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Thème

THE METHOD OF COEFFICIENTS

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Abstract

In this work, based on various articles, we have studied the method of coefficients, which involves simplifying complex identities transforming them into simpler onces, and this is done by using coefficents, and we seen specific appliction of the method of coefficient which is Riordan arrays, allows for the computation of a wide range of combinatorial sums in a uniform and often straightforward manner.

Keywords: Generating function, coefficient extraction, binomial coefficients, identities, Stirling number, Riordan arrays.

Notations

$\mathbb{C}$	The field of complex numbers.
$\mathbb{R}$	The field of real numbers.
$\mathbb{Z}$	The set of entire numbers.
$\mathbb{N}$	The set of natural numbers.
$\mathbb{K}[[t]]$	The ring of formal power series in variable t over a field $\mathbb{K}$.
[n]	The set of $\{1, 2, \ldots, n\}$.
$[t^n]\, f(t)$	The coefficient extraction operator of$f\,(t)$.
$t^{\overline{n}}$	The rising factorial.
$t^{\underline{n}}$	The falling factorial.
$\delta_{n,k}$	The Kronecker delta symbol.
$s(n,k)$	Signed Stirling numbers of the first kind.
$\mid s(n,k) \mid$	The unsigned Stirling numbers of the first kind.
$\binom{n}{k}$	The binomial coefficient.
$\begin{bmatrix} n \\ k \end{bmatrix}$	Unsigned Stirling numbers of the first kind .
$\left\{ {n \atop k} \right\}$	Stirling numbers of the second kind.
$f'(t) = \dfrac{df}{dt}$	The first order derivative of a function.
$\lfloor t \rfloor$	The lower integer part of t.
C_n	The Catalan number.
H_n	The Harmonic number.

Introduction

Introduction

In their book "Mathematics for the Analysis of Algorithms," Greene and Knuth demonstrate a method for determining the closed form of the sum denoted as

$$S_m = \sum_{k=0}^{m} \binom{m}{k} \left(-\frac{1}{2}\right)^k \binom{2k}{k}. \tag{1}$$

They achieve this by employing specific transformations on generating functions and extracting coefficients, as proposed by Ira Gessel.

Egorychev's method is particularly elegant when avoiding the complications of integral representations in proofs. An intriguing instance ([5], p. 28) is his derivation of the Grosswald identity, expressed as follows :

$$\sum_{k=0}^{n-r} (-2)^{-k} \binom{n}{r+k} \binom{n+r+k}{k} = (-1)^{(n-1)/2} 2^{r-n} \binom{n}{\frac{n-r}{r}}. \tag{2}$$

The objective of this thesis is to present the method of coefficients. From our perspective, it can be described roughly as follows: when faced with the task of proving an identity or evaluating an expression (finding a closed or asymptotic formula), instead of directly working with the identity or expression, we seek the generating functions associated with the quantities involved in the problem. We manipulate these functions to obtain a single expression by extracting the coefficient of the relevant power t^n. This provides the solution.

Generating functions have become a popular approach for solving combinatorial problems, especially those in algorithm analysis (see, for example, D. E. Knuth [10]) or (see [1, 2, 7, 8, 9, 15, 16, 20, 21, 22, 23]). Three books by I. P. Goulden and D. M. Jackson, R. Stanley, and H. S. Wilf, offer a clear exposition of this concept. Notably, Wilf explores the snake oil method, closely related to the method of coefficients.

Less well-known is the "coefficient of" functionals, whose notation $[t^n]f(t)$ has only become standard in the past decade or so (for a critical review, see Egorychev ([5], p. 41)). The interpretation of $[t^n]f(t)$ has also posed challenges.

We have studied the article [11], which aims to demonstrate how a simple set of rules from Egorychev ([5], sec. 1.2) can solve various combinatorial problems that can be expressed using generating functions. We begin by mentioning the basic notations (formal power series, generating function, binomail coefficients, Stirling numbers, ...) in chapter one, while we have used the Method of coefficient technique that involves using coefficients to simplify complex identities, making them easier to understand and work with, we give some examples. Furthermore, we have observed a particular application of the coefficient method known as Riordan arrays.

This thesis offers a wealth of examples viewed from a slightly different perspective. We hope that this shift in viewpoint will be instructive for the general reader and especially interesting to those involved in combinatorics and algorithm analysis.

1

Basics notions

In this chapter, we present some basic elements which will be useful for the rest, including a review of rising and falling factorials, the generating function, the explicit form, the recurrence relation, Stirling numbers and their combinatorial properties. For more precise and detailed concepts the reader can refer to [3, 4, 6, 12, 13, 17, 19].

