum_{n\geq 0}\left(p\int_0^1 (1-t)^{p-1}\phi_n(t)dt\right)\frac{x^n}{n!} \\
&= p\int_0^1 \sum_{n\geq 0}(1-t)^{p-1}\phi_n(t)\frac{x^n}{n!}dt \\
&= p\int_0^1 e^{t(e^x-1)}(1-t)^{p-1}dt.
\end{aligned}$$

Corollary 3. *The p−Bell numbers $B_{n,p}$ can also be written as*

$$B_{n,p} = \sum_{k\geq 0}\binom{p+k}{k}^{-1}{}_1F_1\left(\begin{matrix}k+1\\ p+k+1\end{matrix};-1\right)\frac{k^n}{k!}. \tag{2.21}$$

Proof 19. *From (2.20) and (2.9), we have*

$$\begin{aligned}
B_{n,p} &= p\int_0^1 (1-t)^{p-1}e^{-t}\sum_{k\geq 0}\frac{k^n}{k!}t^k dt \\
&= p\sum_{k\geq 0}\frac{k^n}{k!}\int_0^1 (1-t)^{p-1}e^{-t}t^k dt.
\end{aligned}$$

From (1.36), whene we comparing the coefficients, we get the result

$$ {}_1F_1\left(\begin{array}{c} k+1 \\ p+k+1 \end{array};-1\right)=\frac{\Gamma(p+k+1)}{\Gamma(p)\Gamma(k+1)}\int_0^1(1-t)^{p-1}e^{-t}t^k dt$$
$$\int_0^1(1-t)^{p-1}e^{-t}t^k dt=\frac{\Gamma(p)\Gamma(k+1)}{\Gamma(p+k+1)}{}_1F_1\left(\begin{array}{c} k+1 \\ p+k+1 \end{array};-1\right).$$

So

$$\begin{aligned} B_{n,p}&=\sum_{k\geq 0}\frac{k^n}{k!}\frac{p(p-1)!k!}{(p+k)!}{}_1F_1\left(\begin{array}{c} k+1 \\ p+k+1 \end{array};-1\right)\\ &=\sum_{k\geq 0}\frac{k^n}{k!}\frac{p!k!}{(p+k)!}{}_1F_1\left(\begin{array}{c} k+1 \\ p+k+1 \end{array};-1\right)\\ &=\sum_{k\geq 0}\binom{p+k}{k}^{-1}{}_1F_1\left(\begin{array}{c} k+1 \\ p+k+1 \end{array};-1\right)\frac{k^n}{k!}. \end{aligned}$$

Theorem 13. *For $p\geq 1$, the exponential generating function for $p-$Bell numbers is given by*

$$\sum_{n\geq 0}B_{n,p}\frac{z^n}{n!}=\frac{p(\exp(e^z-1))}{(e^z-1)^p}\gamma(p,e^z-1). \tag{2.22}$$

the lower incomplete Gamma function $\gamma(s,x)$ is defined as follows

$$\gamma(s,z)=\int_0^z e^{-t}t^{s-1}dt.$$

Proof 20. *This can be achieved by a simple change of variable $t:=1-\frac{y}{(e^z-1)}$ in (2.19), we get*

$$\begin{aligned} \sum_{n\geq 0}B_{n,p}\frac{z^n}{n!}&=p\int_0^1\exp\left((e^z-1)(\frac{e^z-1-y}{e^z-1})\right)\left(\frac{-y}{e^z-1}\right)^{p-1}(\frac{-1}{e^z-1})dy\\ &=\frac{p}{(e^z-1)(e^z-1)^{p-1}}\int_0^1\exp(e^z-1-y)(y)^{p-1}dy\\ &=\frac{p\exp(e^z-1)}{(e^z-1)^p}\int_0^1 e^{-y}(y)^{p-1}dy\\ &=\frac{p(\exp(e^z-1))}{(e^z-1)^p}\gamma(p,e^z-1). \end{aligned}$$

Theorem 14. *The exponential generating function for $p-$Bell numbers is given by*

$$\sum_{n\geq 0}B_{n,p}\frac{z^n}{n!}=\frac{p!(\exp(e^z-1))}{(e^z-1)^p}-\sum_{k=1}^{p}\frac{p^{\underline{k}}}{(e^z-1)^k}. \tag{2.23}$$

Corollary 4. *For $n,p\geq 0$, we have*

$$B_{n,p}=\sum_{k=0}^{n+p}\binom{n+p}{k}\binom{n+p}{p}^{-1}\phi_{n+p-k}B_k^{(p)}-\sum_{k=1}^{p}\binom{n+k}{k}^{-1}\binom{p}{k}B_{n+k}^{(k)}. \tag{2.24}$$

The following lemma will be useful in the proof of (2.26).

