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Department of Mathematics and Computer Science



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BABAALI Kaouther

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Jury :

Mr. RAHMANI Mourad	Prof	Ensnn	President
Mme. GUETTAI Ghania	M.C.A	U.Y.F.M	Supervisor
Mr. LAISSAOUI Diffalah	Prof	U.Y.F.M	Examiner
Mr. SEBAOUI Madjid	M.C.A	U.Y.F.M	Examiner

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"He who says: I'm up to it... will reach it."
The journey was long and the path was not easy
but with faith and perseverance, I made it.

To the one who honored me with his name, the guiding light of my life,
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Abstract

In this thesis, we focus on the study and analysis of special classes of polynomials, namely the Bernoulli, Euler, and generalized Laguerre polynomials. We explore both classical and generalized forms of these polynomials, presenting their generating functions, explicit expressions, recurrence relations, and various interconnections through special numbers such as Stirling numbers.

A particular emphasis is placed on the generalized versions, including the generalized Bernoulli polynomials, generalized Euler polynomials, and the generalized Laguerre polynomials, highlighting their structure and importance in analytical and combinatorial contexts.

This work also discusses their relations with other special functions like the hypergeometric function and the Riemann zeta function, and provides proofs and applications relevant to both theoretical and applied mathematics.

Keywords: Generating function, explicit formula, recurrence relation, Stirling numbers, hypergeometric function, Riemann zeta function, Bernoulli polynomials, Euler polynomials, generalized Laguerre polynomials, orthogonal polynomials.

ملخص

في هذه المذكرة، نركّز على دراسة وتحليل بعض الأصناف الخاصة من كثيرات الحدود، وهي كثيرات حدود برنولي، وكثيرات حدود أويلر، وكثيرات حدود لاغار المعممة. نستعرض الصيغ الكلاسيكية والمعممة لهذه الكثيرة، حيث نقدم دوال المولدة الخاصة بها، والتعابير الصريحة، والعلاقات التراجعية، بالإضافة إلى مختلف الروابط التي تجمعها بأعداد خاصة مثل أعداد ستيرلينغ.

يعطى اهتمام خاص للإصدارات المعممة، بما في ذلك كثيرات حدود برنولي المعممة، وكثيرات حدود أويلر المعممة، وكثيرات حدود لاغار المعممة، مع تسليط الضوء على بنيتها وأهميتها في السياقين التحليلي والتوافقي.

كما تناقش هذه الدراسة علاقات هذه الكثيرة بدوال خاصة أخرى مثل الدالة فوق الهندسية ودالة زيتا لريمان، وتقدم براهين وتطبيقات ذات صلة بالرياضيات النظرية والتطبيقية.

كلمات مفتاحية: دالة المولدة، صيغة صريحة، العلاقة التراجعية، أعداد ستيرلينغ، الدالة فوق الهندسية، دالة زيتا لريمان، كثيرات حدود برنولي، كثيرات حدود أويلر، كثيرات حدود لاغار المعممة، كثيرات الحدود المتعامدة.

Notations

Sets

$\mathbb{C}$	The field of complex numbers.
$\mathbb{R}$	The field of real numbers.
$\mathbb{Z}$	The set of entire numbers.
$\mathbb{N}$	The set of natural numbers.

Symbols

$[\mathrm{n}]$	The set of $\{1, 2, \ldots, n\}$.
$t^{\overline{n}}$	The rising factorial.
$t^{\underline{n}}$	The falling factorial.
$\binom{n}{k}$	The binomial coefficient.
$\delta_{n,k}$	The Kronecker delta symbol.
$s(n,k)$	Signed Stirling numbers of the first kind.
$\lvert s(n,k)\rvert$	Unsigned Stirling numbers of the first kind .
$\left\{{n \atop k}\right\}$	Stirling numbers of the second kind.
$f'(t) = \dfrac{df}{dt}$	The first order derivative of a function.
$\lfloor n \rfloor$	Floor function
δ_{nm}	Kronecker delta symbol.
$\langle P \mid Q \rangle$	The inner product .

Polynomials and Functions

$W_{n,r}$	r-Wihitney numbers.
B_n	Bernoulli numbers.
$B_n(x)$	Bernoulli polynomials.
$B_n^{\alpha}(x)$	Bernoulli polynomials of Order α.
E_n	Euler numbers.
$E_n(x)$	Euler polynomials.
$E_n^{\alpha}(x)$	Euler polynomials of Order α.
$L_n(x)$	Classical Laguerre polynomials.
$L_n^{\alpha}(x)$	Laguerre polynomials of Order α.
$L_n^m(x)$	Associated Laguerre polynomials.
$a_{n,m}(x)$	The n^{th} associated polynomials.
F_n	The n^{th} Fibonacci number.
$F_n(x)$	The n^{th} Fibonacci polynomials.
$\mathbb{F}_n^m(x)$	The n^{th} m-associated Fibonacci polynomials.
$\zeta(s)$	The Riemann zeta function.
$\Gamma(x)$	The Gamma function, extension of the factorial.
${}_pF_q$	Hypergeometric function.

Introduction

Introduction

Special polynomials such as **Bernoulli**, **Euler**, and **Laguerre** play a fundamental role in various areas of mathematics, including number theory, analysis, and mathematical physics. These polynomials are not only of theoretical interest due to their rich algebraic structures and recurrence relations, but also have practical applications in numerical methods, approximation theory, and solutions of differential equations.

This work is divided into four chapters:

1. The **first chapter** presents essential preliminary concepts and mathematical tools that form the foundation for the subsequent discussions. These include notations, definitions, and classical results related to special functions and polynomials.
2. The **second chapter** is dedicated to **Bernoulli numbers and polynomials**, including their generalizations such as **generalized Bernoulli polynomials of order** α. The chapter covers their generating functions, identities, and key properties that make them useful in various analytical contexts.
3. The **third chapter** explores **Euler polynomials**, following a similar structure to that of the Bernoulli polynomials. Their definition, recurrence relations, and special cases are presented, along with their connections and comparisons to the Bernoulli family.
4. The **fourth chapter** focuses on **Laguerre polynomials**, both in their classical form and in generalized forms such as **Laguerre polynomials of order** α and **associated Laguerre polynomials**. These polynomials are essential in the study of orthogonal systems and frequently appear in the solutions of differential equations in quantum mechanics.

This thesis aims to present a unified view of these families of polynomials, highlighting their definitions, fundamental properties, and the relationships between them. Special attention is given to the role of parameter α in generalizations and to the structure of recurrence relations that define each family.

1

Basics Notions

In this chapter, we present some basic elements which will be useful for the rest, including a review of rising and falling factorials, the generating function, the explicit form, the recurrence relation, Stirling numbers and their combinatorial properties. For more precise and detailed concepts the reader can refer to [5, 9, 12, 13, 18, 26, 28, 30, 32].

1.1 Rising Factorial and Falling Factorial

Definition 1.1 *Let t be a real or complex number and n a positive integer, we define the rising factorial of t of order n, denoted as $t^{\overline{n}}$, by :*

$$t^{\overline{n}} = t(t+1)\dots(t+n-1), \text{ if } n \geqslant 1 \text{ with } t^{\overline{0}} = 1,$$

and the falling factorial of t of order n, denoted as $t^{\underline{n}}$, by:

$$t^{\underline{n}} = t(t-1)\dots(t-n+1), \text{ if } n \geqslant 1 \text{ and } t^{\underline{0}} = 1.$$

The relation between the rising factorial and the falling factorial is given by:

$$t^{\overline{n}} = (-1)^n(-t)^{\underline{n}}.$$

1.2 Gamma Function

Definition 1.2 *$\Gamma(x)$ is the Gamma function, given by the integral:*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\,dt,$$

where

$$\Gamma(m+1) = m!, for \ \ m \in \mathbb{N}.$$

The definition of $t^{\overline{n}}$ can be extended to any integer $n \in \mathbb{Z}$ by the following relation:

$$t^{\overline{n}} = \frac{\Gamma(t+1)}{\Gamma(t-n+1)}.$$

1.3 Binomail Coefficients

Definition 1.3 *The binomial coefficient $\binom{n}{k}$ represents the number of ways to choose a subset with k elements from a set with n elements.*

Definition 1.4 *Let n and k be two non-negative integers. The binomial coefficient of k, denoted as $\binom{n}{k}$, is given by:*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!} \quad (0 \leq k \leq n), \tag{1.1}$$

with $\binom{n}{k} = 0$ for $k > n$, and $\binom{n}{k} = 1$ for $k = 0$ or $k = n$.

We can also define the binomial coefficient in combinatorics by:

Definition 1.5 *The binomial coefficient* $\binom{n}{k}$ *is the number of ways to choose a subset with* k *elements from a set of* n *elements.*

Example 1.1
Replacing n *with* 4 *and* k *with* 2 *in (1.1), we obtain:*

$$\binom{4}{2} = \frac{4!}{(4-2)!2!} = 6.$$

In the set $[4] = \{1, 2, 3, 4\}$, *there are six ways to choose two elements:*

$$\{1, 2\}\,;\ \{1, 3\}\,;\ \{1, 4\}\,;\ \{2, 3\}\,;\ \{2, 4\}\,;\ \{3, 4\}.$$

As a result, $\binom{4}{2}$=6.

1.3.1 Newton's Binomial Formula

Let t and y be two real numbers and n a non-negative integer. So

$$(t+y)^n = \sum_{k=0}^{n} \binom{n}{k} t^k y^{n-k}. \tag{1.2}$$

1.3.2 Pascal's Recurrence Relation

The binomial coefficients satisfying the following relation:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad \text{for every} \quad n \geq k \geq 1. \tag{1.3}$$

1.3.3 Pascal's Triangle

From the recurrence relation (1.3), we can construct Pascal's triangle

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Table 1.1 – Pascal's Triangle Numbers

Property 1.3.1 *The binomial coefficients satisfy nearly an infinite number of identities, including :*

1. *The absorption identity*

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}. \tag{1.4}$$

2. *The symmetry*

$$\binom{n}{k} = \binom{n}{n-k}, \text{ with } n \geqslant k \geqslant 0. \tag{1.5}$$

3. The top summation

$$\binom{n+1}{k+1} = \sum_{j=0}^{n} \binom{k}{j}, \text{ with } n, k \geqslant 0. \tag{1.6}$$

4. The parallel summation

$$\binom{m+n+1}{n} = \sum_{n \geqslant j} \binom{m+j}{j}. \tag{1.7}$$

1.3.4 Generalization of Binomial Coefficients

A generalization of the binomial coefficient $\binom{n}{k}$ consists in replacing n by a complex number t, and to define $\binom{t}{k}$ by posing, $t \in \mathbb{C}$ and $k \in \mathbb{N}$ by:

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!} = \frac{t^{\underline{k}}}{k!} = \frac{(-1)^n t^{\overline{k}}}{k!}, \tag{1.8}$$

and having almost the same properties as $\binom{n}{k}$.
We have

$$\binom{-k}{n} = \binom{k+n-1}{n}(-1)^n. \tag{1.9}$$

1.4 The Generating Function and Some Operations on Power Series

1.4.1 The Generating Function

Definition 1.6 *Let $\mathcal{A}$ be a commutative ring with a unity element. Formal power series with coefficients a_n in $\mathcal{A}$ and indeterminate t are defined as follows:*

$$G(t) = \sum_{n \geq 0} a_n t^n \quad \text{and} \quad H(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!}.$$

The formal series $\sum_{n \geq 0} a_n t^n$ associated to the sequence $((a_n)_{n \geq 0})$ is called the ordinary generating function. The formal series $\sum_{n \geq 0} a_n \frac{t^n}{n!}$ associated to the sequence $((a_n)_{n \geq 0})$ is called the exponential generating function.

We denote by $\mathcal{A}[[t]]$ the set of all formal power series with values in $\mathcal{A}$.

1.4.2 Some Operations on Power Series

Cauchy Product of Series

The product of two ordinary generating functions $\sum_{n \geq 0} a_n t^n$ and $\sum_{n \geq 0} b_n t^n$ is given by:

$$\left(\sum_{n \geq 0} a_n t^n\right)\left(\sum_{n \geq 0} b_n t^n\right) = \sum_{n \geq 0} c_n t^n, \quad \text{where } c_n = \sum_{k=0}^{n} a_k b_{n-k}. \tag{1.10}$$

The product of two exponential generating functions $\sum_{n \geq 0} a_n \frac{t^n}{n!}$ and $\sum_{n \geq 0} b_n \frac{t^n}{n!}$ is given by:

$$\left(\sum_{n \geq 0} a_n \frac{t^n}{n!}\right)\left(\sum_{n \geq 0} b_n \frac{t^n}{n!}\right) = \sum_{n \geq 0} c_n \frac{t^n}{n!}, \quad \text{where } c_n = \sum_{k=0}^{n} a_k b_{n-k} \binom{n}{k}. \tag{1.11}$$

Sum

The sum of two ordinary generating functions $\sum_{n\geq 0} a_n t^n$ and $\sum_{n\geq 0} b_n t^n$ is defined as:

$$\sum_{n\geq 0}(a_n + b_n)t^n = \sum_{n\geq 0} a_n t^n + \sum_{n\geq 0} b_n t^n. \tag{1.12}$$

The sum of two exponential generating functions $\sum_{n\geq 0} a_n \frac{t^n}{n!}$ and $\sum_{n\geq 0} b_n \frac{t^n}{n!}$ is defined as:

$$\sum_{n\geq 0}(a_n + b_n)\frac{t^n}{n!} = \sum_{n\geq 0} a_n \frac{t^n}{n!} + \sum_{n\geq 0} b_n \frac{t^n}{n!}. \tag{1.13}$$

Equipped with these two operations, the set of formal power series $\mathcal{A}[[t]]$ forms a ring.