Preliminairies

1.1 Rising factorial and falling factorial

Definition 1.1.1 *Let t be a real or complex number and n a positive integer, we define the rising factorial of t of order n, denoted as $t^{\overline{n}}$, by :*

$$t^{\overline{n}} = t(t+1)\dots(t+n-1), \text{ if } n \geqslant 1 \text{ and } t^{\overline{0}} = 1$$

and the falling factorial of t of order n, denoted as $t^{\underline{n}}$, by :

$$t^{\underline{n}} = t(t-1)\dots(t-n+1), \text{ if } n \geqslant 1 \text{ and } t^{\underline{0}} = 1.$$

The relation between the rising factorial and the falling factorial is given by :

$$t^{\overline{n}} = (-1)^n(-t)^{\underline{n}}.$$

1.2 Binomail coefficients

Definition 1.2.1 *Let n and k be two non-negative integers. The binomial coefficient of k of order n, denoted as $\binom{n}{k}$, is given by :*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!} \quad (0 \leq k \leq n), \tag{1.1}$$

with $\binom{n}{k} = 0$ for $k > n$, and $\binom{n}{k} = 1$ for $k = 0$ or $k = n$.

We can also define the binomial coefficient in combinatorics by :

Definition 1.2.2 *The binomial coefficient $\binom{n}{k}$ is the number of ways to choose a subset with k elements from a set of n elements.*

Example 1.1
Replacing n with 6 and k with 3 in (1.1), we obtain :

$$\binom{4}{2} = \frac{4!}{(4-2)!2!} = 6.$$

In the set $[4] = \{1, 2, 3, 4\}$, *there are six ways to choose two elements :*

$$\{1, 2\}\,;\ \{1, 3\}\,;\ \{1, 4\}\,;\ \{2, 3\}\,;\ \{2, 4\}\,;\ \{3, 4\}.$$

As a result, $\binom{4}{2}$*=6.*

Newton's binomial formula

Let t and y be two real numbers and n a non-negative integer. So

$$(t+y)^n = \sum_{k=0}^{n} \binom{n}{k} t^k y^{n-k}. \tag{1.2}$$

Pascal's recurrence relation

The binomial coefficients satisfying the following relation :

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad \text{for every} \quad n \geq 1. \tag{1.3}$$

Pascal's triangle

From the recurrence relation (1.3), we can construct Pascal's triangle

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Table 1.1: Pascal's triangle numbers

Property 1.2.1 *The binomial coefficients satisfy nearly an infinite number of identities, including :*

1. *The absorption identity*

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}. \tag{1.4}$$

2. *The symmetry*

$$\binom{n}{k} = \binom{n}{n-k}, \text{ with } n \geqslant k \geqslant 0. \tag{1.5}$$

3. *The top summation*

$$\binom{n+1}{k+1} = \sum_{j=0}^{n} \binom{k}{j}, \text{ with } n, k \geqslant 0. \tag{1.6}$$

4. *The parallel summation*

$$\binom{m+n+1}{n} = \sum_{n \geqslant j} \binom{m+j}{j}. \tag{1.7}$$

5. The trinomial transformation

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}. \tag{1.8}$$

6. The Vandermonde convolution

$$\binom{n+m}{k} = \sum_{j=0}^{n} \binom{n}{j}\binom{m}{k-j}. \tag{1.9}$$

1.2.1 Generalization of binomial coefficients

A generalization of the binomial coefficient $\binom{n}{k}$ consists in replacing n by a complex number t, and to define $\binom{t}{k}$ by posing, $t \in \mathbb{C}$ and $k \in \mathbb{N}$ by :

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!} = \frac{t^{\underline{k}}}{k!} \tag{1.10}$$

and having almost the same properties as $\binom{n}{k}$. One has :

$$\binom{-k}{n} = \binom{k+n-1}{n}(-1)^n. \tag{1.11}$$

1.3 The Kronecker symbol

The Kronecker Symbol is given by :

$$\delta_{n,k} = \begin{cases} 1 & \text{if} \quad n = k \\ 0 & \text{Otherwise.} \end{cases}$$

1.4 Formal power series

The field $\mathbb{K}$ will be always the field of real numbers $\mathbb{R}$, or the field of the complex numbers $\mathbb{C}$, with a characteristic of 0. The notation $\mathbb{K}[[X]]$ denotes the set of all formal power series of the form

$$A(X) \in \mathbb{K}[[X]] : A(X) = \sum_{k=0}^{\infty} a_k X^k, \quad a_k \in \mathbb{K}.$$

