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The Neighbor Locating Chromatic Number of Some Graphs

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Dedication

To those who believed in me, even when I did not believe in myself.

To my late father, who will live forever in my mind and heart.

To my mother, for her constant support and unconditional love.

To whom I love, my family and my friends.

" Mokhtari Halima "

Dedication

To my dear parents who have been there for me every day for their support throughout my education, their love, patience, and encouragement as I have pursued my studies and worked on this memory.

My dear husband, his never-ending backing and encouraging words have been a great help.

My brother and my little sister Ahmed and Manel. All of my family.

My dear partner Halima, who has been really kind and patient with me, did not allow me to fall.

Everyone that helped me to complete this work.

" Meziane Ben Taher Meziane Imane "

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In the name of ALLAH, the Most Gracious and the Most Merciful.

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Abstract

Let $G = (V, E)$ be a simple graph. A proper vertex coloring of G consists in assigning a color to each vertex of G so that two adjacent vertices are colored differently. The minimum number of colors in a proper coloring of vertex of G is called the chromatic number. A neighbor locating coloring is a coloring where any two distinct vertices belonging to the same color class can be distinguished from each other by the colors of their respective neighbors. The minimum cardinality of a neighbor locating coloring of G is the neighborhood locating chromatic number.

Throughout the dissertation, we study the neighbor locating chromatic number and we provide results on the neighbor locating chromatic number of some special graphs, namely, the total graph of the star and wheel graphs and the book graphs.

Résumé

Soit $G = (V, E)$ un graphe simple. Une coloration propre des sommets de G consiste à effectuer une couleur à chaque sommet de G, de sorte que deux sommets adjacents soient colorés différemment. Le nombre minimum de couleurs d'une coloration propre des sommets de G est appelé le nombre chromatique. Une coloration de localisation des voisins est une coloration où deux sommets distincts appartenant à la même classe de couleur peuvent être distingués l'un de l'autre par les couleurs de leurs voisins respectifs. La minimale cardinalité d'une coloration de localisation des voisins de G est le nombre chromatique de localisation du voisinage.

Dans ce mémoire, nous étudions le nombre chromatique de localisation des voisins et nous fournissons des résultats sur le nombre chromatique de localisation des voisins de certains graphes spéciaux. A savoir, le graphe total de l'étoile, le graphe total de la roue et le graphe du livre.

Introduction

Graph theory, the branch of mathematics concerned with the study of graphs, has proven to be a powerful tool for solving real-world problems in a variety of disciplines. Graph theory provides a comprehensive framework for modeling and evaluating complex relationships, from computer science to biology and transportation planning to social network analysis.

Graph coloring is a fundamental topic in graph theory. This involves assigning colors to the corners of a figure in such a way that they meet certain criteria. The goal is to have adjacent vertices have different colors, expressing the concept of distinctness or separation in the graph representation. In 1852, a young Englishman named Francis Guthrie asked if any geological map could be drawn in four different shades if two neighboring countries had different shades. In 1976, K. Appel and W. Haken were the only researchers to respond positively to the four-color combination. Trying to find a solution to this problem has resulted in various graph coloring results. A different approach to graph coloring is neighborhood coloring, which considers the colors of neighboring vertices when assigning colors. Unlike some other shading methods that follow predetermined rules or heuristics, continuous shading allows for adaptability and context awareness in the shading process.

The principle emphasized by this coloring technique is that adjacent vertices should have different colors according to the colors of their neighbors. Behtoei and Anbarloei (2014) developed neighbor search partitions generated from an independent set called adjacency location coloring as a technique for analyzing the location color numbers of

graph connections. Alcon and colleagues coined the term "proximity location coloring" in 2020 (Alcon et al., 2020). Other studies on neighbor locating chromatic numbers have been published, such as (Hernando et al, 2018) and (Alcon et al, 2021). The smallest number of colors in adjacent coloring is the adjacent chromatic number, denoted as $\chi_{NL}(G)$. The purpose of this thesis is to study the neighbor locating chromatic numbers of some well-known graphs. This work consists of three chapters:

The first chapter is divided into three parts. In the first part, we discussed the foundations and ideas of graph theory. Furthermore, we define each concept used in this work. In the second part, we defined some basic chart types. In the last Part, we introduced a number of graphics operations.

The second chapter is divided into three parts. In the first section, we propose the vertex coloring problem, define the vertex coloring and chromatic numbers of graphs, and give some conclusions about the chromatic numbers of some typical graphs. In the second section, we define the positioning chromatic number and then present some important insights. In the last section we define the chromatic number of neighbor locations and we present some previous findings on this parameter.

The last chapter concerns the determinations of the neighbor locating chromatic number for the total graph of star and wheel graphs, and for the book graph.

CHAPTER 1

Concepts and Basics

In this chapter, we give an overview of some basic definitions in graph theory as well as the necessary notations that we will need throughout this work. For any undefined terms the reader can consult any of the references [1], [2] and [3].

1.1 Basic Definitions

In this section we introduce some basic definitions in graph theory.

1.1.1 Graphs

• A **graph** $G = (V(G), E(G))$ is a mathematical structure consisting of two nonempty sets $V(G)$ and $E(G)$. The elements of $V(G)$ are called vertices (or nodes), and the elements of $E(G)$ are called edges. Each edge has a set of two vertices associated to it, which are called its endpoints. For u, v in $V(G)$, we denote the edge between u and v by uv or vu. If $e = uv$ is an edge of G, then e is said to join u and v. A graph's **order** and size are determined by the number of vertices and edges, respectively, in the graph G.

• Two vertices that are joined by an edge are called **adjacent vertices**. Two edges that have an endpoint in common are called **adjacent edges**. A **loop** or **self loop** is

an edge that connects a node to itself. Thus, a loop is an edge of the form v_i, v_j where $v_i = v_j$. A collection of two or more edges having identical endpoints is called **multi-edge**. The number of edges within the multi-edge is called the **edge multiplicity**.

- A **simple graph** is a graph without loops and multiple edges.

Example 1.1.1 *Consider the following simple graph G*

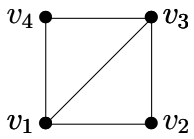


Figure 1.1.1 : The simple graph G

1.1.2 Subgraph

A **subgraph** of a graph G is a graph having all of its vertices and edges in G. If H is a subgraph of G, then G is a **super graph** of H. In other words for a subgraph H of a graph G, if H and G are two graphs with vertex sets $V(H)$, $V(G)$ and edge sets $E(H)$, $E(G)$ respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a subgraph of G and G is a super graph of H.