Lemma 1. *For $p \geq 0$, we have*

$$ {}_1F_1\left(\begin{matrix} p+1 \\ p+2 \end{matrix}; 1-e^z\right)\frac{k^n}{k!} = (-1)^p(p+1)\left(e^{-z}\frac{d}{dz}\right)^p {}_1F_1\left(\begin{matrix} 1 \\ 2 \end{matrix}; 1-e^z\right). \tag{2.25} $$

Theorem 15. *For $p \geq 1$, we have*

$$ \sum_{n\geq 0} B_{n,p}\frac{z^n}{n!} = \exp(e^z-1)(-1)^{p-1}p\left(e^{-z}\frac{d}{dz}\right)^{p-1}\left(\frac{1-\exp(1-e^z)}{e^z-1}\right). \tag{2.26} $$

The next result give us the integral representation of Cesàro type [10].

Theorem 16. *For $n \geq 1$, we have*

$$ B_{n,p} = \frac{2n!p!}{\pi e} Im \int_0^{\pi}\left(\frac{\exp((\exp(e^{i\theta})))}{(\exp(e^{i\theta})-1)^p} - e\sum_{l=0}^{p-1}\frac{(\exp(e^{i\theta})-1)^{l-p}}{l!}\right)\sin(n\theta)d\theta. \tag{2.27} $$

where $Im(z)$ denotes the imaginary part of z.

Recurrence Relations

Now we give a recurrence relations for the $p-Bell$ numbers $B_{n,p}$. The proof is based on (2.19).

Theorem 17. *The $p-Bell$ numbers satisfies the following recurrence relations*

$$ B_{n+1,p} = (n+1)B_{n,p} - \sum_{k=0}^{n-2}\binom{n}{k}(-1)^{n-k}B_{k+1,p} - \frac{p}{p+1}B_{n,p+1}. \tag{2.28} $$

with the initial sequence $B_{0,p} = 1$.

Proof 21. *By differentiation (2.19) with respect to z, we obtain*

$$ \begin{aligned} e^{-z}\frac{d}{dz}f_p(z) &= e^{-z}\frac{d}{dz}p\int_0^1 \exp((e^z-1)t)(1-t)^{p-1}dt \\ &= e^{-z}p\int_0^1 \frac{d}{dz}(\exp(e^z-1)t)(1-t)^{p-1}dt \\ &= e^{-z}p\int_0^1 te^z(\exp(e^z-1)t)(1-t)^{p-1}dt \\ &= p\int_0^1 t\exp((e^z-1)t)(1-t)^{p-1}dt \\ &= p\int_0^1 t\exp((e^z-1)t)(1-t)^{p-1}dt \\ &= -p\int_0^1 -t\exp((e^z-1)t)(1-t)^{p-1}dt \\ &= -p\int_0^1 (1-t-1)\exp(e^z-1)t(1-t)^{p-1}dt \\ &= -p\left(\int_0^1 (1-t)(1-t)^{p-1}\exp((e^z-1)t)(1-t)^{p-1}dt - \int_0^1 (1-t)^{p-1}\exp((e^z-1)t)dt\right) \\ &= -p\int_0^1 (1-t)^p\exp((e^z-1)t)dt + p\int_0^1 (1-t)^{p-1}\exp((e^z-1)t)dt \\ &= \frac{-p}{p+1}f_{p+1}(z) + f_p(z). \end{aligned} $$

Or equivalently

$$e^{-z}\sum_{n\geq 0}B_{n+1,p}\frac{z^n}{n!}=-\frac{p}{p+1}\sum_{n\geq 0}B_{n,p+1}\frac{z^n}{n!}+\sum_{n\geq 0}B_{n,p}\frac{z^n}{n!}.$$

After some rearrangement, we get

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}B_{k+1,p}\right)\frac{z^n}{n!}=-\frac{p}{p+1}\sum_{n\geq 0}B_{n,p+1}\frac{z^n}{n!}+\sum_{n\geq 0}B_{n,p}\frac{z^n}{n!}.$$

The conclusion follows by comparing coefficients of $\frac{z^n}{n!}$.