1.5 The Riemann Zeta Function

Definition 1.7 *The Riemann zeta function $\zeta(s)$ is defined for any complex number s satisfying $\Re(s) > 1$ (i.e., the real part of s is strictly greater than 1), by:*

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}. \tag{1.14}$$

We now present a generalization of the Riemann zeta function, defined by Euler and Zagier [2, 3].

1.6 The Multiple Zeta Function of Euler-Zagier

Definition 1.8 *The Euler-Zagier multiple zeta values of depth p (with $p \geq 1$) are defined by:*

$$\zeta_p(s_1, s_2, \ldots, s_p) = \sum_{0<n_1<n_2<\cdots<n_p} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_p^{s_p}}, \tag{1.15}$$

where $s_1, s_2, \ldots, s_p$ are positive integers ≥ 1, with $s_p > 1$.

Akiyama et al. [2, 3] defined the regularized values of the multiple zeta function at non-positive integers by:

$$\zeta_p(-r_1, -r_2, \ldots, -r_p) = \lim_{s_1\to -r_1} \cdots \lim_{s_p\to -r_p} \zeta_p(s_1, s_2, \ldots, s_p), \quad (r_1, r_2, \ldots, r_p \geq 0). \tag{1.16}$$

1.7 The Stirling Numbers

1.7.1 Stirling Numbers of the First Kind

Definition 1.9 *The signed Stirling numbers of the first kind, denoted as $s(n,k)$, are the coefficients of the falling factorial polynomials in the canonical basis t^n, that's to say:*

$$t^{\underline{n}} = \sum_{k=0}^{n} s(n,k) t^k.$$

Definition 1.10 *The unsigned Stirling numbers of the first kind, denoted as $|s(n,k)|$, are the coefficients of the rising factorial polynomials in the canonical basis t^n, that's to say:*

$$t^{\overline{n}} = \sum_{k=0}^{n} |s(n,k)| t^k,$$

we have:

$$|s(n,k)| = (-1)^{n-k} s(n,k).$$

Definition 1.11 *The Stirling numbre of the first kind $/s(n,k)/$ counts the number of permutations of n objects that consist of exactly k disjoint cycles.*

Example 1.2 *Write $t^{\underline{3}}$ as a polynomial*

$$\begin{aligned} t^{\underline{3}} &= t(t-1)(t-2) \\ &= t^3 - 3t^2 + 2t. \end{aligned}$$

We can say from this example that :
$s(3,0) = 0.$
$s(3,1) = 2.$
$s(3,2) = -3.$
$s(3,3) = 1.$

The Generating Function

The exponential generating function of the signed Stirling numbers of the first kind is given by:

$$\sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!} = \frac{(\ln(1+t))^k}{k!}.$$

The Explicit Formula

The explicit formula for the signed Stirling numbers of the first kind:

$$\mid s(n,k) \mid = \sum_{0 \leqslant j \leqslant h \leqslant n-k}^{\infty} (-1)^{j+h} \binom{h}{j} \binom{n-1+h}{n-k+h}.$$

The explicit formula for the signed Stirling numbers of the first kind:

$$s(n,k) = \frac{n!}{k!} \sum_{i_1 + \cdots + i_k = n} \frac{1}{i_1 \cdots i_k}.$$

The Recurrence Formula

The Stirling numbers of the first kind satisfy the following recurrence relation:

$$\begin{cases} s(n,0) = s(0,k) = 0, \quad \text{except} \quad s(0,0) = 1, \\ s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \quad (n \geqslant k \geqslant 1). \end{cases} \tag{1.17}$$

Table of the Stirling Numbres of the First Kind

Using the triangular recurrence relation (1.17), we can calculate the Stirling numbers of the first kind $s(n,k)$:

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	−1	1					
3	0	2	−3	1				
4	0	−6	11	−6	1			
5	0	24	−50	35	−10	1		
6	0	−120	274	−225	85	−15	1	
7	0	720	−1764	1624	−735	175	−21	1

Table 1.2 – Stirling numbres of the first kind

1.7.2 Stirling Numbers of the Second kind

Definition 1.12 *Stirling numbers of the second kind, denoted as* $\left\{ {n \atop k} \right\}$ *are the coefficients of the polynomial* $t^{\overline{n}}$*, that's to say:*

$$t^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} t^{\underline{k}}.$$

Definition 1.13 *The Stirling numbers of the second kind,* $\left\{ {n \atop k} \right\}$ *counts the number of partitions of the set* $\{1, 2, \ldots, n\}$ *into* k *non-empty subsets (*$0 \leq k \leq n$*).*

Example 1.3 *The set* $[4]$ *contain* 7 *partitions into two non-empty subsets* $\left\{ {4 \atop 2} \right\}$ *represent the partitions into* 2 *subsets:*

$$\{\{1\}\{2,3,4\}\}, \quad \{\{2\}\{1,3,4\}\}, \quad \{\{3\}\{1,2,4\}\}, \quad \{\{4\}\{1,2,3\}, \\ \{1,2\}\{3,4\}\}, \quad \{\{1,3\}\{2,4\}\}, \quad \{\{1,4\}\{2,3\}\}.$$

As a result, $\left\{ {4 \atop 2} \right\} = 7$*.*

The Generating Function

The ordinary generating function of Stirling numbers of the second kind is given by:

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} t^n = \frac{1}{(1-t)(1-2t)\cdots(1-kt)} = \prod_{j=1}^{k} \frac{1}{(1-jt)}.$$

The exponential generating function of Stirling numbers of the second kind is given by:

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{t^n}{n!} = \frac{1}{k!}(e^t - 1)^k.$$

The Explicit Formula

The Stirling numbers of the second kind are given by the following explicit formula:

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

The Recurrence Formula

The Stirling numbers of the second kind satisfy the following recurrence relation:

$$\begin{cases} \left\{ {n \atop 0} \right\} = \left\{ {0 \atop k} \right\} = 1, \quad \text{except} \quad \left\{ {0 \atop 0} \right\} = 1 \\ \\ \left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}, \ n \geq k \geq 0. \end{cases}.$$

Table of Stirling Numbers of Second Kind

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	1					
3	0	1	3	1				
4	0	1	7	6	1			
5	0	1	15	25	10	1		
6	0	1	31	90	65	15	1	
7	0	1	63	301	350	140	20	1

Table 1.3 – Stirling Numbers of the Second Kind

1.8 Generalized Stirling and Whitney Numbers

1.8.1 r-Stirling Numbers of the Second Kind

For any nonnegative integer r, the *r-Stirling numbers of the second kind*, $S_r(n,k)$, are a generalization of the Stirling numbers of the second kind. These numbers count the number of partitions of a set of n elements into exactly k non-empty subsets such that the first r elements are in distinct subsets. Their exponential generating function is given by:

$$\sum_{k\geq 0} \frac{S_r(n+r,k+r)}{n!} t^n = \frac{1}{k!} e^{rt} \left(e^t - 1\right)^k. \tag{1.18}$$

En particular, for $r = 0$ *or* 1, we get the classical Stirling numbers of the second kind.

1.8.2 r-Whitney Numbers

For any positive integer m, the *r-Whitney numbers of the second kind*, $W_{m,r}(n,k)$, are the coefficients in the following expansion:

$$(mx + r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k) x(x-1)\cdots(x-k+1). \tag{1.19}$$

The generating function for these numbers is given by:

$$\sum_{n\geq k} \frac{W_{m,r}(n,k)}{n!} t^n = \frac{1}{m^k k!} e^{rt} \left(e^{mt} - 1\right)^k. \tag{1.20}$$

It is easy to observe that for specific values of r, these numbers reduce to the classical Stirling numbers. Specifically:

$$W_{1,0}(n,k) = S\left(n,k\right), \quad W_{1,r}(n,k) = S_r(n+r,k+r). \tag{1.21}$$

1.8.3 Generalized Stirling Numbers

The *generalized Stirling numbers* $S_n^k(x)$ of the second kind are defined by:

$$S_n^k(x) = \frac{1}{k!} \Delta^k x^n, \tag{1.22}$$

where Δ denotes the forward difference operator is defined by:

$$\Delta f(x) = f(x+1) - f(x).$$

They can also be expressed as:

$$S_n^k(x) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+j)^n. \tag{1.23}$$

Their generating function is given by:

$$\sum_{n\geq k} \frac{S_n^k(x)}{n!} t^n = \frac{1}{k!} e^{xt} \left(e^t - 1\right)^k. \tag{1.24}$$

From this generating function, we deduce the following important relations:

$$S_n^k(x) = \sum_{j=0}^{n} \binom{n}{j} S(j,k) x^{n-j}. \tag{1.25}$$

$$S_n^k(0) = S(n,k). \tag{1.26}$$

$$S_n^k(r) = S_r(n+r, k+r). \tag{1.27}$$

$$m^{n-k} S_n^k\left(\frac{r}{m}\right) = W_{m,r}(n,k). \tag{1.28}$$

These tools will be useful throughout the study of generalized Euler and Bernoulli polynomials. For more details, see references such as [7, 11, 22, 23].

1.9 Fibonacci Numbers and Polynomials

Fibonacci polynomials generalize Fibonacci numbers by using variables. They are defined by a recurrence and have useful properties in mathematics [16, 15, 37].

1.9.1 Fibonacci Numbers

Definition 1.14 *The Fibonacci sequence $(F_n)_{n\in\mathbb{N}}$ is defined by the recurrence:*

$$F_{n+2} = F_{n+1} + F_n, \quad \forall n \geq 0, \quad F_0 = 0, \quad F_1 = 1. \tag{1.29}$$

The first few Fibonacci numbers are given in the following Table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233

Table 1.4 – First Values of Fibonacci Numbers

Remark 1 *Negative-index Fibonacci numbers are given by: $F_{-n} = (-1)^{n+1} F_n$.*

Property 1.9.1 (Binet's Formula)

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}, \quad \forall n \geq 0. \tag{1.30}$$

1.9.2 Connection with the Golden Ratio

Definition 1.15 *A number $\frac{a}{b}$ is called a golden ratio if it satisfies*

$$\frac{a+b}{a} = \frac{a}{b}. \tag{1.31}$$

Letting $\gamma = \frac{a}{b}$, we obtain the quadratic equation

$$\gamma = 1 + \frac{1}{\gamma} \Rightarrow \gamma^2 - \gamma - 1 = 0. \tag{1.32}$$

Its solutions are

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}. \tag{1.33}$$

Remark 2 *The positive solution α is the famous golden ratio, usually denoted by φ.*

Lemma 1

$$\alpha\beta = -1.$$
$$\alpha^2 - \beta^2 = \sqrt{5}.$$
$$\alpha^2 + \beta^2 = 3.$$

Property 1.9.2 *The Fibonacci number F_n can be expressed using the golden ratio as*

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}. \tag{1.34}$$

Property 1.9.3

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha_1. \tag{1.35}$$

1.9.3 Binomial Form of Fibonacci Numbers

Theorem 1.1

$$F_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} 5^k. \tag{1.36}$$

The floor function $\left\lfloor \frac{n-1}{2} \right\rfloor$ *returns the greatest integer less than or equal to* $\frac{n-1}{2}$.

Remark 3 *Fibonacci numbers can also be written using matrices*

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}, \tag{1.37}$$

which leads to ***Cassini's Identity****:*

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n. \tag{1.38}$$

1.9.4 Generating Function

Property 1.9.4 *The generating function for the Fibonacci numbers is*

$$\sum_{n\geq 0} F_n t^n = \frac{t}{1-t-t^2}. \tag{1.39}$$

1.9.5 Fibonacci Polynomials

Definition 1.16 *The sequence of fibonacci polynomials* $F_n(x)$ *is defined by the recurrence*

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \quad \forall n \geq 0, \quad F_0(x) = 0, \quad F_1(x) = 1. \tag{1.40}$$

Using this definition, we can give us the first terms of the Fibonacci polynomials.

n	$F_n(x)$
0	0
1	1
2	x
3	x^2+1
4	x^3+2x
5	x^4+3x^2+1
6	x^5+4x^3+3x
7	$x^6+5x^4+6x^2+1$

Table 1.5 – First Terms of the Fibonacci Polynomials

Property 1.9.5 *The generating function of the Fibonacci polynomials is:*

$$\sum_{n\geq 0} F(x)t^n = \frac{t}{1-xt-t^2}. \tag{1.41}$$

Proof. Let the generating function be

$$f(x,t) = \sum_{n\geq 0} F_n(x)t^n.$$

Using the reccurence relation

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x).$$

Then

$$\begin{aligned} f(x,t) &= \sum_{n\geq 0} \left(F_{n+2}(x) - xF_{n+1}(x)\right) t^n \\ &= \frac{1}{t^2}(f(x,t)-t) - \frac{x}{t} f(x,t) \\ &= \left(1 - \frac{1}{t^2} - \frac{x}{t}\right) f(x,t) \\ &= \frac{t^2+xt-1}{t^2} f(x,t), \end{aligned}$$

then

$$f(x,t) = \frac{t}{1-xt-t^2}.$$

■

1.10 Hypergeometric Functions

The origin of hypergeometric functions dates back to the beginning of the 19^{th} century, when Gauss studied the following differential equation [4, 10]:

$$t(1-t)y'' + (c-(a+b+1)t)y' - aby = 0, \tag{1.42}$$

where $a, b, c \in \mathbb{C}$. This equation is known as Gauss's differential equation. The generalized hypergeometric function has a series representation:

$$\sum_{n\geq 0} C_n,$$

with

$$\frac{C_{n+1}}{C_n},$$

a rational function of n, defined by the following relation:

$$\frac{C_{n+1}}{C_n} = \frac{(n+a_1)(n+a_p)t}{(n+b_1)\cdots(n+b_q)(n+1)}.$$

If $C_0 = 1$, then

$$C_n = \frac{(a_1)^{\overline{n}}(a_2)^{\overline{n}}\cdots(a_p)^{\overline{n}}}{(b_1)^{\overline{n}}(b_2)^{\overline{n}}\cdots(b_q)^{\overline{n}}} \cdot \frac{t^n}{n!}.$$

and the hypergeometric function is then

$${}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix};t\right) = \sum_{n\geq 0} \frac{(a_1)^{\overline{n}}(a_2)^{\overline{n}}\cdots(a_p)^{\overline{n}}}{(b_1)^{\overline{n}}(b_2)^{\overline{n}}\cdots(b_q)^{\overline{n}}} \cdot \frac{t^n}{n!}.$$

If $p = 1$ and $q = 1$, we obtain the confluent hypergeometric function of Kummer:

$${}_1F_1\left(\begin{matrix} a \\ b \end{matrix};t\right),$$

which is defined by:

$$\sum_{n\geq 0} \frac{(a)^{\overline{n}}}{(b)^{\overline{n}}} \cdot \frac{t^n}{n!}. \tag{1.43}$$

If $p = 2$ and $q = 1$, we obtain the Gauss hypergeometric function:

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};t\right) = \sum_{n\geq 0} \frac{(a)^{\overline{n}}(b)^{\overline{n}}}{(c)^{\overline{n}}} \cdot \frac{t^n}{n!},$$

where $|t| < 1$, the solution of the differential equation (1.42).