Example 1.1.2 *Consider the following graphs G and the subgraph* G_1

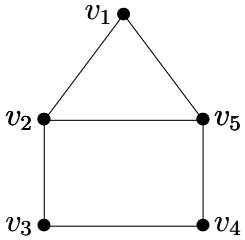


Figure 1.1.2 : The graph G

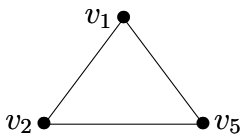


Figure 1.1.3 : Graph G_1 the subgraph of G

1.1.3 Neighborhood and Degree

Let $G = (V, E)$ be a graph. Then

- For a vertex $v \in V$ of G, a vertex that is joined by an edge to a vertex v is said to be a **neighbor** of v.
- The **open neighborhood** of a vertex v in a graph G, denoted by $N(v)$, is the set of all the neighbors of v.
- The **closed neighborhood** of v is given by $N[v] = N(v) \cup \{v\}$.
- For a subset $S \subseteq V(G)$, the **open neighborhood** of S is defined by the following set $N_G(S) = \cup_{v \in S} N_G(v)$ and the **closed neighborhood** is defined by $N_G[S] = N_G(S) \cup S$.
- The **degree** of a vertex v in a graph G is the number of vertices in G that are adjacent to v. Equivalently, the degree of v denoted by deg(v) is the number of vertices in its neighborhood, $deg(v) = |N(v)|$.
- The largest degree among the vertices of G is called the **maximum degree** of G and it is denoted by $\triangle(G)$.
- The smallest degree among the vertices of G is called the **minimum degree** of G and it is denoted by $\delta(G)$.
- A vertex of degree 0 is called an **isolated vertex** and a vertex of degree 1 is an **end-vertex** or a **leaf**. While a vertex of degree $n - 1$ is called a **univ vertex**.
- If v is a vertex of a graph G of order n, then

$$0 \leqslant \delta(G) \leqslant deg(v) \leqslant \triangle(G) \leqslant n - 1$$

.

Example 1.1.3 *Consider the following graph G, we provide an illustration of the above definitions*

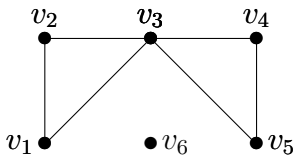


Figure 1.1.4 : A graph G

Note that

1. *The vertex set* $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ *and hence the order of* G *equals* 6.
2. *The edges set* $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_3, v_3v_1\}$ *and hence the size of* G *equals* 6.
3. *The open neighborhood* $N(v_1) = \{v_2, v_3\}$, $N(v_2) = \{v_1, v_3\}$, $N(v_3) = \{v_1, v_2, v_3\}$.
4. *The closed neighborhood* $N[v_2] = \{v_1, v_3, v_2\}$,$N[v_5] = \{v_5, v_4, v_3\}$.
5. v_6 *is the only isolated vertex in* G, *because* $deg(u_6) = 0$.
6. *The maximum degree of* G *is* $\triangle(G) = 4$, *and the minimum degree is* $\delta(G) = 2$.

1.1.4 Walk, Trail and Path

Let $G = (V, E)$ be a graph, and suppose $u, w \in V$.

- A **Walk** in a graph G is an alternative sequance of vertices and edges $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$, such that endpts $(e_i) = v_{i-1}, v_i$ for $i = 1, \ldots, n$.
- A **trail** is a walk with no repeated edges.
- A **path** is a walk that has no repeated vertices.
- A **cycle** is a closed trail having more than one vertex with no repeated vertices except that the first vertex is the last vertex.

1.1.5 Distance and Diameter

Let $G = (V, E)$ be a graph and u, v and w three vertices of G, then:

- The **distance** between the vertices u and v, denoted by $d(u, v)$, is the length of a shortest path between them.

- For a subset S of V, the **distance** between u and S is given by

$$d(u, S) = min\{d(u, x) : x \in S\}$$

- The **diameter** of a graph is defined as the largest shortest path distance in the graph. In other words, it is the maximum value of $d(u, v)$ over all u, v pairs. That is:

$$diam(G) = max_{x,y \in V(G)}\{d(x, y)\}.$$

Example 1.1.4 *Consider the following graph G.*

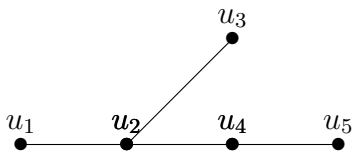


Figure 1.1.5 : The graph G

1. $d(u_1, u_2) = d(u_2, u_3) = d(u_2, u_4) = d(u_4, u_5) = 1$,
   $d(u_1, u_3) = d(u_4, u_3) = d(u_2, u_5) = 2$, $d(u_1, u_5) = d(u_3, u_5) = 3$.

2. *The diameter of this graph is* 3.

1.1.6 Graph Parameters

- A complete subgraph K of a graph G is called a **clique** in G. Such that the order of a it is the number of its vertices. A maximal clique in a graph G is a clique that is not a subset of any other clique in G. A maximum clique in G is a

maximal clique that has the largest order. The order of a maximum clique of G is denoted by $\omega(G)$.

- A **stable** set or **independent** set in a graph G is a set of vertices, all pairwise nonadjacent. The **independence number** of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of the independent sets of G.

1.2 Particular Graphs

In this section, we recall the definitions of certain particular known graphs.

Connected and Disconnected Graphs

A graph $G = (V, E)$ is connected if for each pair of distinct vertices $u, v \in V$, there is a path connecting u to v. A component in G is a maximal connected subgraph of the graph G. A connected graph G has only one component, otherwise, we say that G is disconnected.

Trivial and nontrivial graphs

A graph of order 1 is called a trivial graph. While a graph with two or more vertices is called a nontrivial graph.

Complete graph

A complete graph is a simple graph such that every pair of vertices are connected by an edge. The complete graph on n vertices denoted by K_n.

Example 1.2.1 *Consider the following graph*

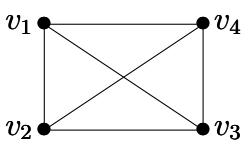


Figure 1.2.1 : Complet graph of order 4, K_4

Theorem 1.2.1 *For a complete graph with n vertices, there are* $(n(n-1))/2$ *edges.*

Complement graph

The complement of a graph G, denoted by $\overline{G}$, is the graph with vertex set $V(G)$ such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G.

Example 1.2.2 *Consider the following graphs*

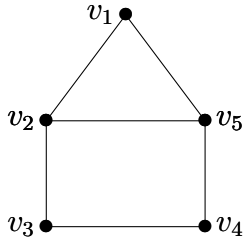


Figure 1.2.2 : A graph G

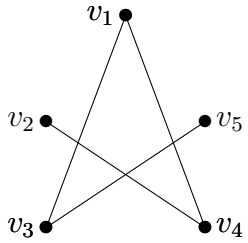


Figure 1.2.3 : $\overline{G}$,The Complement graph of G

Acyclic graph

An acyclic graph is a graph that does not contain a cycle.

Unicyclic graph

A unicyclic graph is a connected graph with exactly one cycle.

Example 1.2.3 *Consider the following graph*

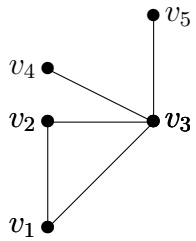


Figure 1.2.4 : Unicyclic graph

Bipartite graph

A graph G is bipartite if $V(G)$ can be partitioned into two sets U and W (called partite sets) so that every edge of G joins a vertex of U and a vertex of W.

Example 1.2.4 *The bipartite graph is the following graph*

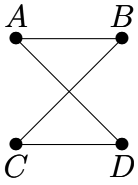


Figure 1.2.5 : A bipartit graph where $U = \{A, C\}$, $W = \{B, D\}$

Complete bipartite graph

A graph G is a complete bipartite graph if $V(G)$ can be partitioned into bipartite sets U and W such that every vertex in one of the bipartite sets is joined to every vertex in the other bipartite set so that uw is an edge of G if and only if $u \in U$ and $w \in W$. If $|U| = s$ and $|W| = t$, then this complete bipartite graph has order $s + t$ and size st and is denoted by $K_{s,t}$ (or $K_{t,s}$).