By using (2.28), we list the $p-$Bell numbers $B_{n,p}$ for $0\leq n\leq 6$ and $0\leq p\leq 3$

$$S_p=\begin{pmatrix}1 & 1 & 1 & 1 & \cdots & B_{0,p}\\ 1 & 1/2 & 1/3 & 1/4 & \cdots & B_{1,p}\\ 2 & 5/6 & 1/2 & 7/20 & \cdots & B_{2,p}\\ 5 & 7/4 & 14/15 & 3/5 & \cdots & B_{3,p}\\ 15 & 68/15 & 13/6 & 179/140 & \cdots & B_{4,p}\\ 52 & 167/12 & 127/21 & 185/56 & \cdots & B_{5,p}\\ 203 & 2057/42 & 235/12 & 8389/840 & \cdots & B_{6,p}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ B_{n,0} & B_{n,1} & B_{n,2} & B_{n,3} & \cdots & B_{n,p}\end{pmatrix}$$

When $p=0$, the formula (2.28) it becomes

Corollary 5. *We have*

$$\phi_{n+1}=(n+1)\phi_n+\sum_{k=1}^{n-1}(-1)^{n-k}\binom{n}{k-1}\phi_k. \tag{2.29}$$

The next result presents different types of recurrence for $p-$Bell numbers $B_{n,p}$. To prove Theorem 18, we need the following lemma.

Lemma 2. *We have*

$${}_1F_1\left(\begin{matrix}1\\ p+1\end{matrix};z\right)=\left(1+\frac{z}{p+1}\right){}_1F_1\left(\begin{matrix}1\\ p+2\end{matrix};z\right)-\frac{z}{p+2}{}_1F_1\left(\begin{matrix}1\\ p+3\end{matrix};z\right). \tag{2.30}$$

Theorem 18. *The $p-$Bell numbers satisfy the following recurrence relation*

$$B_{n+1,p+1}=B_{n+1,p}-\frac{n+1}{p+1}B_{n,p+1}+\frac{n+1}{p+2}B_{n,p+2}. \tag{2.31}$$

with the initial sequence $B_{0,p}=1$ and the final sequence $B_{n,0}=\phi_n$.

Next, we propose a formula based on a three-term recurrence relation to calculate the $p-$Bell numbers. Consider the following sequence $Z_{n,m}$ with two indices by

$$Z_{n,m}(p)=\binom{m+p}{p}\sum_{k=0}^{m}s(m,k)B_{n+k,p}. \tag{2.32}$$

From Theorem 11, we can write

$$Z_{n,m}(p)=\binom{m+p}{p}\sum_{k=0}^{n}\left\{\begin{matrix}n+m\\ k+m\end{matrix}\right\}_m\binom{m+k+p}{p}^{-1}. \tag{2.33}$$

With $Z_{0,m}(p)=1$ and $Z_{n,0}(p)=B_{n,p}$.

Theorem 19. *The $Z_{n,m}(p)$ satisfy the following three-term recurrence relation*

$$Z_{n+1,m}(p) = \frac{m+1}{m+p+1} Z_{n,m+1}(p) + mZ_{n,m}(p). \tag{2.34}$$

with the initial sequence given by $Z_{0,m}(p) = 1$.

Proof 22. *From (2.32) and*

$$s(n+1,k) = s(n,k-1) - ns(n,k); \qquad 1 \le k \le n$$

We have

$$\begin{aligned}
Z_{n,m+1}(p) &= \binom{m+p+1}{p} \sum_{k=0}^{m+1} s(m+1,k) B_{n+k,p} \\
&= \binom{m+p+1}{p} \sum_{k=0}^{m+1} \left(s(m,k-1) - ms(m,k) \right) B_{n+k,p} \\
&= \frac{(m+p)!(m+p+1)}{p!m!(m+1)} \sum_{k=0}^{m} s(m,k) B_{n+k+1,p} - m \sum_{k=0}^{m} \frac{(m+p)!(m+p+1)}{p!m!(m+1)} s(m,k) B_{n+k,p} \\
&= \frac{(m+p+1)}{(m+1)} Z_{n+1,m}(p) - m \frac{m+p+1}{m+1} Z_{n,m}(p).
\end{aligned}$$