1.11 Orthogonal Polynomials

Given two real polynomials in x, $P(x)$ and $Q(x)$, we note:

$$\langle P \mid Q \rangle = \int_a^b P(x)Q(x)\rho(x)\,dx, \tag{1.44}$$

where $\rho(x)$ is defined and integrable function on (a, b), positive or zero, and piecewisz continuous. The integral $\langle P \mid Q \rangle$ is an inner product since:

$$\langle P \mid Q \rangle = \langle Q \mid P \rangle. \tag{1.45}$$

$$\langle P \mid P \rangle = \int_a^b P^2(x)\rho(x)\,dx \quad \begin{cases} > 0 & \text{if } P(x) \neq 0, \\ = 0 & \text{if } P(x) = 0 \end{cases} \tag{1.46}$$

Polynomials P and Q are said to be orthogonal if:

$$\langle P \mid Q \rangle = 0.$$

The family of polynomials $\{P_n(x)\}$, where $P_n(x)$ is a polynomial of degree n, is called a family or sequence of orthogonal polynomials with respect to the weight $\rho(x)$ on the interval (a, b) if:

$$\langle P_n \mid P_m \rangle = 0, \quad \text{for } n \neq m. \tag{1.47}$$

The sequence $\{P_n\}$ is said to be orthonormal if:

$$\langle P_n \mid P_m \rangle = \delta_{mn},$$

where δ_{mn} is the Kronecker delta symbol, defined by:

$$\delta_{mn} \quad \begin{cases} 1 & \text{if } m \neq n, \\ 0 & \text{if } m = n \end{cases} \tag{1.48}$$

2

The Generalized Bernoulli Polynomials

The generalized Bernoulli polynomials are an important extension of the classical Bernoulli polynomials, which originally appeared in number theory. They keep many key properties of the original polynomials but allow for broader applications. Defined using generating functions with extra parameters, they appear in fields like modular forms and finite differences, and are related to other special polynomials such as Euler and Genocchi polynomials. In this chapter, we present their definitions, main properties, and important identities [3, 5, 8, 14, 25, 35, 36].

2.1 Bernoulli Numbers

In 1755, Euler posed the following definition for the Bernoulli numbers, which remains the most common modern definition for the sequence [14].

Definition 2.1 *The Bernoulli numbers are the coefficients of the exponential generating function:*

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!}. \tag{2.1}$$

Explicit Formula

Theorem 2.1 *The explicit formula of the Bernoulli B_n is given by:*

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} \left\{ n \atop k \right\}.$$

Proof. We have:

$$\begin{aligned} t &= \ln(1 + (e^t - 1)) \\ &= \sum_{n \geq 0} (-1)^n \frac{(e^t - 1)^{n+1}}{n+1}. \end{aligned}$$

Using the generating function (2.1), we get:

$$\begin{aligned}\sum_{n\geq 0} B_n \frac{t^n}{n!} &= \frac{t}{e^t-1} \\ &= \sum_{k\geq 0}(-1)^k \frac{(e^t-1)^k}{k+1} \\ &= \sum_{k\geq 0}(-1)^k \frac{k!}{k+1}\cdot\frac{(e^t-1)^k}{k!} \\ &= \sum_{k\geq 0}(-1)^k \frac{k!}{k+1}\sum_{k\geq 0}\left\{{n \atop k}\right\}\frac{t^n}{n!} \\ &= \sum_{k\geq 0}\left(\sum_{k=0}^{n}(-1)^k \frac{k!}{k+1}\left\{{n \atop k}\right\}\right)\frac{t^n}{n!}.\end{aligned}$$

Now, identifying the two power series gives the directly result directly. ∎

Recurrence Relation

The Bernoulli numbers satisfy the following recurrence relation:

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0, \quad with \quad B_0 = 1. \tag{2.2}$$

Using this relation to calculate B_1, B_2, B_3, ...
For example, for n= 1, we have:

$$\begin{aligned}\sum_{k=0}^{1}\binom{2}{k}B_k &= 0, \\ \binom{2}{0}B_0 + \binom{2}{1}B_1 &= 0, \\ B_0 + 2B_1 &= 0, \\ 2B_1 &= -B_0, \\ B_1 &= -\frac{1}{2}.\end{aligned}$$

Numbers B_n are rational. Nemely, we have:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \ldots.$$

and

$$B_{2k+1} = 0, \ for\ k > 0.$$

Theorem 2.2 *The recurrence relation of the Bernoulli numbers is given by:*

$$B_0 = 1, \quad and \quad B_n = \sum_{k=0}^{n}\binom{n}{k}B_k, \quad \forall n \geq 2.$$

Proof. Let us define the generating function as:

$$B(t) = \frac{t}{e^t-1}.$$

Then, we have:

$$e^t B(t) = t + B(t).$$

We know that:

$$e^t = \sum_{n\geq 0} \frac{t^n}{n!}, \quad \text{and} \quad t + B(t) = 1 + \frac{t}{2} + \sum_{n\geq 2} B_n \frac{t^n}{n!}.$$

Now, multiply both series together:

$$\left(\sum_{n\geq 0} \frac{t^n}{n!}\right)\left(\sum_{n\geq 0} B_n \frac{t^n}{n!}\right) = 1 + \frac{t}{2} + \sum_{n\geq 2} B_n \frac{t^n}{n!}.$$

So,

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n} \frac{B_k}{k!(n-k)!}\right) t^n = 1 + \frac{t}{2} + \sum_{n\geq 2} B_n \frac{t^n}{n!}.$$

Now, by comparing both sides term by term, we get:

$$\sum_{k=0}^{n} \frac{B_k}{k!(n-k)!} = \frac{B_n}{n!}.$$

Multiplying both sides by $n!$, we get:

$$B_n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} B_k = \sum_{k=0}^{n-1} \binom{n}{k} B_k.$$

■

Before stating the main result of this section, we need the following Lemma:

Lemma 2 *Let* $f(t) = \sum_{n\geq 0} a_n t^n$, *then*

$$\text{If } f \text{ is even, then } a_{2n+1} = 0, \quad \text{for all } n \geq 0.$$

Proof. Since f is even, we have:

$$f(t) = f(-t).$$

So,

$$\sum_{n\geq 0} a_n t^n = \sum_{n\geq 0} a_n (-t)^n,$$

which gives:

$$\begin{aligned}\sum_{n\geq 0} a_n t^n - \sum_{n\geq 0} a_n (-t)^n &= \sum_{n\geq 0} a_n (t^n - (-t)^n) \\ &= \sum_{n\geq 0} a_n \left(1 - (-1)^n\right) t^n = 0.\end{aligned}$$

Now, notice that:

$$1 - (-1)^n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, we get:

$$\sum_{n\geq 0} \left(1 - (-1)^{2n+1}\right) a_{2n+1} t^{2n+1} = 2 \sum_{n\geq 0} a_{2n+1} t^{2n+1} = 0.$$

This implies that:

$$a_{2n+1} = 0.$$

■

Theorem 2.3 *The Bernoulli numbers B_n with odd indices are zero, that is:*

$$\forall n \geq 1, \quad B_{2n+1} = 0.$$

Proof. We use the previous Lemma 2, knowing that the function $\frac{t}{e^t - 1} + \frac{t}{2}$ is even, and

$$\frac{t}{e^t - 1} + \frac{t}{2} = B_0 + \left(B_1 + \frac{1}{2}\right) t + \sum_{n \geq 2} B_n \frac{t^n}{n!}.$$

So we get:

$$B_1 + \frac{1}{2} = 0 \quad \text{and} \quad B_{2n+1} = 0, \quad \forall n \geq 1.$$

■

2.2 Bernoulli Polynomials

In this part, we study the Bernoulli polynomials [14]. We define them and give some of their main properties. These polynomials are closely related to Bernoulli numbers and appear in many areas of mathematics, especially in calculus.

Definition 2.2 *The Bernoulli polynomials are a sequence of polynomials, $B_n(x)$, defined by the following power series expansion:*

$$\sum_{n \geq 0} \frac{B_n(x) t^n}{n!}. \tag{2.3}$$

The generating function for the Bernoulli polynomials is the generating function for the Bernoulli numbers multiplied by a term of e^{tx}. Our first observation of the Bernoulli polynomials is that the constant term of $B_n(x)$ is in fact B_n. If we set $x = 0$, then

$$\frac{te^{t(0)}}{e^t - 1} = \frac{t}{e^t - 1}$$

and so $B_n(0) = B_n$.
There are further connections: we can use the generating function for the Bernoulli numbers to develop a recurrence relation for the Bernoulli polynomials.

Theorem 2.4 *The Bernoulli polynomials, $B_n(x)$ satisfy the recurrence relation:*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{2.4}$$

Proof. We have:

$$\frac{te^{tx}}{e^t - 1} = \sum_{n \geq 0} B_n(x) \frac{t^n}{n!}.$$

Expanding e^{tx} using its Taylor series:

$$e^{tx} = \sum_{m \geq 0} \frac{(tx)^m}{m!}.$$

Since the generating function of Bernoulli numbers is:

$$\frac{t}{e^t - 1} = \sum_{k \geq 0} B_n \frac{t^n}{n!},$$

multiplying both series:

$$\begin{aligned}\sum_{n\geq 0} B_n \frac{t^n}{n!} \sum_{n\geq 0} \frac{(tx)^n}{n!} &= \sum_{n\geq 0}\sum_{k=0}^{n} B_k \frac{t^k}{k!}\frac{x^{n-k}t^{n-k}}{(n-k)!} \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k} B_k x^{n-k}\right)\frac{t^n}{n!} \\ &= \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}.\end{aligned}$$

Since power series are unique, the coefficients must be equal for each n, giving:

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k} B_k x^{n-k}.$$

■

Table of Bernoulli polynomials

According to the recurrence relation (2.4), we obtain the following Table 2.1:

n	$B_n(x)$
0	1
1	$x - \frac{1}{2}$
2	$x^2 - x + \frac{1}{6}$
3	$x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$
4	$x^4 - 2x^3 + x^2 - \frac{1}{30}$
5	$x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$
6	$x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$

Table 2.1 – Table of Bernoulli Polynomials

Theorem 2.5 *The Bernoulli polynomial $B_n(x)$ satisfies the identity:*

$$B_n'(x) = nB_{n-1}(x). \tag{2.5}$$

Proof.
We start with the generating function of Bernoulli polynomials:

$$\frac{te^{tx}}{e^t - 1} = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}.$$

Differentiating both sides with respect to x:

$$\frac{d}{dx}\left(\frac{te^{tx}}{e^t - 1}\right) = \sum_{n=0}^{\infty} B_n'(x)\frac{t^n}{n!}.$$

Since

$$\frac{d}{dx}e^{tx} = te^{tx},$$

we obtain:

$$\frac{t^2 e^{tx}}{e^t - 1} = t \sum_{n \geq 0} B_n(x)$$

$$\frac{t^n}{n!}$$

$$= \sum_{n \geq 0} B_n(x) \frac{t^{n+1}}{n!}$$

$$= \sum_{n \geq 0} n B_{n-1}(x) \frac{t^n}{n!}$$

$$= \sum_{n \geq 0} B'_n(x) \frac{t^n}{n!}.$$

Comparing coefficients, we conclude:

$$B'_n(x) = n B_{n-1}(x).$$

■

Theorem 2.6 *For all $n \geq 1$, the Bernoulli polynomials satisfy the identity:*

$$B_n(1) - B_n(0) = 0, \quad \textit{for odd } n. \tag{2.6}$$

Proof. We use the defining property of Bernoulli polynomials given by their generating function:

$$\frac{te^{tx}}{e^t - 1} = \sum_{n \geq 0} B_n(x) \frac{t^n}{n!}.$$

Setting $x = 1$ and $x = 0$, we get:

$$\frac{te^t}{e^t - 1} = \sum_{n \geq 0} B_n(1) \frac{t^n}{n!}.$$

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} B_n(0) \frac{t^n}{n!}.$$

By subtracting these two equations:

$$\frac{te^t}{e^t - 1} - \frac{t}{e^t - 1} = \sum_{n \geq 0} \left(B_n(1) - B_n(0) \right) \frac{t^n}{n!}.$$

Factoring out $\frac{t}{e^t - 1}$:

$$\frac{t(e^t - 1)}{e^t - 1} = t = \sum_{n \geq 0} \left(B_n(1) - B_n(0) \right) \frac{t^n}{n!}.$$