Example 1.2.5 *Consider the following complet bipartit graph where* $U = \{A, B, C\}$, $W = \{D, E\}$ *with* $|U| = 3$, $|W| = 2$.

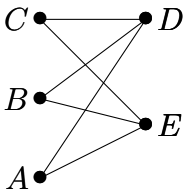


Figure 1.2.6 : A complet bipartit graph $K_{2,3}$.

Star

A star graph S_n of order n ($n > 2$) with maximum diameter 2, has $n - 1$ leaves.

Example 1.2.6 *The star graph S_9 is the following graph*

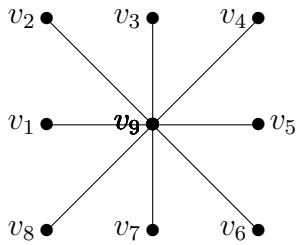


Figure 1.2.7 : A star graph S_9 with central vertex v_9.

Tree

A tree is a connected graph that contains no cycles, denoted by T.

Example 1.2.7 *A tree graph is the following graph*

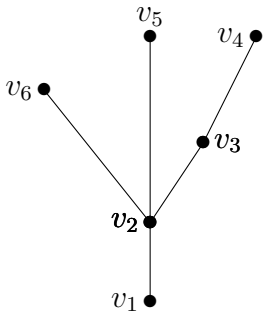


Figure 1.2.8 : A tree graph T_6

Wheel graph

A wheel graph of $n + 1$ vertices, denoted W_n, is a graph that contains a cycle of order n and a single vertex and for which every vertex in the cycle is connected to the single vertex.

Example 1.2.8 *The following graph is a wheel graph W_3*

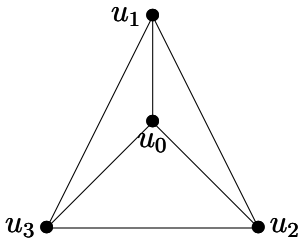


Figure 1.2.9 : Wheel graph denoted by W_3 or $W_{1,3}$

Book graph

The book graph B_n is a graph with the vertex set $V(B_n) = \{u_i, v_i : i = 0, 1, \ldots, n\}$and the edge set $E(B_n) = \{u_i v_i : \ i = 0, 1, \ldots, n\} \cup \{u_0 u_i, v_0 v_i : \ i = 1, 2, \ldots, n\}$.

Example 1.2.9 *Let the following graph be a book graph of order* 3 B_3

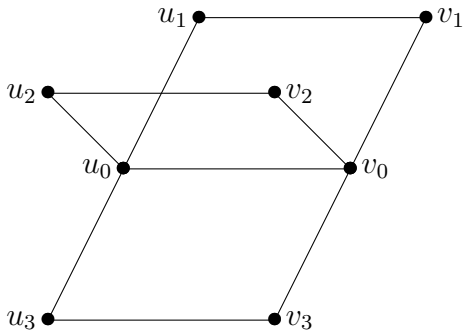


Figure 1.2.10 : Book graph B_3

Total graph of a graph

The total graph $T(G)$ has vertex set $V(G) \cup E(G)$ and two vertices of $T(G)$ are adjacent whenever they are neighbors in G.

1.3 Join Graphs, Union Graphs and Cartesian Product

In this section, we review the join graphs, union graphs and cartesian product of two disjoint graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$.

Join Graphs

The join graphs of the graphs G and H, denoted by $G \vee H$, is the graph with the vertex set $V = V_1 \cup V_2$ and edges set $E = E_1 \cup E_2 \cup E_3$ where $E_3 = \{v_1v_2 : v_1 \in V_1, v_2 \in V_2\}$.

Lemma 1.3.1 *Let G and H be two disjoint graphs. Then $|V(G + H)| = |V(G)| + |V(H)|$. and $|E(G + H)| = |E(G)| + |E(H)| + |V(G)||V(H)|$.*

Example 1.3.1 *The graph in the figure 1.2.9 is the graph join of $K_1 + C_3$*

Union Graphs

The union graphs of G and H, denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$
and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$

Cartesian Product

The cartesian product of two graphs G_1 and G_2, denoted by $G_1 \square G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \square G_2$ if either $u_1 = u_2$ and v_1 adjacent to v_2 in G_2 or $v_1 = v_2$ and u_1 adjacent to u_2 in G_1.

Remark 1.3.1 *The cartesian product of two graphs is commutative, i.e. $G_1 \square G_2 = G_2 \square G_1$.*

CHAPTER 2

Chromatic Number, Locating Chromatic and Neighbor Locating Chromatic Number

In this chapter, we introduce some types of graph coloring which are proper coloring, locating coloring, and the neighbor-locating coloring. Moreover, we present some known results for the neighbor locating chromatic number.

2.1 Historical

The origin of the coloring problem dates back to the 19^{th} century with one of the most famous and productive problems, which is the four-color problem. This problem remained unsolved for more than a century. The four-color problem consisted in deciding the following question: can we color any geographical map so that two neighboring regions do not have the same color by using only four colors at most?

The first results of graph coloring concern almost exclusively planar graphs, it was then a matter of coloring maps. In 1852, the young mathematician F.Guthrie

wondered if it is always possible to color a geographical map using four colors so that neighboring countries have different colors. The first written record of the four color conjecture appears in a letter that A. D. Morgan sent to W.R. C Hamilton in 1852. P J. Heawood showed the five color theorem by taking ideas from A. Kempe. Many works were published during the following century to reduce the number of colors to four, until the final demonstration of two American researchers, K.Appel and W.Haken in 1976. They were able to respond affirmatively to this conjecture of the four colors.

2.2 Proper Coloring

The definitions that are presented in this section can be found in [8].

Definition 2.2.1 *A **vertex coloring** of a graph G means an assignment of colors to the vertices of G, one color to each vertex.*

Definition 2.2.2 *A **proper vertex coloring** of a graph G can be considered as a function $c : V(G) \to \mathbb{N}$(where $\mathbb{N}$ is the set of positive integers) such that $c(u) \neq c(v)$ if u and v are adjacent in G.*

A k-coloring of a graph G is a vertex coloring of G using k colors. A graph G is said to be k-colorable if it can be properly colored using k colors. If V_i, $(1 \leq i \leq k)$ is the set of vertices in G colored by the color i, then each nonempty set V_i is called a **color class** and the nonempty elements of $V_1, V_2, \ldots, V_k$ produce a partition of $V(G)$. Note that the adjacent vertices of G have different colors by c. So, each nonempty color class V_i, $(1 \leq i \leq k)$ is an independent set of $V(G)$.

Definition 2.2.3 *The minimum positive integer k for which G is k-colorable is called the **chromatic number** of G and is denoted by $\chi(G)$. In other words, the chromatic number of G can be considered as the cardinality of a minimum partition of V into independent sets.*

Example 2.2.1 *The following graph G has a proper 3-coloring.*

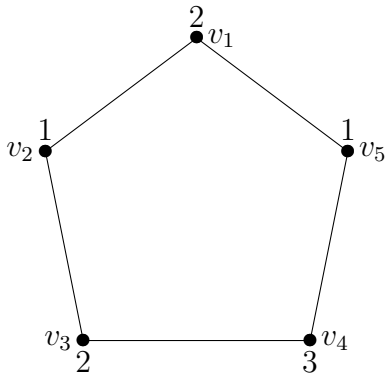


Figure 2.2.1 : Proper coloring of G, $\chi(G) = 3$.