So equivalent to (2.34)

$$Z_{n+1,m}(p) = \frac{m+1}{m+p+1} Z_{n,m+1}(p) + mZ_{n,m}(p).$$

Remark 2.3.1. *Note thate for $p = 0$, the recurrence relation (19) reduce to a known formula in [20].*

2.4 Poly-Bell Numbers

It is well know that the Cauchy formula for repeated integration [19] is given by

$$D^{-n} f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt. \tag{2.35}$$

Where D^{-n} denotes the n-fold integral and f be a continuous function on the real line. Comparing the formula (2.35) with the formula (2.20) we have

Theorem 20. *For $p \ge 1$, we have*

$$\begin{aligned}
B_{n,p} &= p! D^{-p} \phi_n(x)|_{x=1} \\
&= p! \int_0^1 dx_p \int_0^{x_p} dx_{p-1} \cdots \int_0^{x_2} \phi_n(x_1) dx_1.
\end{aligned} \tag{2.36}$$

Proof 23. *Comparing the formula (2.35) with the formula (2.20), where $x = 1$ we obtain*

$$\begin{aligned}
B_{n,p} &= p! D^{-p} \phi_n(x)|_{x=1} \\
&= p! \frac{1}{(p-1)!} \int_0^1 (1-t)^{p-1} \phi_n(x) dt \\
&= \frac{p\Gamma(p)}{\Gamma(p)} \int_0^1 (1-t)^{p-1} \phi_n(x) dt \\
&= p \int_0^1 (1-t)^{p-1} \phi_n(x) dt.
\end{aligned}$$

Using the integral expression (2.36), the poly-Bell numbers $\mathbb{B}_n^{(p)}$ are defined by the following relation

$$\mathbb{B}_n^{(p)} = \frac{B_{n,p}}{p!}. \tag{2.37}$$

The generating function of $\mathbb{B}_n^{(p)}$ is given by

$$\sum_{n\geq 0} \mathbb{B}_n^{(p)} \frac{z^n}{n!} = \sum_{n\geq 0} \frac{(e^z-1)^n}{\Gamma(n+p+1)}. \tag{2.38}$$

Proof 24.

$$\begin{aligned}
\sum_{n\geq 0} \mathbb{B}_n^{(p)} \frac{z^n}{n!} &= \sum_{n\geq 0} \frac{B_{n,p}}{p!} \frac{z^n}{n!} \\
&= \sum_{n\geq 0} \left(\sum_{k\geq 0} \binom{k+p}{k}^{-1} \left\{ n \atop k \right\} \right) \frac{z^n}{n!} \frac{1}{p!} \\
&= \sum_{k\geq 0} \binom{k+p}{k}^{-1} \sum_{n\geq k} \left\{ n \atop k \right\} \frac{z^n}{n!} \frac{1}{p!} \\
&= \sum_{k\geq 0} \frac{k!p!}{(k+p)!p!} \frac{(e^z-1)^k}{k!} \\
&= \sum_{k\geq 0} \frac{(e^z-1)^k}{(k+p)!} \\
&= \sum_{k\geq 0} \frac{(e^z-1)^k}{\Gamma(k+p+1)}.
\end{aligned}$$

For p negative we can write

$$\sum_{n\geq 0} \mathbb{B}_n^{(-p)} \frac{z^n}{n!} = \sum_{n\geq p} \frac{(e^z-1)^n}{\Gamma(n-p+1)}. \tag{2.39}$$

We have get the explicit formula of the poly-Bell numbers at a negative upper index p by the generating function .