Observing the right-hand side, we see that only the $n = 1$ term remains nonzero, meaning:

$$B_n(1) - B_n(0) = 0, \quad \text{for } n \geq 1, \text{ and } n \text{ odd}.$$

■

Theorem 2.7 *For $n = 1, 2, \ldots$, the following identity holds:*

$$B_n(x + 1) - B_n(x) = n x^{n-1}. \tag{2.7}$$

Proof. To prove this identity, we start from the exponential generating function of the Bernoulli polynomials, which is given by:

$$\frac{te^{xt}}{e^t-1} = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}. \tag{2.8}$$

Now, consider replacing x with $x+1$ in the generating function:

$$\frac{te^{(x+1)t}}{e^t-1} = \sum_{n\geq 0} B_n(x+1)\frac{t^n}{n!}. \tag{2.9}$$

Using the identity $e^{(x+1)t} = e^{xt}e^t$, Equation (2.9) becomes:

$$\frac{te^{xt}e^t}{e^t-1} = e^t \cdot \frac{te^{xt}}{e^t-1}.$$

Now subtract Equation (2.8) from this expression:

$$e^t \cdot \frac{te^{xt}}{e^t-1} - \frac{te^{xt}}{e^t-1} = \left(e^t-1\right)\frac{te^{xt}}{e^t-1} = te^{xt}.$$

Therefore, we obtain:

$$\sum_{n\geq 0} \left(B_n(x+1) - B_n(x)\right)\frac{t^n}{n!} = te^{xt}.$$

On the other hand, we can expand the right-hand side as a power series:

$$te^{xt} = \sum_{n\geq 1} x^{n-1}\frac{t^n}{(n-1)!} = \sum_{n\geq 1} nx^{n-1}\frac{t^n}{n!}.$$

Now, by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the equality, we conclude that:

$$B_n(x+1) - B_n(x) = nx^{n-1} \quad \text{for all } n \geq 1.$$

■

Theorem 2.8 *For $n = 1, 2, \ldots$, the Bernoulli polynomials satisfy the following identity:*

$$B_n(1-x) = (-1)^n B_n(x). \tag{2.10}$$

Proof. We begin with the exponential generating function for Bernoulli polynomials:

$$\frac{te^{xt}}{e^t-1} = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}. \tag{2.11}$$

Let us replace x with $1-x$ in the generating function:

$$\frac{te^{(1-x)t}}{e^t-1} = \sum_{n\geq 0} B_n(1-x)\frac{t^n}{n!}. \tag{2.12}$$

Note that:

$$e^{(1-x)t} = e^t e^{-xt}.$$

So Equation (2.12) becomes:

$$\frac{te^t e^{-xt}}{e^t-1} = e^t\frac{te^{-xt}}{e^t-1}.$$

Now, we observe that:

$$\frac{te^{-xt}}{e^t-1} = \sum_{n\geq 0} B_n(-x)\frac{t^n}{n!}.$$

But this function is related to $B_n(x)$ via the identity:

$$B_n(-x) = (-1)^n B_n(x).$$

Thus,

$$\frac{te^{(1-x)t}}{e^t - 1} = e^t \sum_{n\geq 0} B_n(-x)\frac{t^n}{n!} = \sum_{n\geq 0}(-1)^n B_n(x)\frac{t^n}{n!}e^t.$$

Since both sides are generating functions for $B_n(1-x)$ and $(-1)^n B_n(x)$, we can equate the coefficients of $\frac{t^n}{n!}$, and we obtain:

$$B_n(1-x) = (-1)^n B_n(x) \quad \text{for all } n \geq 0.$$

■

Theorem 2.9 *For $n = 0, 1, 2, \ldots$, the even-indexed Bernoulli polynomials satisfy:*

$$B_{2n}(1-x) = B_{2n}(x). \tag{2.13}$$

Proof. We recall the general reflection identity for Bernoulli polynomials:

$$B_n(1-x) = (-1)^n B_n(x) \quad \text{for all } n \geq 0.$$

Now, let us consider the case where $n = 2k$ is even. Then:

$$B_{2k}(1-x) = (-1)^{2k} B_{2k}(x) = B_{2k}(x),$$

because $(-1)^{2k} = 1$ for any integer k. This proves that all even-indexed Bernoulli polynomials are symmetric with respect to the line $x = \frac{1}{2}$, and thus:

$$B_{2n}(1-x) = B_{2n}(x) \quad \text{for all } n \geq 0.$$

■

Property 2.2.1 *Let $B_n(t)$ be the Bernoulli polynomial of degree n, and let $x, y \in \mathbb{R}$. Then:*

$$\int_x^y B_n(t)\,dt = \frac{B_{n+1}(y) - B_{n+1}(x)}{n+1}. \tag{2.14}$$

2.3 Applications

2.3.1 Applications in Analysis

The Bernoulli numbers appear in the Taylor series expansion of the tangent functions (circular and hyperbolic), in the Euler-Maclaurin formula, as well as in expressions for certain values of the Riemann zeta function, etc [5, 14].

1. **Sums of Powers of Integers**
   The Bernoulli numbers first appeared in the sums of powers $S_k(n) = \sum_{i=1}^n i^k$, specifically in Faulhaber's formula as follows:
   For all $k \geq 0$ and $n \geq 1$, we have:

$$S_k(n) = \sum_{j=0}^{k} \binom{k}{j} B_j \frac{n^{k+1-j}}{k+1-j}.$$

2. **Taylor Series Expansions**
   The Taylor series expansions of the functions $\tan x$ and $\cos x$ around 0 are given by:

$$\tan x = \sum_{n=1}^{\infty} (-1)^{n-1}(2^{2n}-1)2^{2n}B_{2n}\frac{x^{2n-1}}{(2n)!}.$$

$$\cos x = \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^n 2^{2n} B_{2n}\frac{x^{2n-1}}{(2n)!}.$$

3. **The Euler-Maclaurin Formula**
   The Bernoulli numbers appear in the Euler-Maclaurin formula as follows:

$$\sum_{k=0}^{n-1} f(k) \approx \int_0^n f(x)\,dx - \frac{1}{2}(f(n)-f(0)) + \frac{1}{6}\cdot\frac{f'(n)-f'(0)}{2!} - \frac{1}{30}\cdot\frac{f^{(3)}(n)-f^{(3)}(0)}{4!} + \frac{1}{42}\cdot\frac{f^{(5)}(n)-f^{(5)}(0)}{6!} + \cdots$$

4. **Relation with the Riemann Zeta Function ζ**
   In 1735, Leonhard Euler determined the exact value of $\zeta(2)$ using the formula:

$$\zeta(2) = \sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

   He also proved that the Riemann zeta function at positive even integers is related to the Bernoulli numbers:

$$\zeta(2n) = \sum_{k\geq 1} \frac{1}{k^{2n}} = (2\pi)^{2n} \cdot \frac{(-1)^{n+1} B_{2n}}{2(2n)!}.$$

5. **Relation with the Euler-Zagier Multiple Zeta Function**
   Akiyama et *al.* *[3] defined an explicit formula for $\zeta_k(0,\cdots,0,-n)$, related to the Bernoulli numbers:*

$$\zeta_k(0,\cdots,0,-n) = -\frac{1}{(k-1)!(n+1)} \sum_{j=1}^{k} s(k,j) B_{n+j} + (-1)^k \delta_n, \tag{2.15}$$

   where δ_n is a function defined by:

$$\delta_n = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

2.3.2 Arithmetic Applications

Applications to Prime Numbers
This result was obtained in 1840 by Karl Georg Christian von Staudt and Thomas Clausen.

Theorem 2.10 (Von Staudt-Clausen Theorem) *Let $P \subset \mathbb{N}$ be the set of prime numbers. For $n = 1$ and for every even integer $n \geq 2$, B_n can be written as:*

$$B_n = -\sum_{p\in P,\ p-1|n} \frac{1}{p} + C_n, \quad (C_n \text{ is an integer}), \tag{2.16}$$

where $p-1 \mid n$ means that $p-1$ divides n.

This theorem states the following: If one adds to B_n the inverses $\frac{1}{p}$ of each prime number p such that $p-1$ divides n, one obtains an integer whenever $n = 1$ or n is a non-zero even number.
This property can also be expressed as: If $p-1 \mid 2m$, then $pB_{2m} \equiv p$. (The notation $a \equiv b \mod p$ means that p divides the numerator of $a-b$ but not its denominator.)
Moreover, this theorem allows an immediate characterization of the denominators of the Bernoulli numbers, given in the following result:

Theorem 2.11 *For even n, the denominator of B_n is the product of the prime numbers p such that $p-1 \mid n$.*

2.4 Bernoulli Polynomials of Order α

This section focuses on the Bernoulli polynomials of order α (called the generalized Bernoulli polynomials in [8]). We present their definition, and mention their basic properties. These polynomials generalize the classical Bernoulli polynomials and are useful in many mathematical problems [8, 25, 35, 36].

Definition 2.3 *The Bernoulli polynomials of order α $B_n^{\alpha}(x)$ are defined using the generating function below:*

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n\geq 0} \frac{B_n^{\alpha}(x)t^n}{n!}. \tag{2.17}$$

Using the generalized Stirling numbers $S_n^k(x)$ of the second kind, we obtain the following result related to the generating function:

Theorem 2.12 *The following identity holds:*

$$B_n^{\alpha}(x) = \sum_{k=0}^{n}(-1)^k \binom{n+k}{k}^{-1} \binom{n+\alpha}{n-k} \binom{\alpha+k-1}{k} S_{n+k}^k(x). \tag{2.18}$$

Proof. We proceed as in proof of the explicit formula of Srivastava and Todorov [[35], p. 511, Eq. (10)]:

$$\begin{aligned} B_n^{\alpha}(x) = &\sum_{k=0}^{n}(-1)^k \binom{\alpha+k-1}{k} \\ &\times \sum_{p=0}^{n-k} \binom{n}{p+k} \frac{(p+k)!}{(p+2k)!} x^{n-k-p} S_{p+2k}^k(0)k! \\ &\times \sum_{l=0}^{p} \binom{\alpha+k+l-1}{l}. \end{aligned}$$

Now, applying the identity:

$$\sum_{l=0}^{n-k} \binom{\alpha+k+l-1}{l} = \binom{n+\alpha}{n-k},$$

and rearranging terms, we get:

$$B_n^{\alpha}(x) = \sum_{k=0}^{n}(-1)^k \frac{\binom{\alpha+k-1}{k}\binom{n+\alpha}{n-k}}{\binom{n+k}{k}} \sum_{j=0}^{n+k} \binom{n+k}{j} x^{n+k-j} S_j^k(0).$$

Now, using (1.25), we get the desired result (2.18) asserted by Theorem 2.12. ∎

Remark 4 *If we set $x = 0$ in Theorem 2.12, we obtain a closed-form formula for the Bernoulli numbers B_n^{α} of order α [[36], p. 665, Eq. (3)].*

Remark 5 *By making the substitutions $x \to x+1$ and $\alpha \to x+1$ in the generating function (2.17), we get the Stirling polynomials $\mathfrak{G}_n(x)$, defined as:*

$$\left(\frac{t}{1-e^{-t}}\right)^{x+1} = \sum_{n=0}^{\infty} \mathfrak{G}_n(x)\frac{t^n}{n!},$$

which have the explicit expression:

$$\mathfrak{G}_n(x) = \sum_{k=0}^{n}(-1)^k \binom{n+k}{k}^{-1} \binom{n+x+1}{n-k} \binom{x+k}{k} S_{n+k}^k(x+1).$$

Remark 6 *If we choose* $x = \frac{r}{m}$ *and* $\alpha = -k$*, where* k *is a non-negative integer, in Equation* (2.18)*, we find:*

$$B_n^{-k}\left(\frac{r}{m}\right) = \sum_{j=0}^{n}(-1)^j \binom{n+j}{j}^{-1} \binom{n-k}{j-k} \binom{-k+j-1}{-k-1} S_{n+j}^{j}\left(\frac{r}{m}\right).$$

Also, using Equations (1.28) *and* (1.8)*, we get:*

$$B_n^{-k}\left(\frac{r}{m}\right) = \frac{1}{m^n}\binom{n+k}{k}^{-1} W_{m,r}(n+k,k).$$

This result was also proven by Mihoubi and Tiachachat [[25], Eq.8].

Remark 7 *By taking* $x = r$ *and* $\alpha = 1$ *in Theorem 2.12, we obtain:*

$$B_n(r) = \sum_{k=0}^{n}(-1)^k \binom{n+k}{k}^{-1} \binom{n+1}{k+1} S_r(n+k+r,k+r).$$

For $r = 0$*, this formula reduces to Gould's identity for the classical Bernoulli numbers* B_n*.*

3

The Generalized Euler Polynomials

Euler polynomials are fundamental tools in mathematical analysis, contributing to the study of various mathematical phenomena and applications. In this chapter, we focus on the generalized Euler polynomials, which represent an extension of the traditional form to include a broader range of mathematical cases. The definitions, properties, and key results related to these polynomials will be discussed, along with their applications in fields such as combinatorics and series theory [8, 14, 24, 29, 33, 34, 35].

3.1 Euler Numbers and Polynomials

In this part, we talk about Euler numbers and Euler polynomials [14]. We give their definitions and write some of their basic formulas. These numbers and polynomials are important in mathematics and appear in many algebra problems [14].