In the next sections, we will review other types of coloring which are locating coloring and neighbor-locating coloring.

2.3 The Locating-Coloring

Let G be a connected graph and C be a proper k-coloring of G.
Let $W = (C_1, C_2, \ldots, C_k)$ be an ordered partition of $V(G)$ into the resulting color classes C_i, $1 \leqslant i \leqslant k$.

Definition 2.3.1 *For any vertex v of a graph G, the color code of the vertex v with respect to W is defined to be the ordered k-tuple $c_W(v) = (d(v, C_1), d(v, C_2), \ldots, d(v, C_k))$, where $d(v, C_i)$ is the distance between the vertex v and the set C_i, $1 \leqslant i \leqslant k$.*

Definition 2.3.2 *The locating coloring is a proper coloring c so that, distinct vertices of G have distinct color codes.*

Definition 2.3.3 *The minimum number of colors used to form a locating coloring is called the **locating-chromatic number** of G, denoted by $\chi_L(G)$.*

Example 2.3.1 *The following graph H has a locating 4 coloring:*

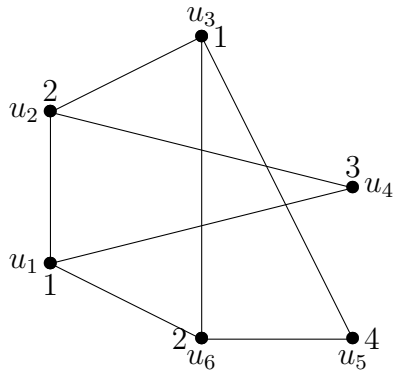


Figure 2.3.1 : A locating coloring of H.

We obtain a partition $W = \{S_1, S_2, S_3, S_4\}$ *of* $V(H)$*, where* $S_1 = \{u_1, u_3\}$*,* $S_2 = \{u_2, u_6\}$*,* $S_3 = \{u_4\}$ *and* $S_4 = \{u_5\}$*. The color code of each vertex is* $c_W(u_1) = (0, 1, 1, 2)$*,* $c_W(u_2) = (1, 0, 1, 2)$*,* $c_W(u_3) = (0, 1, 2, 1)$*,* $c_W(u_4) = (1, 1, 0, 3)$*,* $c_W(u_5) = (1, 1, 3, 0)$*,* $c_W(u_6) = (1, 0, 2, 1)$*. Observe that, distinct vertices of* H *have distinct color codes.*

Theorem 2.3.1 *Let* c *be a locating coloring in a connected graph* G*. If* u *and* v *are distinct vertices of* G *such that* $d(u, t) = d(v, t)$ *for all* $t \in V(G) - \{u, v\}$*, then* $c(u) \neq c(v)$*. In particular, if* u *and* v *are non-adjacent vertices of* G *such that* $N(u) = N(v)$*, then* $c(u) \neq c(v)$*.*

2.4 The Neighbor-Locating Coloring

The aim of this section is to review the definition of the neighbor-locating chromatic number and present some important results about it.

Definition 2.4.1 *Let* G *be a graph. A coloring* $\Pi_i = \{S_1, \ldots, S_k\}$ *of a graph* G *is called a* ***neighbor-locating coloring****, an* ***NL-coloring*** *for short, if for every pair of different vertices* u *and* v *belonging to the same color set* S_i*, the set of colors of the neighborhood of* u *is different from the set of colors of the neighborhood of* v*, that is*

$$\{j : N(u) \cap S_j \neq \oslash\} \neq \{j : N(v) \cap S_j \neq \oslash\}.$$

Definition 2.4.2 *The neighbor-locating chromatic number $\chi_{NL}(G)$, the* ***NLC-number*** *for short, is the minimum cardinality of an NL-coloring of G.*

Theorem 2.4.1 *[6] Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a partition of the set of vertices $V(G)$ for a graph G. For every vertex $x \in V(G)$, the tuple $c_\Pi(x) = (x_1, x_2, \ldots, x_k)$ represented as follows:*

$$x_i = \begin{cases} 0 & \text{if } x \in S_i \\ 1 & \text{if } x \in N(S_i) \setminus S_i \\ 2 & \text{if } x \notin N[S_i] \end{cases} \tag{2.4.1}$$

Then, the partition Π is a neighbor-locating coloring if and only if distinct vertices have different color code.

Example 2.4.1 *The following graph G has a 4-NL-coloring.*

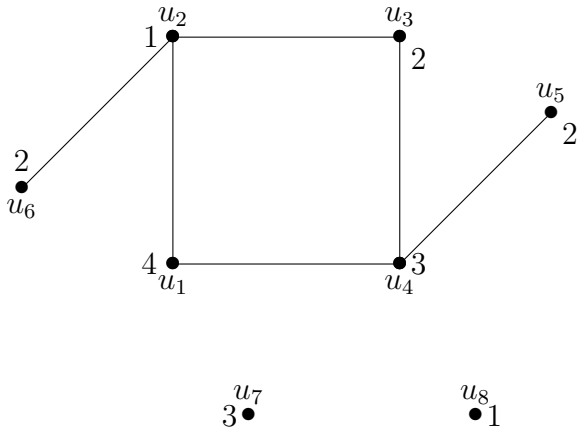


Figure 2.4.1 : A 4-NL-coloring of the graph G.

▶ *The graph coloring in Figure 2.4.1, is given by $\Pi = \{S_1, S_2, S_3, S_4\}$, where $S_1 = \{u_2, u_8\}$, $S_2 = \{u_3, u_5, u_6\}$, $S_3 = \{u_4, u_7\}$ and $s_4 = \{u_1\}$, is an NL-coloring. Since for the vertices $u_3, u_5 \in S_2$, $d(u_3, S_1) = 1$ but $d(u_5, S_1) = 2 \neq 1$, And the isolated vertices u_7 and u_8 have different colors.*

▶ *In Figure 2.4.1 we have the tuples:* $c_\Pi(u_1) = (1,2,1,0)$, $c_\Pi(u_2) = (0,1,2,1)$, $c_\Pi(u_3) = (1,0,1,2)$, $c_\Pi(u_4) = (2,1,0,1)$, $c_\Pi(u_5) = (2,0,1,2)$, $c_\Pi(u_6) = (1,0,2,2)$, $c_\Pi(u_7) = (2,2,0,2)$ *and* $c_\Pi(u_8) = (0,2,2,2)$.

Using the above presentation for the vertices, Definition 2.4.1 can be rewritten as follows:

A partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ is an NL-coloring if and only if the sets $S_1, S_2, \ldots, S_k$ are independent and $c_\Pi(x) \neq c_\Pi(y)$, for every pair of distinct vertices x and y.

Notice that $c_\Pi(x)$ has exactly one component equals to 0. In fact, S_i contains exactly all the vertices $x \in V(G)$ such that the i-th component of $c_\Pi(x)$ is equal to 0. Moreover, if Π is an NL-coloring of G and $x \in V(G)$ is a non-isolated vertex, then the tuple $c_\Pi(x)$ has at least one component equals to 1.