We define K as $\mathbb{K}[[t]]$, which denotes the ring of formal power series with coefficients in K, using the indeterminate t. Suppose $f(t) = \sum_{k=0}^{\infty} f_k t^k$ belongs to K, and r is the smallest integer for which $f_r \neq 0$. In such a case, r is referred to as the order of $f(t)$. The collection of formal power series with an order of r is denoted as K_r. Specifically, K_0 represents the set of invertible formal power series, which implies that for a series $f(t)$ in K, there exists another series $f^{-1}(t)$ in K such that $f(t)f^{-1}(t) = 1$. The ring K is an integral domain, ensuring the well-definedness of its quotient field denoted as L. The elements of L are known as formal Laurent series.

1.4.1 Operations on formal power series

We define some operations on $\mathbb{K}[[X]]$.

Definition 1.4.1 *(Addition)*
Let f and g be tow formal power series noted by f.p.s :

$$f(X) = \sum_{n=0}^{\infty} a_n X^n, g(X) = \sum_{n=0}^{\infty} b_n X^n,$$

then the power series addition is defined by :

$$c(X) = f(X) + g(X) = \sum_{n=0}^{\infty} c_n X^n, \tag{1.12}$$

where

$$c_n = a_n + b_n, \quad n \geqslant 0.$$

Definition 1.4.2 *(Multiplication)*
Power series multiplication is defined by :

$$d(X) = a(X) \cdot b(X) = \sum_{n=0}^{\infty} d_n X^n, \tag{1.13}$$

where

$$d_n = a_0 b_n + \cdots + a_n b_0, \quad n \geqslant 0.$$

Definition 1.4.3 *(Multiplication by scalar)*
Let $f \in \mathbb{K}[[X]]$ and $\lambda \in \mathbb{R}$, then

$$\lambda f(X) = \lambda \left(\sum_{n=0}^{\infty} a_n X^n \right) = \sum_{n=0}^{\infty} \lambda a_n X^n. \tag{1.14}$$

Definition 1.4.4 *(Composition)*
Let $f(X) = \sum_{k=0}^{\infty} a_n X^n$ and $g(X) = \sum_{k=0}^{\infty} b_n X^n$ be elements of $\mathbb{K}[[X]]$ with $\omega(g) > 0$. We denote $f(X)$

$$f(X) = a_0 + a_1 X + a_2 X^2 + \cdots$$

and we similarly denote $g(X)$

$$g(X) = b_1 X + b_2 X^2 + \cdots$$

We define the composition of $f(X)$ composed with $g(X)$, denoted $f \circ g$, to be the formal power series $h(X)$ in $\mathbb{K}[[X]]$ such that

$$\begin{aligned} h(X) &= f \circ \ g \\ &= f(g(X)) \\ &= a_1(g(X)) + a_2(g(X))^2 + \cdots \\ &= a_1(b_1 X + b_2 X^2 + \cdots) + a_2(b_1 X + b_2 X^2 + \cdots)^2 + \cdots \\ &= \sum_{n=1}^{\infty} c_n X^n, \end{aligned}$$

where

$$c_n = \sum_{k \in \mathbb{N}, j_1 + \cdots + j_k = n} a_k b_{j_1} b_{j_2} \ldots b_{j_k}.$$

Definition 1.4.5 *(Derivative)*
Let $f(X) = \sum_{n=0}^{\infty} a_n X^n$; by definition, the derived series $f'(X)$ is given by the formula

$$f'(X) = \sum_{n=1}^{\infty} n a_n X^{n-1}. \tag{1.15}$$

1.5 Generating function

Definition 1.5.1 *Let $(a_n)_{n\geq 0}$ be a sequence belonging to $\mathbb{K}$. The ordinary generating function (O.G.F) (or the formal power series) of the sequence $(a_n)_{n\geq 0}$ is the series*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots$$

Another type of generating functions is

Definition 1.5.2 *The Exponential generating function (E.G.F)*

$$\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} = a_0 + a_1 \frac{t}{1!} + a_2 \frac{t^2}{2!} + \cdots$$

Definition 1.5.3 *The notation $\mathbb{K}[[t]]$ represents the set of all generating functions over the field $\mathbb{K}$.*

Example 1.2 *Consider the following generating function :*

$$\sum_{n=0}^{\infty} a^n t^n = 1 + at + (at)^2 + \cdots + (at)^n + \cdots = \frac{1}{1-at}, \ \ a \in \mathbb{C}.$$