Definition 3.1 *The Euler numbers $\widetilde{E}_n$ are defined by the expansion:*

$$\frac{2}{e^t + e^{-t}} = \sum_{n\geq 0} \widetilde{E}_n \frac{t^n}{n!}. \tag{3.1}$$

From this, we have:

$$\widetilde{E}_0 = 1, \quad \widetilde{E}_2 = -1, \quad \widetilde{E}_4 = 5, \quad \widetilde{E}_6 = -61, \quad \widetilde{E}_8 = 1385, \ \textit{etc.}$$

Also, we observe that:

$$\widetilde{E}_{2k-1} = 0, \quad \textit{for all } k \geq 1.$$

Definition 3.2 *Euler polynomials $E_n(x)$ are defined by the generating function:*

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n\geq 0} E_n(x) \frac{t^n}{n!}. \tag{3.2}$$

Theorem 3.1 *The Euler polynomials can be expressed as:*

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\widetilde{E}_k}{2^k} \left(x - \frac{1}{2}\right)^{n-k}. \tag{3.3}$$

Proof. We start with the generating function:

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n\geq 0} E_n(x) \frac{t^n}{n!}.$$

$$\frac{2e^{tx}}{e^t+1} = \frac{2}{e^{\frac{t}{2}}+e^{-\frac{t}{2}}} e^{(x-\frac{1}{2})t}.$$

Note that:

$$\sum_{n\geq 0} E_n(x)\frac{t^n}{n!} = \left(\sum_{n\geq 0} \frac{\widetilde{E}_n}{2^n}\frac{t^n}{n!}\right)\left(\sum_{n\geq 0}(x-\frac{1}{2})^n\frac{t^n}{n!}\right),$$

which implies the equality

$$E_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{\widetilde{E}_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}.$$

■

For $x=\frac{1}{2}$, we get the equality

$$\widetilde{E}_n = 2^n E_n\left(\frac{1}{2}\right). \tag{3.4}$$

Theorem 3.2 *The derivative of the Euler polynomials $E_n(x)$ satisfies the following identity:*

$$E_n'(x) = nE_{n-1}(x). \tag{3.5}$$

Proof. We start from the generating function of the Euler polynomials:

$$\sum_{n\geq 0}\frac{E_n(x)}{n!}t^n = \frac{2e^{tx}}{e^t+1}.$$

Differentiating both sides with respect to x:

$$\frac{d}{dx}\left(\frac{2e^{tx}}{e^t+1}\right) = \sum_{n=0}^{\infty} E_n'(x)\frac{t^n}{n!}.$$

Since

$$\frac{d}{dx}e^{tx} = te^{tx},$$

we obtain

$$\frac{2te^{tx}}{e^t+1} = \sum_{n=0}^{\infty} E_n'(x)\frac{t^n}{n!}.$$

Shifting the index in the summation

$$\sum_{n\geq 0} E_n(x)\frac{t^{n+1}}{n!} = \sum_{n\geq 0} E_n'(x)\frac{t^n}{n!}.$$

Comparing coefficients, we conclude

$$E_n'(x) = nE_{n-1}(x).$$

■

Theorem 3.3 *Euler polynomials satisfy the identity:*

$$E_n(x+1)+E_n(x) = 2x^n. \tag{3.6}$$

Proof. We use the generating function of Euler polynomials (3.2):

$$\sum_{n\geq 0} E_n(x)\frac{t^n}{n!} = \frac{2e^{tx}}{e^t+1}.$$

Replace x by $x+1$, we get:

$$\sum_{n\geq 0} E_n(x+1)\frac{t^n}{n!} = \frac{2e^{t(x+1)}}{e^t+1} = e^t \cdot \frac{2e^{tx}}{e^t+1}.$$

Then,

$$\sum_{n\geq 0} E_n(x+1)\frac{t^n}{n!} + \sum_{n\geq 0} E_n(x)\frac{t^n}{n!} = \frac{2e^{tx}(e^t+1)}{e^t+1} = 2e^{tx}.$$

Expanding the right-hand side as a power series:

$$2e^{tx} = \sum_{n\geq 0} 2x^n \frac{t^n}{n!}.$$

Equating the coefficients of $\frac{t^n}{n!}$ gives:

$$E_n(x+1) + E_n(x) = 2x^n.$$

■

Theorem 3.4 *The Euler polynomials satisfy the symmetry identity:*

$$E_n(1-x) = (-1)^n E_n(x). \tag{3.7}$$

Proof. We begin with the generating function for Euler polynomials:

$$\sum_{n\geq 0} E_n(x)\frac{t^n}{n!} = \frac{2e^{tx}}{e^t+1}.$$

Substituting $1-x$ in place of x, we get:

$$\sum_{n\geq 0} E_n(1-x)\frac{t^n}{n!} = \frac{2e^{t(1-x)}}{e^t+1} = \frac{2e^{-tx}}{e^{-t}+1}.$$

Since $\frac{2e^{-tx}}{e^{-t}+1} = \sum_{n\geq 0} E_n(x)\frac{(-t)^n}{n!}$, we compare the two series and obtain:

$$E_n(1-x) = (-1)^n E_n(x).$$

■

3.2 Euler Polynomials of Order α

In this section, we study the Euler polynomials of order α (called the generalized Euler polynomials in [8]). We start by defining them and then show and main formulas. These polynomials are a generalization of the classical Euler polynomials and have interesting properties [8, 24, 29, 33, 35].

Definition 3.3 *The Euler polynomials of order α $E_n^\alpha(x)$ are defined by the following generating function:*

$$\left(\frac{2}{e^t+1}\right)^\alpha e^{xt} = \sum_{n\geq 0} \frac{E_n^\alpha(x)t^n}{n!}.$$

In this continuation of the work by Srivastava and Todorov [35], we aim to study explicit formulas for the *Euler polynomials of order α $E_n^\alpha(x)$*).
We focus especially on a family of modified Euler polynomials of order α. This study leads to a useful and interesting method to compute the Euler polynomials of order α based on a **three-term recurrence relation**.

To build this algorithm, we define an infinite matrix $\left[\mathbb{E}_{n,p}^\alpha(x)\right]$, where $n, p \geq 0$.

— The **first row** of the matrix is called the *initial sequence*, and it is defined by:

$$\mathbb{E}^{\alpha}_{0,p}(x) := 1$$

— The entries of the matrix satisfy the recurrence relation:

$$\mathbb{E}^{\alpha}_{n+1,p}(x) = (x+p)\mathbb{E}^{(\alpha)}_{n,p}(x) - \frac{1}{2}(p+\alpha)\mathbb{E}^{\alpha}_{n,p+1}(x).$$

— The **first column** of the matrix gives the Euler polynomials of order α:

$$\mathbb{E}^{\alpha}_{n}(x) := \mathbb{E}^{\alpha}_{n,0}(x).$$

Theorem 3.5 *The following relationship holds true:*

$$E^{\alpha}_{n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k}(\alpha)^{\overline{k}} S^k_n(x). \tag{3.8}$$

Proof. We aim to prove the following equation:

$$\begin{aligned}\sum_{n\geq 0}\sum_{k=0}^{n} \frac{(-1)^k}{2^k}(\alpha)^{\overline{k}} S^k_n(x)\frac{t^n}{n!} &= \sum_{n\geq 0} \frac{(-1)^k}{2^k}(\alpha)^{\overline{k}} \sum_{n=k}^{n} S^k_n(x)\frac{t^n}{n!} \\ &= e^{xt}\sum_{k\geq 0} \frac{(-1)^k}{2^k}(\alpha)^{\overline{k}}\frac{1}{k!}(e^x-1)^k.\end{aligned}$$

We begin by using identity (1.8). Consider the following double sum:

$$\begin{aligned}\sum_{n\geq 0}\sum_{k=0}^{n} \frac{(-1)^k}{2^k}(\alpha)^{\overline{k}} S^k_n(x)\frac{t^n}{n!} &= e^{xt}\sum_{k\geq 0}\binom{\alpha+k-1}{k}\left(\frac{1-e^x}{2}\right)^k \\ &= e^{xt}\left(1-\frac{1-e^x}{2}\right)^{-\alpha} \\ &= e^{xt}\left(\frac{2}{e^x+1}\right)^{\alpha} \\ &= \left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} \\ &= \sum_{n\geq 0} E^{\alpha}_n(x)\frac{t^n}{n!}.\end{aligned}$$

The assertion (3.8) of Theorem 3.5 now follows by comparing the coefficients of $\frac{t^n}{n!}$. The proof of Theorem 3.5 is thus completed.

$$E^{\alpha}_{n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k}(\alpha)^{\overline{k}} S^k_n(x).$$

■

In particular, for $x = 0$, we have:

$$E^{\alpha}_{n}(0) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k}k!\binom{\alpha+k-1}{k} S(n,k),$$

The special case of (3.8) when $x = \frac{r}{m}$ reduces immediately to a known result [[24], Theorem 3] . Moreover, upon setting $\alpha = 1$ in (3.8), we get the formula (3.5). In the year 1983, Srivastava et *al.* *[33] proved the following addition theorem for the Euler polynomials of order α:*

$$E_n^{\alpha+\gamma\beta}(x+\gamma y)=\sum_{k=0}^{n}\binom{n}{k}\frac{\gamma+n}{\gamma+k}E_k^{\alpha-k\beta}(x-ky)E_{n-k}^{k\beta+\gamma\beta}(ky+\gamma y)$$

For $\Re(\gamma)>0$. By setting $\gamma=1$, $y=0$, and $x,\beta\to p$, for all nonnegative integers p, in the addition theorem, we obtain:

$$E_n^{\alpha+p}(p)=\sum_{k=0}^{n}\binom{n+1}{k+1}E_k^{\alpha-kp}(p)E_{n-k}^{kp+p}(0).$$

Now, if we let

$$\mathbb{E}_{n,p}^{\alpha}(0):=E_n^{\alpha+p}(p),$$

we can deduce the following generating function:

$$\sum_{n\geq 0}\mathbb{E}_{n,p}^{\alpha}(0)\frac{t^n}{n!}=e^{pt}\left(\frac{2}{e^t+1}\right)^{\alpha+p},$$

and the following explicit formula:

$$\mathbb{E}_{n,p}^{\alpha}(0)=\sum_{k=0}^{n}(-1)^k\frac{2^k}{k!}\binom{\alpha+k+p-1}{k}S_p(n+p,k+p).$$

In Theorem 3.5 below, we obtain an explicit formula via the Stirling number $s(n,k)$ of the first kind.

Theorem 3.6 *For a nonnegative integer p,*

$$\mathbb{E}_{n,p}^{\alpha}(0)=\frac{(-1)^p2^p}{p!}\binom{\alpha+p-1}{p}\sum_{k=0}^{p}s(p,k)E_{n+k}^{\alpha}(0),\tag{3.9}$$

Proof. *The explicit formula (3.9) can be derived from a known result in [[29], p. 681, Corollary 1] for the Stirling transform upon specializing the initial sequence $\left\{a^{\overline{p}}\right\}_{p=0}^{\infty}$ as follows:*

$$a^{\overline{p}}=\frac{(-1)^p p!}{2p}\binom{\alpha+p-1}{p}$$

■

We now propose an algorithm which is based on a three-term recurrence relation for calculating the generalized Euler numbers $E_n^{\alpha}(0)$.

Theorem 3.7 *The $\mathbb{E}_{n,p}^{\alpha}(0)$ satisfies the following three-term recurrence relation:*

$$\mathbb{E}_{n+1,p}^{\alpha}(0)=p\mathbb{E}_{n,p}^{\alpha}(0)-\frac{1}{2}(\alpha+p)\mathbb{E}_{n,p+1}^{\alpha}(0)\tag{3.10}$$

with the initial sequence given by:

$$\mathbb{E}_{0,p}^{\alpha}(0)=1.$$

Proof. *We have the following expression for $\mathbb{E}_{n,p+1}^{\alpha}(0)$:*

$$\begin{aligned}E_{n,p+1}^{\alpha}(0)&=\frac{(-1)^{p+1}2^{p+1}}{(p+1)!}\binom{\alpha+p}{p+1}^{-1}\sum_{k=0}^{p+1}s(p+1,k)E_{n+k}^{\alpha}(0)\\&=\frac{(-1)^{p+1}2^{p+})}{(p+1)!}\binom{\alpha+p}{p+1}^{-1}\sum_{k=1}^{p+1}\left[s(p,k-1)-ps(p,k)\right]E_{n+k}^{\alpha}(0).\end{aligned}$$

After some manipulations, we find that:

$$\mathbb{E}^{\alpha}_{n,p+1}(0) = -\frac{2}{p+\alpha}\mathbb{E}^{\alpha}_{n+1,p}(0) + \frac{2p}{p+\alpha}\mathbb{E}^{\alpha}_{n,p}(0).$$

■

Finally, we consider the polynomials $\mathbb{E}^{\alpha}_{n,p}(x)$ obtained by first setting $\gamma = 1$, $x, \beta \to p$, and then interchanging x and y in the addition theorem. We have:

$$\mathbb{E}^{\alpha}_{n,p}(x) := E^{\alpha+p}_{n}(p+x).$$

By using the generating function, we obtain:

$$\sum_{n\geq 0} E^{\alpha}_{n,p}(x)\frac{t^n}{n!} = e^{(p+x)t}\left(\frac{2}{e^t+1}\right)^{\alpha+p}.$$

So that explicitly, we have

$$\begin{aligned}\mathbb{E}^{\alpha}_{n,p}(x) &= \sum_{k=0}^{n}\binom{n}{k}\mathbb{E}^{\alpha}_{n,p}(0)x^{n-k} \\ &= \sum_{k=0}^{n}\frac{(-1)^k}{2^k}k!\binom{\alpha+p+k-1}{k}S^k_n(x+p).\end{aligned}$$

Theorem 3.8 *The polynomials $\mathbb{E}^{\alpha}_{n,p}(x)$ satisfy the following three-term recurrence relation:*

$$\mathbb{E}^{\alpha}_{n+1,p}(x) = (p+x)\mathbb{E}^{\alpha}_{n,p}(x) - \frac{1}{2}(p+\alpha)E^{\alpha}_{n,p+1}(x)$$

with the initial sequence given by:

$$\mathbb{E}^{\alpha}_{0,p}(x) = 1.$$

Proof. *It is readily seen that:*

$$\begin{aligned}\frac{d}{dx}\mathbb{E}^{\alpha}_{n,p}(x) &= \sum_{k=0}^{n}(n-k)\binom{n}{k}E^{\alpha}_{k,p}(0)x^{n-k-1} \\ &= \frac{n}{x}\sum_{k=0}^{n}\binom{n}{k}\mathbb{E}^{\alpha}_{k,p}(0)x^{n-k} - \frac{n}{x}\sum_{k=0}^{n-1}\binom{n-1}{k}\mathbb{E}^{\alpha}_{k,p}(0)x^{n-k-1}.\end{aligned}$$

Now, we get:

$$x\frac{d}{dx}\mathbb{E}^{\alpha}_{n,p}(x) = n\mathbb{E}^{\alpha}_{n,p}(x) - p\sum_{k=0}^{n-1}\binom{n-1}{k}\mathbb{E}^{\alpha}_{k,p}(0)x^{n-k-1} + \frac{1}{2}n(p+\alpha)\sum_{k=0}^{n-1}\binom{n-1}{k}\mathbb{E}^{\alpha}_{k,p+1}(0)x^{n-k-1}.$$

After some rearrangement, we find that:

$$xn\mathbb{E}^{\alpha}_{n-1,p}(x) = n\mathbb{E}^{\alpha}_{n,p}(x) - pn\mathbb{E}^{\alpha}_{n-1,p}(x) + \frac{1}{2}(p+\alpha)nE^{\alpha}_{n-1,p+1}(x).$$

This completes the proof.