2.4.1 Some Results of the Neighbor Locating Chromatic Number

Many researchers studied the neighbor locating chromatic number and they provided some interesting results for some classes of graphs and bounds for others. In the following theorems, we will mention some of these results.

Theorem 2.4.2 *[4] Let* $\Pi = \{S_1, \ldots, S_k\}$ *be a* k*-NL-coloring of a graph* G *order* n *and maximum degree* Δ. *For every* $1 \leq i \leq k$, *there are at most* $\binom{k-1}{j}$ *vertices in* S_i *of color-degree* j, *where* $1 \leq j \leq k-1$ *and consequently*

$$| S_i | \leq \sum_{j=1}^{\Delta} \binom{k-1}{j}$$

Theorem 2.4.3 *[4] Let* G *be a connected graph of order* $n \geq 3$. *Then:*

1. *If* $diam(G) \leq 2$, *then* $\chi_L(G) = \chi_{NL}(G)$.

2. *If* $diam(G) \geq 3$, *then* $\chi_{NL}(G) \leq n-2$.

Remark 2.4.1 *If G is nontrivial connected graph, then*

$$2 \leq \chi(G) \leq \chi_L(G) \leq \chi_{NL}(G).$$

Example 2.4.2 *Consider the following example*

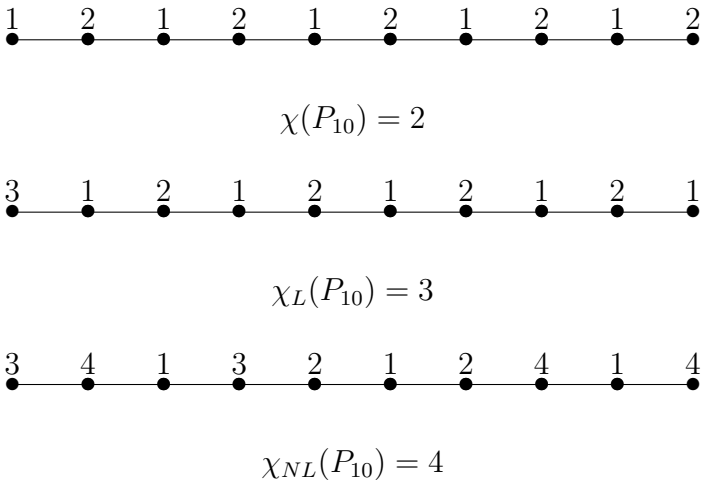


Figure 2.4.3 : Three types of coloring for the Graph G

Theorem 2.4.4 *[4] If G is a graph of order $n > 3$, then $\chi_{NL}(G) = n$ if and only if G either a complete multipartite graph or a null graph.*

Theorem 2.4.5 *[4] For a positive integer $n \geqslant 2$,*

$$\chi_{NL}(P_n) = m, where\ m = \min\{k : k \in \mathbb{N},\ n \leqslant \frac{(k^3 - k^2)}{2}\}$$

.

Theorem 2.4.6 *[4] Let $l = min\{k :\ k \in \mathbb{N},\ 9 \leqslant n \leqslant \frac{(k^3 - k^2)}{2}\}$, then*

$$\chi_{NL}(C_n) = \begin{cases} l+1 & if\ n = \frac{1}{2}(l^3 - l^2) - 1, \\ l & otherwise. \end{cases}$$

In this part, we are going to present the neighbor-locating chromatic number for some graph operations: the join, the union and the Cartesian product.

Theorem 2.4.7 *[5] Let G_1 and G_2 be two graphs. Then*

$$\chi_{NL}(G_1 \vee G_2) = \chi_{NL}(G_1) + \chi_{NL}(G_2).$$

Example 2.4.3 $\chi_{NL}(P_4 \vee P_{10}) = \chi_{NL}(P_4) + \chi_{NL}(P_{10})$ *by Theorem* 2.4.7, *so* $\chi_{NL}(P_4 \vee P_{10}) = 3 + 4 = 7$.

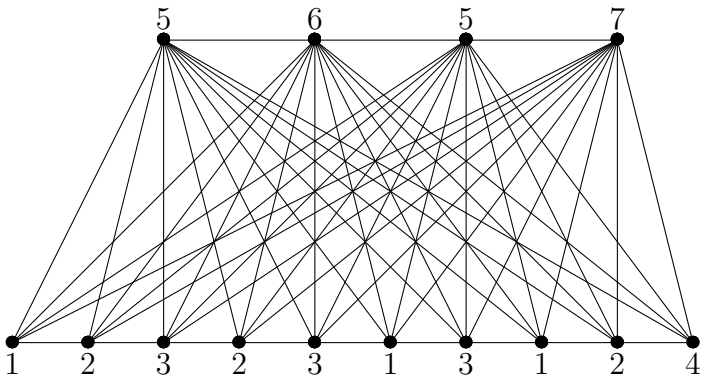


Figure 2.4.4 : $\chi_{NL}(P_4 \vee P_{11}) = \chi_{NL}(P_4) + \chi_{NL}(P_{10}) = 7$.

Any NL-coloring of the union of two graphs H and G induces an NL-coloring of G and an NL-coloring of H. So, we have the following results.

Theorem 2.4.8 *[5] Let G and H be two graphs with $\chi_{NL}(G) = k$, $\chi_{NL}(H) = h$. Then $max\{h, k\} \leq \chi_{NL}(G + H)$.*

Example 2.4.4 *To see the lower bound in Theorem 2.4.8 is tight consider the following:*

1. *For $max\{h, k\} = 2$, $\chi_{NL}(P_2 + \overline{K}_2) = 2$.*
2. *For $max\{h, k\} = 3, \chi_{NL}(2P_3) = 3$.*
3. *For $max\{h, k\} \geq 4$, $\chi_{NL}(K_{1,k-1} + K_{1,h-1}) = max\{h, k\}$.*

Theorem 2.4.9 *[5] Let G be a graph with $\chi_{NL}(G) = k$. If G contains a universal vertex, then $\chi_{NL}(2G) \leq k + 1$.*

For the cartesian product we have the following theorems:

Theorem 2.4.10 *[7] Let G and H be two connected graphs. Then*

$$\chi_{NL}(G\Box H) \leq \chi_{NL}(G) + \chi_{NL}(H)$$

.

Remark 2.4.2 *The bound in Theorem 2.4.10 is tight, to see this consider the graph P_2. Then $\chi_{NL}(P_2\Box P_2) = \chi_{NL}(C_4) = 4 = \chi_{NL}(P_2) + \chi_{NL}(P_2)$.*

Theorem 2.4.11 *[7] If G is a graph with a universal vertex and H is a connected graph, then $\chi_{NL}(G\Box H) \leq \chi_{NL}(G) + \mid V(H) \mid -1$.*

Remark 2.4.3 *The bound in the previous theorem is tight. Consider the graphs K_3, P_2. Then $\chi_{NL}(K_3) = 3$, $\chi_{NL}(P_2) = 2$ and $\chi_{NL}(K_3\Box P_2) = 4$ as we can see in the following figure.*

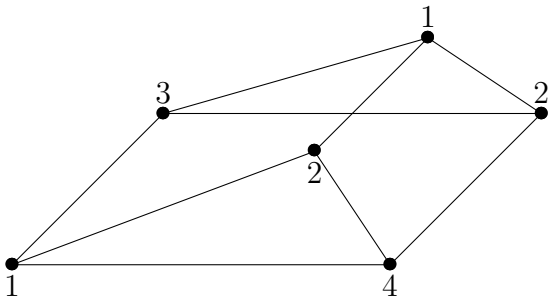


Figure 2.4.5 : The NL-coloring of $(K_3\Box P_2)$

CHAPTER 3

The Neighbor-Locating Chromatic Number of Some Graphs

The contribution in this chapter is the determination of the neighbor locating-chromatic number of three different graphs. This chapter is organized as follows: In section 3.1 we determined the neighbor locating-chromatic number of the total graph of star and wheel graphs. Then, in section 3.2 , we determine this parameter for the book graph.