Theorem 21. *The poly-Bell numbers $\mathbb{B}_n^{(-p)}$ are positive and we have*

$$\mathbb{B}_n^{(-p)} = \sum_{k=p}^{n} \frac{k!}{(k-p)!} \left\{ n \atop k \right\}. \tag{2.40}$$

Proof 25.

$$\begin{aligned}
\sum_{n\geq 0} \mathbb{B}_n^{(-p)} \frac{z^n}{n!} &= \sum_{n\geq p} \frac{(e^z-1)^n}{\Gamma(n-p+1)} \\
&= \sum_{n\geq p} \frac{n!(e^z-1)^n}{n!(n-p)!} \\
&= \sum_{n\geq p} \frac{n!}{(n-p)!} \frac{(e^z-1)^n}{n!} \\
&= \sum_{n\geq p} \frac{n!}{(n-p)!} \sum_{n\geq 0} \left\{ n \atop k \right\} \frac{z^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\sum_{n\geq 0}\frac{z^n}{n!}$, we get the result

$$\mathbb{B}_n^{(-p)} = \sum_{n\geq p}\frac{n!}{(n-p)!}\left\{ n \atop k \right\}.$$

Corollary 6. *For $p \geq 1$, we have*

$$\begin{aligned}\mathbb{B}_n^{(-p)} &= \frac{d^p}{dx^p}\phi(x)|_{x=1} \\ &= p!\sum_{j=p}^{n}\binom{n}{j}\left\{ j \atop p \right\}\phi_{n-j}.\end{aligned} \tag{2.41}$$

Proof 26. *From (2.6), we have*

$$\begin{aligned}\frac{d^p}{dx^p}\phi_n(x) &= \frac{d^p}{dx^p}\left(\sum_{k=0}^{n}\left\{ n \atop k \right\}x^k\right) \\ &= \sum_{k=0}^{n}\left\{ n \atop k \right\}\frac{d^p}{dx^p}x^k \\ &= \sum_{k=0}^{n}\left\{ n \atop k \right\}k^{\underline{n}}x^{k-p} \\ &= p!\sum_{k=0}^{n}\left\{ n \atop k \right\}\binom{k}{p}x^{k-p}.\end{aligned}$$

On the other hand, from(2.7), we have

$$\begin{aligned}\sum_{n\geq 0}\frac{d^p}{dx^p}\phi_n(x)\frac{z^n}{n!} &= \frac{d^p}{dx^p}exp(x(e^z-1)) \\ &= (e^z-1)^p exp(x(e^z-1)) \\ &= \frac{p!(e^z-1)^p}{p!}\sum_{n\geq 0}\phi_n(x)\frac{z^n}{n!} \\ &= p!\sum_{n\geq 0}\left\{ n \atop p \right\}\frac{z^p}{p!}\sum_{n\geq 0}\phi_n(x)\frac{z^n}{n!} \\ &= p!\sum_{n\geq 0}\frac{z^n}{n!}\left(\sum_{j=0}^{n}\binom{n}{j}\left\{ j \atop p \right\}\phi_{n-j}(x)\right).\end{aligned}$$

Equating coefficients of $\sum_{n\geq 0}\frac{z^n}{n!}$ and setting $x = 1$, we get the result

$$\frac{d^p}{dx^p}\phi_n(x)\sum_{j=0}^{n}\binom{n}{j}\left\{ j \atop p \right\}\phi_{n-j}(x).$$

The question now is that the Bernoulli multiple numbers have the property of duality [2, 18], so what can we say about the poly-Bell numbers?

Theorem 22. *The double generating function of $\mathbb{B}_n^{(-p)}$ is given by*

$$\sum_{p\geq 0}\sum_{n\geq p}\mathbb{B}_n^{(-p)}\frac{z^n}{n!}\frac{y^p}{p!} = \exp((y+1)(e^z-1)). \tag{2.42}$$

Proof 27. *By using (2.39) for gives the double generating function of* $\mathbb{B}_n^{(-p)}$

$$\begin{aligned}
\sum_{n\geq 0}\mathbb{B}_n^{(-p)}\frac{z^n}{n!} &= \sum_{n\geq p}\frac{(e^z-1)^n}{\Gamma(n-p+1)}\\
&= \sum_{p\geq 0}\sum_{n\geq 0}\mathbb{B}_n^{(-p)}\frac{z^n}{n!}\frac{y^p}{p!}\\
&= \sum_{p\geq 0}\sum_{n\geq p}\frac{(e^z-1)^{n+p-p}}{n-p}\frac{y^p}{p!}\\
&= \sum_{p\geq 0}y^p(e^z-1)^p\frac{1}{p}\sum_{i\geq 0}\frac{(e^z-1)^i}{i!}\\
&= \sum_{p\geq 0}\frac{(y(e^z-1))^p}{p!}\sum_{i\geq 0}\frac{(e^z-1)}{i!}\\
&= \exp(y(e^z-1))\exp(e^z-1)\\
&= \exp((e^z-1)(y+1)).
\end{aligned}$$

Using the above theorem, we can answer the question in the negative, the duality property does not hold for poly-Bell numbers.
As a consequence of Theorem 22, we derive the generating function for the poly-Bell numbers $\mathbb{B}_n^{(-p)}$ at a negative upper index p.