■

3.2.1 Relation with the Euler Polynomials

The Euler polynomials $E_n(x)$ can be computed from the Bernoulli polynomials using the following relation:

$$E_n(x) = \frac{1}{2}\sum_{k=0}^{n-1}\binom{n}{k}\left(2B_k(x) - E_k(x)\right).$$

The Euler polynomials satisfy the following integral identities:

$$\int_0^1 E_n(x)E_m(x)\,dx = (-1)^n\frac{4(2^{n+m+2}-1)n!\ m!}{(n+m+2)!}B_{n+m+2}.$$

4

The Generalized Laguerre Polynomials

Laguerre polynomials are special functions that appear in many areas of mathematics and physics. They are orthogonal and defined as solutions to a second-order differential equation and are especially important in quantum mechanics. In this chapter, we study the classical Laguerre polynomials, their generalization with a parameter α, and the associated Laguerre polynomials. We present their definitions, main formulas, and recurrence relations [1, 6, 14, 19, 21, 31].

4.1 The Classical Laguerre Polynomials

In this section, we present the classical Laguerre polynomials $L_n(x)$, focusing on their main definitions and important properties. We will explore their generating functions, recurrence relations, and key identities that characterize these polynomials. These properties form the foundation for further generalizations and applications studied in later sections [6, 21].

Definition 4.1 *The classical Laguerre polynomials $L_n(x)$ [21] are commonly introduced through the following generating function:*

$$\frac{1}{1-t}\exp\left(-\frac{xt}{1-t}\right)=\sum_{n\geq 0}L_n(x)\,t^n, \tag{4.1}$$

and can be computed explicitly by the Rodrigue's formula [6]

$$L_n(x)=\frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right). \tag{4.2}$$

The explicit form can be rewritten differently, as shown in this theorem

Theorem 4.1

$$L_n(x)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^k}{k!}, \tag{4.3}$$

with

$$L_0(x)=1.$$

Proof. *Using (4.1) and applying the Leibniz formula, we get*

$$\begin{aligned}
\frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right) &= \frac{e^x}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{d^k}{dx^k}e^{-x}\right)\left(\frac{d^{n-k}}{dx^{n-k}}x^n\right)\\
&= \frac{e^x}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k e^{-x}(n)(n-1)(n-2)...(n-(n-k-1))x^k\\
&= \frac{e^x}{n!}e^{-x}\sum_{k=0}^{n}\binom{n}{k}(-1)^k n^{\underline{n-k}}x^k\\
&= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{n!}{(n-n+k)!}x^k\\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{n!}{k!n!}x^k\\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^k}{k!}.
\end{aligned}$$

This completes the proof of Theorem 4.1.

4.1.1 Some explicit expressions of Laguerre polynomials

We have an explicit series for $L_n(x)$ given by Relation (4.3). For the first Laguerre polynomials, this gives:

$$\begin{aligned}
L_0(x) &= 1.\\
L_1(x) &= -x+1.\\
L_2(x) &= \frac{1}{2!}(x^2-4x+2).\\
L_3(x) &= \frac{1}{3!}(-x^3+9x^2-18x+6).\\
L_4(x) &= \frac{1}{4!}(x^4-16x^3+72x^2-96x+24).
\end{aligned}$$

4.1.2 Laguerre Differential Equation and Its Solutions

The Laguerre differential equation is written in the form:

$$xy''(x)+(1-x)y'(x)+ny(x)=0, \tag{4.4}$$

where $n=0,1,2,\ldots$
The solutions of this equation are the Laguerre polynomials, which are expressed by the Formula (4.2).

Now, we give some expression of Laguerre polynomials for specifiс values.

Theorem 4.2 *We have:*

$$L_n(0)=1. \tag{4.5}$$

$$L_n'(0)=-n. \tag{4.6}$$

$$L_n''(0)=\frac{1}{2}n(n-1). \tag{4.7}$$

$$\int_0^\infty e^{-tx}L_n(x)\,dx=\frac{1}{t}\left(1-\frac{1}{t}\right)^n. \tag{4.8}$$

Proof. *Using the Definition 4.1, we prove (4.5).*
For $x = 0$

$$\begin{aligned}\frac{e^0}{1-t} &= \frac{1}{1-t}\\ &= \sum_{n\geq 0} L_n(0)t^n\\ &= \sum_{n\geq 0} t^n.\end{aligned}$$

Then

$$L_n(0) = 1.$$

We now prove the Formula (4.6).
The expression of $L_n(x)$ *satisfies the Laguerre Equation (4.4), therefore, we have:*

$$x\frac{d^2}{dx^2}L_n(x) + (1-x)\frac{d}{dx}L_n(x) + nL_n(x) = 0.$$

By setting $x = 0$ *in this equation and using (4.5), we obtain*

$$\begin{aligned}L_n'(0) + nL_n(0) &= L_n'(0) + n\\ &= 0.\end{aligned}$$

Thus

$$L_n'(0) = -n.$$

Will be proved (4.7).
In the expression of $L_n(x)$ *defined by the Relation (4.3), only the term in* x *contributes to* $L_n''(0)$.
Therefor, we have:

$$L_n''(0) = \sum_{k=2}^{n}(-1)^k\binom{n}{k}k(k-1)\frac{x^{k-2}}{k!}.$$

For $k = 2, \ldots, n$. *We have:*

$$L_n''(x) = (-1)^2\binom{n}{2}2\frac{x^0}{2!} + (-1)^3\binom{n}{3}6\frac{x^1}{3!} + \cdots + (-1)^n\binom{n}{n}n(n-1)\frac{x^{n-2}}{n!}.$$

$$\begin{aligned}L_n''(0) &= (-1)^2\binom{n}{2}2(1)\frac{1}{2!} + 0\\ &= \frac{n!}{4(n-2)!}2\\ &= \frac{n(n-1)}{2}.\end{aligned}$$

Let us show that (4.8).
Replacing the expression of $L_n(x)$ *defined by (4.3) into the integral, we get:*

$$\begin{aligned}\int_0^\infty e^{-tx}L_n(x)\,dx &= \int_0^\infty e^{-tx}\sum_{k=0}^{n}(-1)^k\frac{n!}{(k!)^2(n-k)!}x^k\,dx\\ &= \sum_{k=0}^{n}(-1)^k\frac{n!}{(k!)^2(n-k)!}\int_0^\infty x^k e^{-tx}\,dx.\end{aligned}$$

Set $z = tx$, so that $dz = t\,dx$. Then:

$$\begin{aligned}
\int_0^\infty e^{-tx} L_n(x)\,dx &= \sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2(n-k)!} \int_0^\infty \left(\frac{z}{t}\right)^k e^{-z} \frac{dz}{t} \\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2(n-k)!} \frac{1}{t^{k+1}} \int_0^\infty z^k e^{-z}\,dz \\
&= \sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2(n-k)!} \frac{1}{t^{k+1}} \Gamma(k+1) \\
&= \frac{1}{t} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \left(\frac{1}{t}\right)^k \qquad (\Gamma(k+1) = k!) \\
&= \frac{1}{t} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{-1}{t}\right)^k \\
&= \frac{1}{t} \left[\binom{n}{0} + \binom{n}{1} \left(\frac{-1}{t}\right)^1 + \binom{n}{2} \left(\frac{-1}{t}\right)^2 + \cdots + \binom{n}{n} \left(\frac{-1}{t}\right)^n \right] \\
&= \frac{1}{t} \left[1 - \frac{1}{t}\right]^n .
\end{aligned}$$

■

4.2 The Laguerre Polynomials of Order α

In this section, we study the generalized Laguerre polynomials of order α, denoted by $L_n^\alpha(x)$ [21]. We present their main definitions, key properties, and recurrence relations. These polynomials extend the classical Laguerre polynomials by including the parameter α, which allows for a broader range of applications in analysis and physics .

Definition 4.2 *Let $\alpha > -1$ be a real parameter. The* ***Laguerre polynomial of order*** *α, denoted as $L_n^\alpha(x)$ is defined by the following generating function:*

$$\frac{1}{(1-t)^{\alpha+1}} \exp\left(-\frac{xt}{1-t}\right) = \sum_{n\geq 0} L_n^\alpha(x)\, t^n, \quad |t| < 1, \tag{4.9}$$

or by the Rodrigue's formula:

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^{n+\alpha}\right), \quad n \geq 0. \tag{4.10}$$

In particular*, for $\alpha = 0$ in (4.9) and (4.10), we have the classical Laguerre polynomials as follows:*

$$\begin{aligned}
L_n^0(x) &= \frac{x^0 e^x}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^{n+0}\right) \\
&= \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n) \\
&= L_n(x).
\end{aligned}$$

In the following theorem, we will see a new way to write $L_n^\alpha(x)$.

Theorem 4.3 *The Laguerre polynomial of order α, $L_n^\alpha(x)$ can be expressed as:*

$$L_n^\alpha(x) = \sum_{k=0}^{n} (-1)^k \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+k+1)} \frac{x^k}{k!(n-k)!}. \tag{4.11}$$

Proof. *We prove the Theorem 4.3 using the Rodrigue's Formula (4.10).*

$$\begin{aligned}
L_n^{\alpha}(x) &= \frac{x^{-\alpha}e^{x}}{n!}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}) \\
&= \frac{x^{-\alpha}e^{x}}{n!}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{d^n}{dx^n}e^{-x}\right)\left(\frac{d^{n-k}}{dx^{n-k}}x^{n+\alpha}\right) \\
&= \frac{x^{-\alpha}e^{x}}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k e^{-x}(n+\alpha)(n+\alpha-1)...(n+\alpha-(n-k-1))x^{k+\alpha} \\
&= \frac{x^{-\alpha}e^{x}}{n!}e^{-x}\sum_{k=0}^{n}\binom{n}{k}(-1)^k(n+\alpha)^{\underline{n}}x^{k+\alpha} \\
&= \frac{x^{-\alpha}}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{(n+\alpha)!}{(\alpha+k)}x^{k+\alpha} \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{(n+\alpha)!}{(\alpha+k)!}\frac{x^{k+\alpha-\alpha}}{n!} \\
&= \sum_{k=0}^{n}(-1)^k\frac{(n+\alpha)!}{(\alpha+k)!}\frac{x^k}{n!}\frac{n!}{k!(n-k)!} \\
&= \sum_{k=0}^{n}(-1)^k\frac{(n+\alpha)!}{(\alpha+k)!}\frac{x^k}{k!(n-k)!} \\
&= \sum_{k=0}^{n}(-1)^k\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+k+1)}\frac{x^k}{k!(n-k)!}.
\end{aligned}$$

■

In partuclar*, for $\alpha = 0$, we obtain the Theorem 4.1:*

$$\begin{aligned}
L_n^{0}(x) &= \sum_{k=0}^{n}(-1)^k\frac{\Gamma(0+n+1)}{\Gamma(0+k+1)}\frac{x^k}{k!(n-k)!} \\
&= \sum_{k=0}^{n}(-1)^k\frac{\Gamma(n+1)}{\Gamma(k+1)}\frac{x^k}{k!(n-k)!} \\
&= \sum_{k=0}^{n}\frac{n!}{k!}\frac{x^k}{k!(n-k)!} \\
&= \sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{k!} \\
&= L_n(x).
\end{aligned}$$

Theorem 4.4 *The Laguerre polynomial of order α, $L_n^{\alpha}(x)$ is given by:*

$$L_n^{\alpha}(x) = \frac{(\alpha+1)^{\overline{n}}}{n!}\sum_{k=0}^{n}\frac{(-n)^{\overline{k}}}{(\alpha+1)^{\overline{k}}}\frac{x^k}{k!}. \tag{4.12}$$

Proof. *We prove the Theorem 4.4 using the Theorem 4.3.*

$$\begin{aligned}
L_n^{\alpha}(x) &= \sum_{k=0}^{n}(-1)^k \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+k+1)} \frac{x^k}{k!(n-k)!} \\
&= \sum_{k=0}^{n}(-1)^k \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+k+1)} \frac{x^k}{n!} \frac{n!}{k!(n-k)!} \\
&= \sum_{k=0}^{n} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)} \frac{(-1)^k (n)^{\underline{k}} x^k}{k!n!} \\
&= \sum_{k=0}^{n} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+k+1)} \frac{(-n)^{\overline{k}}}{n!} \frac{x^k}{k!} \\
&= \sum_{k=0}^{n} \frac{(\alpha+1)^{\overline{n}}}{n!} \sum_{k=0}^{n} \frac{(-n)^{\overline{k}}}{(\alpha+1)^{\overline{k}}} \frac{x^k}{k!} \\
&= \frac{(\alpha+1)^{\overline{n}}}{n!} \sum_{k=0}^{n} \frac{(-n)^{\overline{k}}}{(\alpha+1)^{\overline{k}}} \frac{x^k}{k!}.
\end{aligned}$$

This completes the proof of Theorem 4.4.