3.1 The NLC Number of the Total Graph of Star and Wheel Graphs

The neighbor-locating chromatic number of the total graph of star and wheel graphs.

3.1.1 The Total Graph of the Star Graph

Let S_n be the n-star graph, where $V(S_n) = \{v_0, v_1, v_2, \ldots, v_n\}$ and $E(S_n) = \{v_0v_1, v_0, v_2, \ldots, v_0v_n\}$. The total graph of the n-star graph is given by subdividing

each edge exactly once and joining all the middle vertices of adjacent edges of S_n and all the old adjacent vertices of S_n.

Denote the middle vertex corresponding to edge v_0v_i by u_i where $1 \leqslant i \leqslant n$.

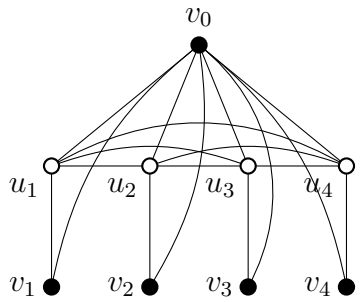


Figure 3.1.1 : Total graph of S_5

The following theorem proves the exact value of the locating chromatic number for the total graph of the star graph $T(S_n)$.

Theorem 3.1.1 *Let $T(S_n)$ be a Total graph of the star S_n for $n \geq 3$. Then the Neighbor-locating chromatic number of $T(S_n)$ is $\chi_{NL}(T(S_n)) = n$.*

Proof. First, we determine the lower bound of the neighbor locating chromatic number for the total graph of the star S_n. For $n \geq 3$. Since the total graph of the star graph contains a clique of order n, then we have $\chi_{NL}(T_n) \geq n$. Now, let c be a neighbor-locating coloring. To show that n is an upper bound for the locating chromatic number of the total graph of star $T(S_n)$ it suffices to prove the existence of an optimal neighbor-locating coloring $c : V(S_n) \to \{1, 2, \ldots, n\}$.
For $n \geq 3$ we construct the function c in the following way:

$$c(v_0) = n,$$

$$c(u_i) = i, \quad 1 \leq i \leq n,$$

$$c(v_n) = 1,\ c(v_i) = i + 1, \quad 1 \leq i \leq n.$$

By using the coloring c, we obtain the color codes of $V(T(S_n))$ as follows:

$$c_{\Pi}(v_0) = (1, \ldots, 1, 0)$$

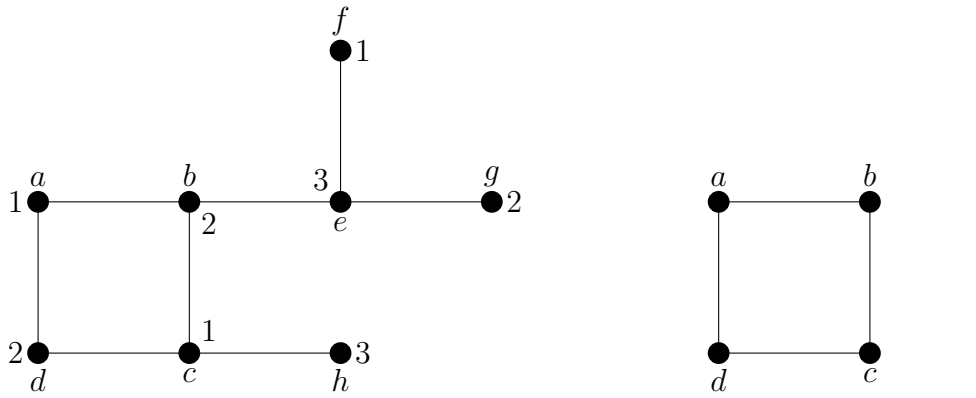
$$c_{\Pi}(u_i) = \begin{cases} 0 & \text{for } i^{th}component, 1 \leq i \leq n, \\ 1 & \text{otherwise.} \end{cases} \tag{3.1.1}$$

$$c_{\Pi}(v_i) = \begin{cases} 0 & \text{for } (i+1)^{th}component, 1 \leq i \leq n-1, \\ 1 & \text{for } i^{th}component, 1 \leq i \leq n-1, \\ 2 & \text{otherwise.} \end{cases} \tag{3.1.2}$$

$$c_{\Pi}(v_n) = \begin{cases} 0 & \text{for } 1^{st}component, \\ 1 & \text{for } n^{th}component, \\ 2 & \text{otherwise.} \end{cases} \tag{3.1.3}$$

Since all vertices in $V(T(S_n))$ have distinct color codes, then based on the Theorem 2.4.1 we conclude that c is an n-neighbor locating coloring of $T(S_n)$. Thus $\chi_{NL}(T(S_n)) = n$. ■

Remark For any graph G certainly $\omega(G) \leq \chi_{NL}(G)$. The situation is more complicated for any subgraph (not a clique). For example, the next figure presents a graph G and a subgraph H of G, where $\chi_{NL}(G) < \chi_{NL}(H)$.



(a) The graph G $\chi_{NL}(G)$=3.

(b) H=C_4 Subgraph of G $\chi_{NL}(C_4)$=4.

Figure 3.1.2 : Graph has a subgraph such that $\chi_{NL}(H) \geqslant \chi_{NL}(G)$

3.1.2 The Total Graph of the Wheel Graph

Let W_n be the n-wheel graph. $V(W_n) = \{u_0, u_1, u_2, \ldots, u_n\}$, u_0 be the center vertex and the other vertices on the rim, $E(W_n) = \{u_0u_i, 1 \leqslant i \leqslant n\} \cup \{u_iu_{i+1}, 1 \leqslant i \leqslant n-1\} \cup \{u_nu_1\}$. The total graph of wheel $T(W_n)$ is given by subdividing each edge only once and joining all the middle vertices of adjacent edges and all the older adjacent vertices of (W_n). Denote the middle vertex on the edge u_0u_i where $1 \leqslant i \leqslant n$ by v_i, the middle vertex of the edge u_iu_{i+1} where $1 \leqslant i \leqslant n-1$ by v'_i and the middle vertex of the edge u_nu_1 by v'_n.