Corollary 7. *We have*

$$\sum_{p\geq 0}\mathbb{B}_n^{(-p)}\frac{y^p}{p!} = \phi(1+y).$$

Here is a small table of the numbers $\mathbb{B}_n^{(p)}$, $(-4\leq p\leq 1, 0\leq n\leq 9)$.

$p\diagdown n$	0	1	2	3	4	5	6	7	8	9
-1	0	1	3	10	37	151	674	3263	17007	94828
-2	0	0	2	12	62	320	1712	9604	56674	351792
-3	0	0	0	6	60	450	3120	21336	147756	1048830
-4	0	0	0	0	24	360	3720	33600	287784	2424744

Table 2.3: The smal table of the numbers $\mathbb{B}_n^{(p)}$

2.4.1 The p-Bell Polynomials

For $p\geq 0$, let us consider the p-Bell polynomials $B_{n,p}(x)$ defined by means of the following generating function

$$F_p(x) = \sum_{n\geq 0}B_{n,p}(x)\frac{z^n}{n!} = {}_1F_1\left(\begin{array}{c}1\\ p+1\end{array};e^z-1\right)e^{xz}. \tag{2.43}$$

The explicit formula for the p-Bell polynomial is easily obtained from (2.43)

$$B_{n,p}(x) = \sum_{k=0}^{n}\binom{n}{k}B_{k,p}x^{n-k}. \tag{2.44}$$

We have

$$\begin{aligned}\sum_{n\geq 0} B_{n,p}\frac{z^n}{n!} &= {}_1F_1\left(\begin{array}{c}1\\ p+1\end{array};e^z-1\right)e^{xz}\\ &= \left(\sum_{n\geq 0} B_{n,p}\frac{z^n}{n!}\right)\left(\sum_{n\geq 0}\frac{(xz)^n}{n!}\right)\\ &= \sum_{n\geq 0} n!\left(\sum_{k=0}^{n}\frac{B_{k,p}}{k!}\frac{x^{n-k}}{(n-k)!}\right)\frac{z^n}{n!}\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}B_{k,p}x^{n-k}\right)\frac{z^n}{n!}.\end{aligned}$$

For each integer p, we also define the following class of polynomials $\mathbb{B}_n^{(p)}(x)$, which we call poly-Bell polynomials, is defined by

$$\mathbb{B}_n^{(p)}(x) = \frac{B_{n,p}(x)}{p!}.$$

The first few p-Bell polynomials are

$$\begin{aligned}B_{0,p}(x) &= 1\\ B_{1,p}(x) &= x+\frac{1}{p+1}\\ B_{2,p}(x) &= x^2+\frac{2x}{p+1}+\frac{p+4}{(p+1)(p+2)}\\ B_{3,p}(x) &= x^3+\frac{3x^2}{p+1}+\frac{3(p+4)x}{(p+1)(p+2)}\frac{p^2+11p+30}{(p+1)(p+2)(p+3)}\\ B_{4,p}(x) &= x_4+\frac{4x_3}{p+1}+\frac{6(p+4)x_2}{(p+1)(p+2)}+\frac{4(p^2+11p+30)x}{(p+1)(p+2)(p+3)}+\frac{p^3+23p^2+160p+360}{(p+1)(p+2)(p+3)(p+4)}.\end{aligned}$$

We present here another expression for the explicit formula of $B_{n,p}(x)$), which includes weighted Stirling numbers $S_n^k(x)$ of the second kind [9].
These numbers are related to r-Stirling numbers of the second kind $\left\{{n \atop k}\right\}_r$ and Whitney numbers of the second kind [21] $W_{m,r}(n,k)$.