Theorem 4.5 *The Laguerre polynomial of order α, $L_n^{\alpha}(x)$ can be expressed in terms of the confluent hypergeometric function as follows:*

$$L_n^{\alpha}(x) = \frac{(\alpha+1)^{\overline{n}}}{n!} {}_1F_1(-n;\ \alpha+1;\ x). \tag{4.13}$$

Proof. *We will prove this Theorem 4.5 using Theorem 4.4.*

$$\begin{aligned}
L_n^{\alpha}(x) &= \frac{(\alpha+1)^{\overline{n}}}{n!} \sum_{k=0}^{n} \frac{(-n)^{\overline{k}}}{(\alpha+1)^{\overline{k}}} \frac{x^k}{k!}, \\
&= \frac{(\alpha+1)^{\overline{n}}}{n!} {}_1F_1(-n;\ \alpha+1;\ x),
\end{aligned}$$

where

$${}_1F_1(-n;\ \alpha+1;\ x) = \sum_{k=0}^{n} \frac{(-n)^{\overline{k}}}{(\alpha+1)^{\overline{k}}} \frac{x^k}{k!}.$$

The values of k do not go to infinity when the first (upper) parameter in the hypergeometric function ${}_1F_1$ is a negative integer. For example, when the parameter is $-n$ with $n \in \mathbb{N}$, we have:

$$(-n)^{\overline{k}} = (-1)^k \frac{n!}{(n-k)!},$$

which becomes zero for $k > n$. Therefore, the summation terminates at $k = n$, and the series becomes finite. ∎

Recurrence Relation for Laguerre Polynomials of Order α

Theorem 4.6 *For $n \geq 1$, the Laguerre polynomials of order α satisfy the identity:*

$$\frac{d}{dx} L_n^{\alpha}(x) - \frac{d}{dx} L_{n-1}^{\alpha}(x) + L_{n-1}^{\alpha}(x) = 0. \tag{4.14}$$

Proof.
We begin by recalling the recurrence relation for the derivatives of generalized Laguerre polynomials:

$$\frac{d}{dx} L_n^{\alpha}(x) = -L_{n-1}^{\alpha+1}(x), \tag{4.15}$$

and

$$\frac{d}{dx}L_{n-1}^{\alpha}(x) = -L_{n-2}^{\alpha+1}(x). \tag{4.16}$$

From(4.15) and (4.16), we obtain:

$$\begin{aligned}
\frac{d}{dx}L_n^{\alpha}(x) - \frac{d}{dx}L_{n-1}^{\alpha}(x) &= -\sum_{k=0}^{n-1}(-1)^{k+1}\frac{(n+\alpha)!}{(n-1-k)!(k+1+\alpha)!k!}x^k \\
&\quad + \sum_{k=0}^{n-2}(-1)^{k+1}\frac{(n-1+\alpha)!}{(n-2-k)!(k+1+\alpha)!k!}x^k \\
&= (-1)^n\frac{(n+\alpha)!}{0!(\alpha+n)!(n-1)!}x^{n-1} \\
&\quad + \sum_{k=0}^{n-2}(-1)^{k+1}\left(\frac{(n-1+\alpha)!}{(n-2-k)!(k+1+\alpha)!k!} - \frac{(n+\alpha)!}{(n-1-k)!(k+1+\alpha)!k!}\right)x^k \\
&= (-1)^n\frac{1}{(n-1)!}x^{n-1} \\
&\quad + \sum_{k=0}^{n-2}(-1)^{k+1}\frac{(n-1+\alpha)!}{(k+1+\alpha)!k!}\left(\frac{1}{(n-2-k)!} - \frac{n+\alpha}{(n-1-k)!}\right)x^k \\
&= (-1)^n\frac{1}{(n-1)!}x^{n-1} + \sum_{k=0}^{n-2}(-1)^{k+1}\frac{(n-1+\alpha)!}{(n-1-k)!(k+\alpha)!k!}x^k.
\end{aligned}$$

$$\frac{d}{dx}L_n^{\alpha}(x) - \frac{d}{dx}L_{n-1}^{\alpha}(x) = -\sum_{k=0}^{n-1}(-1)^{k}\frac{(n-1+\alpha)!}{(n-1-k)!(k+\alpha)!k!}x^k. \tag{4.17}$$

We substitute (4.17) into the Equation(4.14), we get

$$\begin{aligned}
\frac{d}{dx}L_n^{\alpha}(x) - \frac{d}{dx}L_{n-1}^{\alpha}(x) + L_{n-1}^{\alpha}(x) &= -\sum_{k=0}^{n-1}(-1)^{k}\frac{(n-1+\alpha)!}{(n-1-k)!(k+\alpha)!k!}x^k \\
&\quad + \sum_{k=0}^{n-1}(-1)^{k}\frac{(n-1+\alpha)!}{(n-1-k)!(k+\alpha)!k!}x^k \\
&= 0.
\end{aligned}$$

■

Theorem 4.7 *Let $L_n^{\alpha}(x)$ be the Laguerre polynomial of order α. Then it satisfies the second-order linear differential equation:*

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{4.18}$$

Proof.
Let us denote:

$$\begin{aligned}
y = L_n^{\alpha}(x) &= \sum_{k=0}^{n}(-1)^{k}\frac{(n+\alpha)!}{(n-k)!(k+\alpha)!k!}x^k. \\
y' = \frac{d}{dx}L_n^{\alpha}(x) &= \sum_{k=1}^{n}(-1)^{k}\frac{(n+\alpha)!}{(n-k)!(k+\alpha)!(k-1)!}x^{k-1}. \\
y'' = \frac{d^2}{dx^2}L_n^{\alpha}(x) &= \sum_{k=2}^{n}(-1)^{k}\frac{(n+\alpha)!}{(n-k)!(k+\alpha)!(k-2)!}x^{k-2}.
\end{aligned}$$

$$xy'' = \sum_{k=2}^{n}(-1)^k \frac{(n+\alpha)!}{(n-k)!(k+\alpha)!(k-2)!}x^{k-1}.$$

$$(\alpha+1-x)y' = (\alpha+1)\sum_{k=1}^{n}(-1)^k \frac{(n+\alpha)!}{(n-k)!(k+\alpha)!(k-1)!}x^{k-1} - \sum_{k=1}^{n}(-1)^k \frac{(n+\alpha)!}{(n-k)!(k+\alpha)!(k-1)!}x^k.$$

$$ny = n\sum_{k=0}^{n}(-1)^k \frac{(n+\alpha)!}{(n-k)!(k+\alpha)!k!}x^k.$$

We substitute into the Equation(4.18), we get:

$$\begin{aligned}
xy'' + (\alpha+1-x)y' + ny &= \sum_{k=1}^{n-1}(-1)^{k+1}\frac{(n+\alpha)!}{(n-1-k)!(k+1+\alpha)!(k-1)!}x^k \\
&\quad + \sum_{k=0}^{n-1}(-1)^{k+1}\frac{(n+\alpha)!(\alpha+1)}{(n-1-k)!(k+1+\alpha)!k!}x^k - \sum_{k=1}^{n}(-1)^k\frac{(n+\alpha)!}{(n-k)!(k+\alpha)!(k-1)!}x^k \\
&\quad + \sum_{k=0}^{n}(-1)^k\frac{n(n+\alpha)!}{(n-k)!(k+\alpha)!k!}x^k \\
&= \sum_{k=0}^{n}(-1)^k\left[\frac{-(n+\alpha)!}{(n-1-k)!(k+1+\alpha)!k!} + \frac{-(n+\alpha)!(\alpha+1)}{(n-1-k)!(k+1+\alpha)!k!}\right. \\
&\quad \left. + \frac{(n+\alpha)!}{(n-k)!(k+\alpha)!(k-1)!} + \frac{n(n+\alpha)!}{(n-k)!(k+\alpha)!k!}\right]x^k \\
&= \sum_{k=0}^{n}(-1)^k\left[\frac{-(n+\alpha)!(\alpha+2)}{(n-1-k)!(k+1+\alpha)!k!} + \frac{(n+\alpha)!(k+n)}{(n-k)!(k+\alpha)!k!}\right]x^k \\
&= \sum_{k=0}^{n}(-1)^k\frac{(n+\alpha)!}{(n-k)!(k+\alpha)!k!}\left[-(\alpha+2)(n-k)(k+\alpha+1)+(k+n)\right]x^k \\
&= 0
\end{aligned}$$

■

4.3 The Associated Laguerre Polynomials

In this section, we study the associated Laguerre polynomials, denoted by $L_n^m(x)$, where the parameter m replaces the order α from the generalized Laguerre polynomials. We focus on their definitions, important properties, and recurrence relations, emphasizing their role in extending the classical Laguerre polynomials and their applications in physics and mathematics [1, 19, 20, 21, 31].

Definition 4.3 *The associated Laguerre polynomials $L_n^m(x)$ [1] are defined by*

$$L_n^m(x) = (-1)^m \frac{d^m}{dx^m}L_{n+m}(x), \tag{4.19}$$

or by generating function:

$$\frac{1}{(1-t)^{m+1}}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n\geq 0}L_n^m(x)\,t^n, \quad |t|<1. \tag{4.20}$$

In particular*, for $\alpha = m$ in (4.10), we have the Rodrigue's Formula for the associated Laguerre polynomials:*

$$L_n^m(x) = \frac{x^{-m}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+m}\right), \quad n\geq 0. \tag{4.21}$$

The associated Laguerre polynomials [19] can be computed recursively by

$$(n+1)\,L_{n+1}^{m-1}(x) = (m+n)\,L_n^{m-1}(x) - xL_n^m(x).$$

These polynomials can be arranged [19] into an infinite matrix $\mathcal{L} = (L_n^m(x))_{n,m\geq 0}$, where the first column, $L_n^0(x) = L_n(x)$, corresponds to the classical Laguerre polynomials.

$$\mathcal{L} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1-x & 2-x & 3-x & 4-x & \cdots \\ \frac{1}{2}x^2-2x+1 & \frac{1}{2}x^2-3x+3 & \frac{1}{2}x^2-4x+6 & \vdots & \\ -\frac{1}{6}x^3+\frac{3}{2}x^2-3x+1 & -\frac{1}{6}x^3+2x^2-6x+4 & \vdots & & \\ \frac{1}{24}x^4-\frac{2}{3}x^3+3x^2-4x+1 & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ L_n^0(x) := L_n & L_n^1(x) & L_n^2(x) & & \end{pmatrix}. \tag{4.22}$$

The next theorem useful recurrence relations involving $L_n^m(x)$.

Theorem 4.8

$$\begin{aligned}
&L_{n-1}^m(x) + L_n^{m-1}(x) = L_n^m(x).\\
&(n+1)L_{n+1}^m(x) = (2n+m+1-x)L_n^m(x) - (n+1)L_{n-m}^m(x).\\
&x\frac{d}{dx}L_n^m(x) = nL_n^m(x) - (n+m)L_{n-1}^m(x).\\
&\frac{d}{dx}L_n^m(x) = -\sum_{k=0}^{n-1} L_k^m(x).\\
&\frac{d}{dx}L_n^m(x) = -L_{n-1}^{m-1}(x).\\
&L_n^{m+1}(x) = \sum_{k=0}^{n} L_k^m(x).
\end{aligned}$$

4.4 The Associated Polynomials

The goal of this part is to study the properties of the associated polynomials given in [19], denoted by $a_{n,m}(x)$, which satisfy the following conditions:

$$a_{n,m}(x) = (-1)^m \frac{d^m}{dx^m} a_{n+m}(x) \tag{4.23}$$

and

$$a_{n+1,m}(x) = \left(1 + \frac{m}{n+1}\right) a_{n,m}(x) - \frac{x}{n+1} a_{n,m+1}(x). \tag{4.24}$$

Where $(a_n(x))_{n\geq 0}$ be any sequence of polynomials of the form

$$a_n(x) = \sum_{k=0}^{n} \binom{n}{k} \lambda_k x^k,$$

and $(\lambda_k)_{k\geq 0}$ is a sequence that depends on the index k.