Example 3.1.1 *The total graph of the 4-wheel graph is the following graph.*

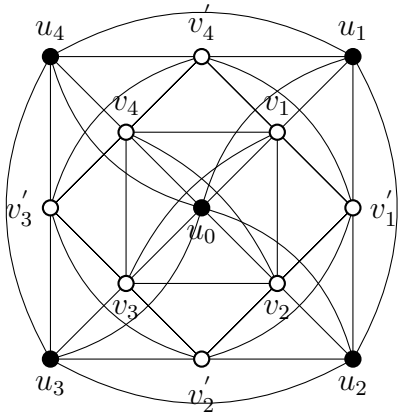


Figure 3.1.3 : The total graph of the 4-wheel graph

Theorem 3.1.2 *For $n = 3$, $\chi_{NL}(T(W_3)) = 6$*

Proof. Let $T(W_3)$ be the total graph of the wheel graph W_3 and c be the NL-coloring of $T(W_3)$. We can clearly note that the vertices u_0, v_1, v_2, v_3 induce a 4-clique, so $\chi_{NL} \geqslant 5$. Now assume that $\chi_{NL} = 4$. The graph has more than two 4-cliques, thus those 4-cliques have the same set of colors, which means there exist at least two vertices have the same color code, a contradiction. As a consequence, $\chi_{NL}(T(W_3)) \geqslant 5$. Now let c be a NL-5 coloring of $T(W_3)$ and $\Pi = \{S_1, S_2, S_3, S_4, S_5\}$ be a partition in the NL-coloring. Assume that the color used for u_0 is used only once (in u_0), since any other colors can be used at most twice we end up with only nine vertices can be colored which contradicts the fact that $|V(T(W_3))| = 10$ only which is a contradiction. Thus $c(u_0)$ must be used in one of the vertices v_i', $i = 1, 2, 3$ and any other color used twice so, for every vertex $x_i \in V(T(W_3))$ we get $d(x_i, s_i) = 1$, which means all vertices have the same color code, a contradiction. As a result we get $\chi_{NL}(T(W_3)) \geqslant 6$.

Now let $S_1 = \{v_1, v_2'\}$, $S_2 = \{u_1, v_2\}$, $S_3 = \{u_2, v_3\}$, $S_4 = \{u_0, v_3'\}$, $S_5 = \{v_1'\}$, $S_6 = \{u_3\}$. Then the coloring c is defined by $c : v(T(W_3)) \longrightarrow \{1, \ldots, 6\}$ such that $c(x) = j$ for all $x \in S_j$ is an NL-6 coloring of $T(W_3)$.

■

Theorem 3.1.3 *For $n = 4$, $\chi_{NL}(T(W_4)) = 6$.*

Proof. Let $T(W_4)$ be the total graph of wheel and c be the NL-coloring of $T(W_4)$. We can clearly see that the vertices u_0, v_1, v_2, v_3, v_4 form a clique of order 5, so $\chi_{NL} \geqslant 5$. Assume that $\chi_{NL} = 5$, then the neighbors of v'_i, $i = 1, 2, 3, 4$ are colored by three or four colors since each two 4-cliques share the same vertex v'_i. In case there is v'_i so that its neighbor colors set is $\{1, 2, 3, 4, 5\} \setminus \{C(v'_i)\}$ then $c_\Pi(v'_i) = c_\Pi(v_i)$ or $c_\Pi(v'_i) = c_\Pi(u_0)$. For that reason the neighbors of v'_i must be colored by only three different colors. Since there exist exactly four 4-cliques and each one of them contains two vertices of v'_i then all 4-cliques must be colored by the same set of colors. Meanwhile, each clique contains exactly one vertex from the v_i vertices which means all the four cliques must be colored by the same set of colors $\{(c(v_i)), 1 \leqslant i \leqslant 4\}$, so there exist at least two vertices u_k and v'_i that share the same color and $c(N(u_k)) = c(N(v'_i))$ and the distance between each one of them and u_0 is two, so they have the same color code. And that's a contradiction. As a result we get $\chi_{NL}(T(W_4)) \geqslant 6$. Let $S_1 = \{u_3, v_4\}$,$S_2 = \{u_4, v_1, v'_2\}$, $S_3 = \{v_2, v'_4\}$, $S_4 = \{u_2, v_3\}$, $S_5 = \{u_0, v'_1, v'_3\}$, $S_6 = \{u_1\}$. Then the coloring c is defined by $c : v(T(W_4)) \longrightarrow \{1, \ldots, 6\}$ such that $c(x) = j$ for all $x \in S_j$ is an NL-6 coloring of $T(W_4)$.

■

Theorem 3.1.4 *for $n \geqslant 5$, $\chi_{NL}(T(W_n)) = n + 1$*

Proof. Since the vertices $u_0, v_1, \ldots, v_n$ induce an $n + 1$-clique, $\chi_{NL}(T(W_n)) \geqslant n + 1$ Let $c : v(T(W_n)) \longrightarrow \{1, 2, \ldots, n + 1\}$ the coloring function in the following way:

$c(u_0) = n + 1$, $\quad c(v_i) = i$, $\quad 1 \leqslant i \leqslant n$.

$c(u_1) = n$, $\quad c(u_{i+1}) = i$, $\quad 1 \leqslant i \leqslant n - 1$.

$c(v'_1) = n - 1$, $\quad c(v'_2) = n$, $\quad c(v'_{i+2}) = i$, $\quad 1 \leqslant i \leqslant n - 2$

By using the coloring c, we obtain the following color codes of $V(T(W_n))$ as follows :

$$c_\Pi(u_0) = (1, 1, 1, \ldots, 0)$$

$$c_{\Pi}(v_i) = \begin{cases} 0 & \text{for } i^{th}component, \quad 1 \leq i \leq n \\ 1 & \text{otherwise} \end{cases} \tag{3.1.4}$$

$$c_{\Pi}(v_i) = \begin{cases} 0 & \text{for } (j)^{th}component, \quad j \equiv i-1 \ (mod\ n) \\ 1 & \text{for } j^{th}component, \quad j \equiv i,\ i-2 \ or \ i-3 \ (mod\ n) \\ 2 & \text{otherwise} \end{cases} \tag{3.1.5}$$

$$c_{\Pi}(v_i') = \begin{cases} 0 & \text{for } j^{th}component, \quad j \equiv i-2 \ (mod\ n) \\ 1 & \text{for } j^{th}component, \quad j \equiv i,\ i-1 \ or \ i-3 \ (mod\ n) \\ 2 & \text{otherwise} \end{cases} \tag{3.1.6}$$

As a consequence, the coloring c is an $n+1$-neighbor locating coloring of $T(W_n)$. ■

The NLC Number of the Book graph

In this section, we try to determine the neighbor locating chromatic number for the book graph. Whereas, we were inspired by the finding presented in Paper[12].

Let B_n be a book graph, with the vertex sets $V(B_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and the edge set $E(B_n) = \{u_i v_i : 1 \leq i \leq n\} \cup \{u_0 u_i, v_0 v_i : 1 \leq i \leq n\}$.

Example 3.1.2 *The book graph B_3 is the following graph.*

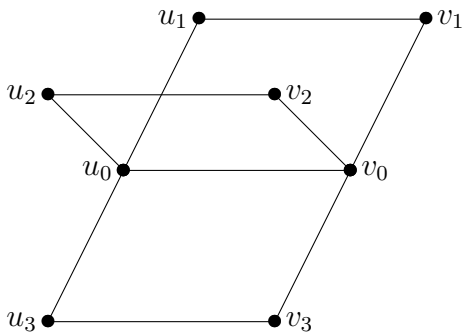


Figure 3.1.4 : Book graph B_3

Remark 3.1.1 *Let c be a NL-coloring of the book graph B_n, $n \geq 1$ then:*

1. $c(u_i) \neq c(v_i)$ *for* $1 \leq i \leq n$.