Another example deduced from the previous one, by putting $a = 3$, is

$$1 + 3t + 9t^2 + \cdots + 3^n t^n + \cdots = \frac{1}{1-3t}.$$

Example 1.3 *The simplest Exponential generating function is certainly*

$$e^{at} = \sum_{n=0}^{\infty} a^n \frac{t^n}{n!}.$$

1.5.1 Operations on generating functions

Let $f, g, h \in \mathbb{K}[[t]]$. Let us write

$$f := f(t) = \sum_{n=0}^{\infty} a_n t^n, g := g(t) = \sum_{n=0}^{\infty} b_n t^n, h := h(t) = \sum_{n=0}^{\infty} c_n t^n.$$

We define some operations on $\mathbb{K}[[t]]$.

Definition 1.5.4 *(Equality)*
The generating functions f and g are equal equivalence to $a_n = b_n, \forall n \in \mathbb{N}$

Definition 1.5.5 *(Multiplication by a scalar)*
Let $f \in \mathbb{K}[[t]]$ and $\lambda \in \mathbb{K}$, then

$$\lambda f(t) = \lambda \left(\sum_{n=0}^{\infty} a_n t^n \right) = \sum_{n=0}^{\infty} \lambda a_n t^n. \tag{1.16}$$

Definition 1.5.6 *(Sum)*
The sum $(f + g)$ of f and g, is defined by

$$f(t) + g(t) = \sum_{n=0}^{\infty} (a_n + b_n) t^n. \tag{1.17}$$

Example 1.4 *Consider the two generating functions :*

$$f(t) = \sum_{n=0}^{\infty}(2)^n t^n = 1 + 2t + 4t^2 + 8t^3 + \cdots = \frac{1}{1-2t},$$
$$g(t) = \sum_{n=0}^{\infty}(-2)^n t^n = 1 - 2t + 4t^2 - 8t^3 + \cdots = \frac{1}{1+2t},$$

the sum of f and g is then :

$$f(t) + g(t) = 2 + 8t^2 + 36t^4 + \cdots = \frac{1}{1-2t} + \frac{1}{1+2t},$$

after an easy calculation, we obtain :

$$f(t) + g(t) = 2\left(\sum_{n=0}^{\infty}(4)^n t^{2n}\right).$$

Definition 1.5.7 *The subtraction of f and g is defined by :*

$$f(t) - g(t) = \sum_{n=0}^{\infty}(a_n - b_n)t^n. \tag{1.18}$$

Example 1.5 *Let g be defined by :*

$$\begin{aligned} g(x) &= 1 + 3t + 6t^2 + 10t^3 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}t^n \\ &= \frac{1}{(1-t)^3}, \end{aligned}$$

and h defined by

$$\begin{aligned} h(t) &= 1 + 4t + 9t^2 + 16t^3 + 25t^4 + \cdots \\ &= \sum_{n=0}^{\infty}(1+n)^2 t^n \\ &= \frac{(1+t)}{(1-t)^3}. \end{aligned}$$

We have

$$h(t) - g(t) = 0 + t + 3t^2 + 6t^3 + \cdots = \sum_{n=0}^{\infty}(1+n)t^n = \frac{t}{(1-t)^3}.$$

Definition 1.5.8 *(Product)*
The product of f and g is defined by

$$f(t)\cdot g(t) = \left(\sum_{n=0}^{\infty}a_n t^n\right)\left(\sum_{n=0}^{\infty}b_n t^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_k b_{n-k}\right)t^n. \tag{1.19}$$

Example 1.6 *Let the two generating functions be given by*

$$f(t) = \sum_{n=0}^{\infty}3nt^n \quad \text{and} \quad g(t) = \sum_{n=0}^{\infty}6t^n.$$

The product of these two generating functions is then :

$$\begin{aligned} h(t) = f(t) \cdot g(t) &= 3(0 + 1t + 2t^2 + \cdots) \times (1 + 6t + 36t^2 + \cdots) \\ &= (0 \times 1)t^0 + (0 \times 6 + 3 \times 1)t + (0 \times 36 + 3 \times 6 + 6 \times 1)t^2 + \cdots \\ &= 3t + 24t^2 + 153t^3 + \cdots \\ &= \frac{3}{(1-t)^2(1-5t)}. \end{aligned}$$

Definition 1.5.9 *Shifting (right)*

Shifting (right) a generating function by k terms means to multiply it by t^k, we then obtain a generating function, such that $a_i = 0$ for $0 \leq i \leq k-1$.