Remark 8 *It is important to observe that if $a_{n,m}(x)$ satisfies the condition (4.23), then the initial sequence α_m is independent of x.*

More precisely, we construct an infinite matrix $\mathcal{M} := (a_{n,m}(x))_{n,m\geq 0}$, where the first row, $\alpha_m := a_{0,m}$, represents the initial sequence and the first column, $\beta_n := a_{n,0}(x)$, represents the final sequence. Specifically, for a nonzero complex number x, we aim to explore the three-term recurrence relation

(4.24), or more generally, to extend the following Laguerre transformation.
Recall that the Laguerre transform [31] of a sequence $(\alpha_n)_{n\geq 0}$ is the sequence $(\beta_n(x))_{n\geq 0}$, given by:

$$\beta_n\ (x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!} \alpha_k \tag{4.25}$$

and the inverse transform is

$$\alpha_n = n! \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{-n} \beta_k\,(x)\,.$$

4.4.1 The Generalized Laguerre transform

Theorem 4.9 *Given the initial sequence $\alpha_m := a_{0,m}$, we define the matrix $\mathcal{M}$ associated with the initial sequence through the recurrence relation (4.24), and then*

1. *The elements of the matrix $\mathcal{M}$ are expressed as*

$$a_{n,m}\,(x) = \sum_{k=0}^{n} (-1)^k \binom{n+m}{n-k} \frac{x^k}{k!} \alpha_{m+k}. \tag{4.26}$$

2. *Assume that the initial sequence α_{m+s} has the exponential generating function $A_s(t) = \sum_{k\geq 0} \alpha_{k+s} \frac{t^k}{k!}$. Then, the sequence $(a_{n,s}(x))_{n\geq 0}$ corresponding to the s-th column of the matrix $\mathcal{M}$ has the ordinary generating function $B_s(t;x) = \sum_{n\geq 0} a_{n,s}(x) t^n$, which is given by*

$$B_s\,(t;x) = \frac{1}{(1-t)^{s+1}} A_s \left(-\frac{xt}{1-t}\right). \tag{4.27}$$

Proof. *From*

$$a_{n,m}\,(x) = (-1)^m \frac{d^m}{dx^m} a_{n+m}\,(x)\,.$$

By applying the Laguerre transform (4.25), we obtain

$$\begin{aligned} a_{n,m}\,(x) &= (-1)^m \frac{d^m}{dx^m} \sum_{k=0}^{n+m} \binom{n+m}{k} \frac{(-x)^k}{k!} \alpha_k \\ &= (-1)^m \sum_{k=0}^{n+m} (-1)^k \binom{n+m}{k} \frac{x^{k-m}}{(k-m)!} \alpha_k. \end{aligned}$$

After rearranging the terms, we arrive at (4.26).
Now by using (4.26), we obtain

$$\begin{aligned} B_s\,(t;x) &= \sum_{n\geq 0} \left(\sum_{k=0}^{n} (-1)^k \binom{n+s}{n-k} \frac{x^k}{k!} \alpha_{s+k} \right) t^n \\ &= \sum_{k\geq 0} (-1)^k \frac{x^k}{k!} \alpha_{k+s} \sum_{n\geq 0} \binom{n+s}{n-k} t^n. \end{aligned}$$

Since

$$\sum_{n\geq 0} \binom{n+s}{n-k} t^n = \frac{t^k}{(1-t)^{s+k+1}},$$

we get

$$B_s\,(t;x) = \frac{1}{(1-t)^{s+1}} \sum_{k\geq 0} \frac{1}{k!} \alpha_{k+s} \left(-\frac{xt}{1-t}\right)^k.$$

This clearly concludes the proof of the theorem. ∎

Theorem 4.10 *Given the final sequence $\beta_n := a_{n,0}(x)$, we define the matrix $\mathcal{M}$ associated with this final sequence as follows:*

$$a_{n,m+1}(x) = \left(\frac{n+m+1}{x}\right) a_{n,m}(x) - \left(\frac{n+1}{x}\right) a_{n+1,m}(x), \tag{4.28}$$

then

1. *The elements of the matrix $\mathcal{M}$ are described by:*

$$a_{n,m}(x) = x^{-m} \sum_{k=0}^{m} (-1)^k \frac{(n+m)!}{n!} \binom{m}{m-k} \beta_{n+k}(x). \tag{4.29}$$

2. *Let's assume that the final sequence β_{n+s} has the following ordinary generating function:*

$$\mathcal{B}_s(t) = \sum_{k\geq 0} \beta_{k+s} t^k.$$

Then, the sequence $(a_{s,m}(x))_{m\geq 0}$ corresponding to the s-th row of the matrix $\mathcal{M}$ has an exponential generating function $\mathcal{A}_s(t;x)$:

$$\mathcal{A}_s(t;x) = \sum_{m\geq 0} a_{s,m}(x) \frac{t^m}{m!},$$

which is given by:

$$\mathcal{A}_s(t;x) = \frac{(-1)^s x}{s!(x-t)^{s+1}} \left((x-t)^2 \frac{d}{dt}\right)^s \left(\left(\frac{t}{t-x}\right)^s \mathcal{B}_s\left(\frac{t}{t-x}\right)\right). \tag{4.30}$$

Proof. *The verification of the expression (4.29) can be carried out through induction on m. Starting with the given expression in (4.29), we have:*

$$\mathcal{A}_s(t;x) = \sum_{m\geq 0} \left(x^{-m} \sum_{k=0}^{m} (-1)^k \frac{(s+m)!}{s!} \binom{m}{m-k} \beta_{s+k}\right) \frac{t^m}{m!}.$$

We can then rewrite this as:

$$\mathcal{A}_s(t;x) = \sum_{k\geq 0} (-1)^k \beta_{s+k} \sum_{m\geq 0} \frac{(s+m)!}{s!m!} \binom{m}{m-k} \left(\frac{t}{x}\right)^m.$$

Which simplifies further to:

$$\mathcal{A}_s(t;x) = \sum_{k\geq 0} (-1)^k \beta_{s+k} \binom{s+k}{s} \sum_{m\geq 0} \binom{s+m}{s+k} \left(\frac{t}{x}\right)^m.$$

Using the known identity:

$$\sum_{m\geq 0} \binom{s+m}{s+k} \left(\frac{t}{x}\right)^m = \frac{x^{s+1}}{(x-t)^{k+s+1}} t^k$$

and applying:

$$\binom{k+s}{s} t^k = \frac{1}{s!} \frac{d^s}{dt^s} t^{k+s}.$$

We can now derive:

$$\mathcal{A}_s(t;x) = \frac{x^{s+1}}{(x-t)^{s+1}} \sum_{k\geq 0} \beta_{s+k} \binom{s+k}{s} \left(-\frac{t}{x-t}\right)^k$$

This can be rewritten as:

$$\mathcal{A}_s(t;x) = \frac{x^{s+1}}{(x-t)^{s+1}} \sum_{k\geq 0} \beta_{s+k} \frac{1}{(-x)^s s!} \left((x-t)^2 \frac{d}{dt}\right)^s \left(\frac{t}{t-x}\right)^{k+s}.$$

Finally, we arrive at:

$$\mathcal{A}_s(t;x) = \frac{(-1)^s x}{s!(x-t)^{s+1}} \left((x-t)^2 \frac{d}{dt}\right)^s \left[\left(\frac{t}{t-x}\right)^s \mathcal{B}_s\left(\frac{t}{t-x}\right)\right],$$

which completes the proof.

■

Corollary 1 *For $n, m \geq 0$, we have*

$$\sum_{k=0}^{n} (-1)^k \binom{n+m}{n-k} \frac{x^k}{k!} \alpha_{m+k} = x^{-m} \sum_{k=0}^{m} (-1)^k \frac{(n+m)!}{n!} \binom{m}{k} \beta_{n+k}\,. \tag{4.31}$$

Remark 9 *The identity in (4.31) represents the generalized Laguerre transform. It can be simplified in two special cases: For $m = 0$, it reduces to the standard Laguerre transform of the sequence α_n. For $n = 0$, it becomes the inverse Laguerre transform of the sequence β_m.*

Example 4.1 By setting the initial sequence $a_{0,m} = 1$ in (4.24), we obtain (4.22). Since $A_m(t) = \exp(t)$, it follows from (4.27) that the final sequence has an ordinary generating function given by:

$$\sum_{n\geq 0} L_n^m(x)\, t^n = \frac{1}{(1-t)^{m+1}} \exp\left(-\frac{xt}{1-t}\right). \tag{4.32}$$

Now, from Corollary 1, we have

$$\begin{aligned} L_n^m(x) &= \sum_{k=0}^{n} (-1)^k \binom{n+m}{n-k} \frac{x^k}{k!} \\ &= x^{-m} \sum_{k=0}^{m} (-1)^k \frac{(n+m)!}{n!} \binom{m}{k} L_{n+k}(x)\,. \end{aligned}$$

Theorem 4.11 *We consider a matrix $\mathcal{L} = \left(L_n^m(x)\right)_{n,m\geq 0}$ of order $n+1$ derived from (4.24) with the initial sequence $a_{0,m} = 1$. Then, we have:*

$$\det \mathcal{L} = 1,$$

where det *denotes the determinant.*

Proof. *Let C_j represent the j-th column of $\mathcal{L}$. By repeatedly applying the following operation for $1 \leq r \leq n$, we obtain:*

$$C_{n-j} \longleftarrow C_{n-j} - C_{n-j-1} \quad (0 \leq j \leq n-r)\,.$$

We obtain a lower triangular matrix where the diagonal elements are all equal to 1.

$$\mathcal{L} = \left((-1)^k \binom{i}{k+j} \frac{x^k}{k!}\right)_{0\leq i,j\leq n}.$$

Since adding a linear combination of the other columns to any given column does not affect the value of the determinant, we obtain the desired result.

■

4.5 Applications

Laguerre polynomials appear in many physical and numerical contexts, especially in quantum mechanics (solutions of Schrödinger's equation for the hydrogen atom) and in numerical analysis for evaluating improper integrals using Gauss-Laguerre quadrature [17, 27].

4.5.1 Gauss-Laguerre Quadrature

For integrals of the form

$$\int_0^\infty e^{-x} f(x)\, dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

where x_i are the roots of $L_n(x)$, and w_i are the corresponding weights.

4.5.2 Radial Wave Function in the Hydrogen Atom

The radial solution of Schrödinger's equation for the hydrogen atom is expressed as:

$$R_{n,l}(r) = N_{n,l}\, r^l e^{-\frac{r}{na_0}} L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right),$$

where $L_n^l(x)$ is the associated Laguerre polynomial. and a_0 is the Bohr radius (the natural unit of length for the hydrogen atom), $N_{n,l}$ is the normalization constant ensuring the wavefunction is normalized, and $R_{n,l}(r)$ is the radial wavefunction describing the probability amplitude of finding the electron at distance r.

4.5.3 Orthogonality of Laguerre Polynomials

Theorem 4.12

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\, dx = \delta_{nm}.$$

4.5.4 On the Associated Fibonacci Polynomials

The classical Fibonacci numbers F_n [20] are defined by the ordinary generating function:

$$\frac{t}{1-t-t^2} = \sum_{n\geq 0} F_n t^n. \tag{4.33}$$

The general term can be written as follows:

$$F_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right), \tag{4.34}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.
By setting the initial sequence as $a_{0,m} := -m!F_m$ in (4.24), we obtain the following matrix:

$$\mathcal{F} = \begin{pmatrix} 0 & -1 & -2 & -12 & -72 \\ x & 2x-2 & 12x-6 & 72x-48 & \cdots \\ -x^2+2x & -6x^2+6x-3 & -36x^2+48x-12 & \vdots & \\ 2x^3-3x^2+3x & 12x^3-24x^2+12x-4 & \vdots & & \\ -3x^4+8x^3-6x^2+4x & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ \mathbb{F}_n^0(x) := \mathbb{F}_n(x) & \mathbb{F}_n^1(x) & \mathbb{F}_n^2(x) & & \end{pmatrix}.$$

Let the final sequence be denoted as $a_{n,0} := \mathbb{F}_n(x)$. Now, we can introduce the m-associated Fibonacci polynomials.

Definition 4.4 The m-associated Fibonacci polynomials $\mathbb{F}_n^m(x)$ are defined by the generating function:

$$\frac{m!}{\alpha-\beta}\left(\frac{\beta^m}{(1-t+\beta xt)^{m+1}}-\frac{\alpha^m}{(1-t+\alpha xt)^{m+1}}\right)=\sum_{n\geq 0}\mathbb{F}_n^m(x)\,t^n.$$

For $x=1$ and $m=0$, we obtain the classical Fibonacci numbers given by Equation (4.33). Now, based on Corollary 1, we derive the following explicit formulas:

Theorem 4.13 *For $m\geq 0$*

$$\begin{aligned}\mathbb{F}_n^m(x)&=(-1)^m\frac{d^m}{dx^m}\mathbb{F}_{n+m}(x)\\&=\sum_{k=0}^{n}(-1)^{k+1}\binom{n+m}{n-k}\frac{x^k}{k!}(m+k)!F_{m+k}\\&=x^{-m}\sum_{k=0}^{m}(-1)^k\frac{(n+m)!}{n!}\binom{m}{k}\mathbb{F}_{n+k}(x).\end{aligned}$$

Next, we examine the case when $m=0$. It is straightforward to demonstrate that $\mathbb{F}_n(x)$ satisfies the recurrence relation:

$$\mathbb{F}_{n+2}(x)=(2-x)\,\mathbb{F}_{n+1}(x)+\left(x^2+x-1\right)\mathbb{F}_n(x),\tag{4.35}$$

The initial conditions for $\mathbb{F}_n(x)$ are given by $\mathbb{F}_0(x)=0$ and $\mathbb{F}_1(x)=x$. Using Equation (4.34), we derive the Binet's formula for $\mathbb{F}_n(x)$.

$$\mathbb{F}_n(x)=\frac{1}{\sqrt{5}}\left(\beta^n(x)-\alpha^n(x)\right),$$

Let $\alpha(x)=1-\alpha x$ and $\beta(x)=1-\beta x$ represent the roots of the characteristic equation of Equation (4.35).

4.5.5 Associated Laguerre Polynomials

The solution of the following differential equation is the associated Laguerre polynomials:

$$x\frac{d^2y}{dx^2}+(m+1-x)\frac{dy}{dx}+ny=0.$$

4.5.6 Orthogonality Relation of the Associated Laguerre Polynomials

Theorem 4.14

$$\int_0^\infty e^{-x}x^mL_n^m(x)L_s^m(x)\,dx=\frac{(n+m)!}{n!}\,\delta_{ns}.$$

Conclusion

Conclusion

In this work, we studied some special and generalized polynomials, such as the Bernoulli, Euler, and Laguerre polynomials, with a focus on their generating properties, recurrence relations, and representations using special numbers like Stirling and Whitney numbers.
Throughout the chapters, we aimed to unify the structural properties of these families within a common mathematical framework and highlighted different approaches to analyze them, whether through series expansions, recurrence relations, or matrix representations.
This study can serve as a foundational step for further research into other types of generalized polynomials or for exploring their potential applications in fields such as functional analysis, differential equations, or combinatorics.

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