Theorem 23. *For $p \geq 0$, we have*

$$B_{n,p}(x) = \sum_{k=0}^{n}\binom{k+p}{k}^{-1} S_n^k(x). \tag{2.45}$$

Proof 28. *From the right-hand side of (2.45) and (1.63) We have*

$$\begin{aligned}
\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{k+p}{p}^{-1}S_n^k(x)\right)\frac{z^n}{n!} &= \sum_{k=0}^{\infty}\binom{k+p}{p}^{-1}\sum_{n\geq 0}S_n^k(x)\frac{z^n}{n!}\\
&= \sum_{k=0}^{\infty}\binom{k+p}{p}^{-1}\frac{1}{k!}e^{xz}(e^z-1)^k\\
&= \sum_{k=0}^{\infty}\frac{p!k!}{(p+k)!}\frac{1}{k!}e^{xz}(e^z-1)\\
&= e^{xz}\sum_{k=0}^{\infty}\frac{p!k!}{(p+k)!}\frac{(e^z-1)^k}{k!}\\
&= e^{xz}{}_1F_1\left(\begin{array}{c}1\\ p+1\end{array};e^z-1\right)\\
&= \sum_{n\geq 0}B_{n,p}(x)\frac{z^n}{n!}.
\end{aligned}$$

Obviously, this is the end of the proof of the theorem.

In particular, we have

$$B_{n,p}(r)=\sum_{k=0}^{n}\binom{k+p}{p}^{-1}\left\{\begin{matrix}n+r\\ k+r\end{matrix}\right\}_r.$$

And

$$B_{n,p}(\frac{r}{m})=\sum_{k=0}^{n}\frac{1}{m^{n-k}}\binom{k+p}{k}^{-1}W_{m,r}(n,k).$$

Now, we want to generalize Theorem 19 to the polynomial case.
Let us consider the polynomials $Z_{n,m}(z;p)$ which is defined by

$$Z_{n,m}(x;p)=\sum_{k=0}^{n}\binom{n}{k}Z_{k,m}(p)x^{n-k}, \tag{2.46}$$

with $Z_{0,m}(x;p)=1$ and $Z_{n,0}(x;p)=B_{n,p}(x)$.

Theorem 24. *The polynomials $Z_{n,m}(x;p)$ satisfy the following three-term recurrence relation*

$$Z_{n+1,m}(x;p)=\frac{m+1}{m+p+1}Z_{n,m+1}(x;p)+(m+x)Z_{n,m}(x;p). \tag{2.47}$$

where the initial sequence is given by the following formula

$$Z_{0,m}=1.$$

Theorem 25. *The p-Bell polynomials $B_{n,p}(x)$ satisfy the following integral representation*

$$B_{n,p}(x)=p\int_0^1(1-t)^{p-1}G_{n,x}(t)dt. \tag{2.48}$$

where

$$G_{n,x}(t)=\sum_{n=0}^{k}\binom{n}{k}\phi_k(t)x^{n-k}.$$

Note that the polynomials $G_{n,x}(t)$ are a special case of the more general polynomials considered by Corcino et al. in [13]. Now, using the properties of $G_{n,x}(t)$, we obtain the following corollaries

Corollary 8. ***(Dobinski's formula)*** *For $p \geq 1$, we have*

$$B_{n,p}(x) = p \sum_{k\geq 0} \frac{(x+k)^n}{k!} {}_1F_1\left(\begin{matrix} k+1 \\ p+k+1 \end{matrix} ; -1 \right). \tag{2.49}$$

Corollary 9. ***(Recurrence relation)*** *For $p \geq 0$, we have*

$$B_{n+1,p}(x) = xB_{n,p}(x) - \sum_{k=0}^{n} \binom{n}{k} \left(\frac{p}{p+1} B_{k,p+1}(x) - B_{k,p}(x) \right). \tag{2.50}$$

CONCLUSION

In this work, we have delved into the realm of enumerative combinatorics, recognizing several sequences of numbers and polynomials (Stirling, Bell, Bernoulli, ...), along with their combinatorial and analytical properties. Specifically, we study a generalization for Bell numbers and polynomials called $p-$Bell numbers and polynomials.

We also hope to continue our studies in this field in the future and further develop the results we have explored concerning this class of numbers and polynomials.

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