2. $c(u_0) \neq c(u_i)$ *and* $c(v_0) \neq c(v_i) for\ 1 \leq i \leq n$.

Lemma 3.1.1 *Let c be a NL-coloring of the book graph B_n, $n \geq 1$.*

Then $(c(u_i), c(v_i)) \neq (c(u_j), c(v_j))$ for $1 \leq i \neq j \leq n$.

Proof. Assume that $(c(u_i), c(v_i)) = (c(u_j), c(v_j))$ for some i and j, $i \neq j$. This means that $c(u_i) = c(u_j) = c_3$ and $c(v_i) = c(v_j) = c_4$ such that $N(u_i) = \{u_0, v_i$ and $N(u_j) = \{u_0, v_j\}$. That means $c(N(u_i)) = \{c_1, c_3\}$ and $c(N(u_j)) = \{c_1, c_3\}$, a contradiction. ■

Lemma 3.1.2 *Let n,k be two positive integers ($k \geq 4$ and $n \geq 1$) and c be a neighbor locating k-coloring of the Book graph B_n. Then*

$$n \leqslant 2\binom{k-2}{2} + 2(k-1) - 1.$$

Proof. First let $c(u_0) = 1$ and $c(v_0) = 2$ and from remark we have $c(u_i) \neq 1$ and $c(v_i) \neq 2$, for $1 \leq i \leq n$. We can distinguish three cases:

case 1 suppose that $\{c(u_a), c(v_a)\}$ different then 1 and 2. Therefor, We have left $k-2$ colors, and every outer part of the sheets requires two different colors from the total color we still have, which means $\binom{k-2}{2}$ possibilities. Since $c(u_a)$ and $c(v_a)$ are reversible the total possibilities are $2\binom{k-2}{2}$.

case 2 If $c(v_a) = 1$ then $c(u_a) \in \{2, \ldots, k\}$, so the total colors that can be used is $k-1$.

case 3 If $c(u_a) = 2$ then $c(v_a) \in \{1, 3, 4, \ldots, k\}$, so the total colors that can be used are $k-1$.

Since the coloring $c(u_a) = 1$ and $c(v_a) = 2$ is repeated in case 2 and case 3, then the coloring is not reusable, and thus the total coloring is $(k-1)-1$. If uses k colors, then $\chi_{NL}(B_n)$ must satisfy the following:

$$n \leqslant 2\binom{k-2}{2} + 2(k-1) - 1.$$

■

Theorem 3.1.5 *For a positive integer $n \geqslant 2$, $\chi_{NL}(B_n) = m$, where*

$$m = min\left\{k : n \leqslant 2\binom{k-2}{2} + 2(k-1) - 1\right\}$$

$$for\ k \geqslant 4.$$

Proof. Since

$$2\binom{(m-1)-2}{2} + 2((m-1)-1) - 1 < n \leqslant 2\binom{m-2}{2} + 2(m-1) - 1,$$

if we give a neighbor locating m-coloring of B_n, then Lemma 3.2.2 implies that $\chi_{NL}(B_n) = m$. Now, define the coloring function $c : V(B_n) \longrightarrow \{1, 2, \ldots, m\}$ as follows:

$c(u_0) = 1,\ \ c(v_0) = 2,$

$c(u_i) = 2,\ \ c(v_i) \in \{1, 3, 4, \ldots, m\}$ for $1 \leqslant i \leqslant m-1,$

$c(v_i) = 1,\ \ c(u_i) \in \{3, 4, \ldots, m\}$ for $m \leqslant i \leqslant 2m-3,$

$c(u_i) = l$ with $l \in \{3, \ldots, m\}$ and $c(v_i) = t$ with $t \in \{3, \ldots, m\}$ such that $t \neq l$ for $2m-2 \leqslant i \leqslant n$.

By using the coloring c, we obtain the color codes of $V(B_n)$ as follows:

$$c_\Pi(u_0) = (0, 1, 1, \ldots, 1), \qquad c_\Pi(v_0) = (1, 0, 1, \ldots, 1)$$

1. If $c(u_i) = 2$ then $c(v_i) \in \{1, 3, \ldots, m\}$ for $1 \leqslant i \leqslant m-1$,

$$c_{\Pi}(u_i) = \begin{cases} 0 & \text{for } 2^{nd}component, \\ 1 & \text{for } 1^{st}component,\ And\ (i+1)^{th}component, \text{ For } 2 < i \leqslant m-1, \\ 2 & \text{otherwise.} \end{cases} \tag{3.1.7}$$

$$c_{\Pi}(v_i) = \begin{cases} 0 & \text{for } 1^{st}component,\ And\ (i+1)^{th}component, \text{ For } 2 < i \leqslant m-1, \\ 1 & \text{for } 2^{nd}component, \\ 2 & \text{otherwise.} \end{cases} \tag{3.1.8}$$

2. If $c(v_i) = 1$ then $c(u_i) \in \{3, \ldots, m\}$ for $m \leqslant i \leqslant 2m-3$

$$c_{\Pi}(u_i) = \begin{cases} 0 & \text{for } (i-2)^{th}component, \text{ For } m \leqslant i \leqslant 2m-3, \\ 1 & \text{for } 1^{st}component, \\ 2 & \text{otherwise.} \end{cases} \tag{3.1.9}$$

$$c_{\Pi}(v_i) = \begin{cases} 0 & \text{for } 1^{st}component, \\ 1 & \text{for } 2^{nd}component,\ And\ (i-2)^{th}component, \text{ For } m \leqslant i \leqslant 2m-3, \\ 2 & \text{otherwise.} \end{cases} \tag{3.1.10}$$

3. If $c(u_i) = l$ with $l \in \{3, \ldots, m\}$ then $c(v_i) = t$ with $t \in \{3, \ldots, m\}$ such that $t \neq l$ for $2m-2 \leqslant i \leqslant n$

$$c_{\Pi}(u_i) = \begin{cases} 0 & \text{for } l^{th}component,\ \text{For } 3 \leqslant l \leqslant m \\ 1 & \text{for } 1^{st}component,\ And\ t^{th}component, \text{ For } 3 \leqslant t \leqslant m, t \neq l, \\ 2 & \text{otherwise.} \end{cases} \tag{3.1.11}$$

$$c_{\Pi}(v_i) = \begin{cases} 0 & \text{for } t^{th}component, \\ 1 & \text{for } 2^{nd}component,\ And\ l^{th}component, \text{ For } 3 \leqslant l \leqslant m, \\ 2 & \text{otherwise.} \end{cases} \tag{3.1.12}$$

As a consequence, the coloring c is a NL-coloring of B_n. ■

Conclusion

During this dissertation, we were interested in the concept of the neighbor-locating chromatic number of some graphs. The determination of the neighbor-locating chromatic number is a complicated topic in graph theory, in view of the fact that there is no general form to determine it for any graph. As a result and after having a collection of the basic concepts used in this study we presented some results concerning this latter. The main objective of this dissertation was to determine the exact value of the neighbor-locating chromatic number for the book graph, the total graph of the star, and the total graph of the wheel graph.

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