For example if

$$f(t) = \sum_{n=0}^{\infty} a_n t^n,$$

then the shifted series by k terms is

$$\hat{f}(t) = t^k f(t) = \sum_{n=0}^{\infty} a_n t^{n+k}. \tag{1.20}$$

1.5.2 Important operations with generating functions

Theorem 1.1 *Generating functions are closed under multiplication and addition.*

Theorem 1.2 *The multiplication of generating functions is distributive with respect to the addition that is :*

$$f(t) \cdot (g(t) + h(t)) = f(t) \cdot g(t) + f(t) \cdot h(t) = (g(t) + h(t)) \cdot f(t).$$

Identities and inverses

The set of the generating functions, also have identity elements. First we want to find $I(t)$ such that :

$$f(t) + I(t) = f(t) = I(t) + f(t).$$

Here, obviously, $I(t) = 0$. For the product the identity element $I(t)$ is such that

$$f(t)I(t) = I(t)f(t) = f(t).$$

Here, obviously, $I(t) = 1$.

Example 1.7

$$\begin{aligned} f(t) &= -(1-t) - \frac{(1-t)^2}{2} - \frac{(1-t)^3}{3} - \cdots = \sum_{n=1}^{\infty} -\frac{(1-t)^n}{n}. \\ g(t) &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t. \end{aligned}$$

We have

$$\begin{aligned} f(g(t)) &= -\frac{\left(1 - \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\right)}{1} - \frac{\left(1 - \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\right)^2}{2} - \cdots \\ &= \ln(e^t) \\ &= t. \end{aligned}$$

Let us define the inverse of f, if it exists by g such that

$$f(t) \cdot g(t) = g(t) \cdot f(t) = 1.$$

The element g is denoted by f^{-1}.

Theoreme 1.3 *The generating function*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n,$$

has an inverse if and only if $a_0 \neq 0$.

Derivatives and integration

Definition 1.5.10 *(Derivative of a generating function). We define the derivative of the generating function* f *by*

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty} a_n t^n\right) = \sum_{n=1}^{\infty} n a_n \cdot t^{n-1}. \tag{1.21}$$

The second derivative of f is defined similarly, by considering the generating series $\sum_{n=0}^{\infty} b_n t^n$, where $b_i = i a_i$. The higher derivatives are defined by an obvious way.

We use the notation $\frac{d^k}{dt^k}$ for the k^{th} derivative in respect to t.

Example 1.8

$$f(t) = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + t^3 + t^4 + \cdots = \frac{1}{1-t},$$

then

$$\frac{d}{dt} f(t) = 1 + 2t + 3t^2 + 4t^3 + \cdots = \sum_{n=1}^{\infty} n t^{n-1} = \frac{1}{(1-t)^2}.$$

Definition 1.5.11 *We define the integral of the generating function* f *as*

$$\int_0^t f(t)dt = \int_0^t \sum_{n=0}^{\infty} a_n t^n dt = \sum_{n=0}^{\infty} \frac{a}{n+1} t^{n+1}.$$

Almost all the rules known in Calculus for derivatives and integrals are valid with generating functions too.

Example 1.9 *We have* $g(t)$ *the generating functions :*

$$g(t) = 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{720}t^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} = \cos(t),$$

so

$$\int_0^t g(t)dt = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \cdots$$

now we note that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} = \sin(t).$$

Compositions

Definition 1.5.12 *Let be f and g be the two generating functions :*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad g(t) = \sum_{n=0}^{\infty} b_n t^n,$$

the composition of f and g is defined by

$$f(g(t)) = \sum_{n=0}^{\infty} a_n \left(\sum_{m=0}^{\infty} b_m t^m \right)^n . \tag{1.22}$$

Example 1.10 *Consider the two generating functions :*

$$f(t) = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n = \frac{1}{(1-t)},$$
$$g(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = e^t,$$

so

$$f(g(t)) = 1 + \sum_{n=0}^{\infty} \frac{t^n}{n!} + \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \right)^2 + \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \right)^3 + \cdots = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{t^m}{m!} \right)^n ,$$

this summation is just

$$\frac{1}{1-e^t}.$$

In the following tables, we provide a compilation of fundamental ordinary generating functions and some operations on it

$G(t)$	Summation Notation	Expanded Notation
$\frac{1}{1-t}$	$\sum_{k=0}^{\infty} t^k$	$1+t+t^2+t^3+\cdots$
$\frac{1}{1+t}$	$\sum_{k=0}^{\infty} (-1)^k t^k$	$1-t+t^2-t^3+\cdots$
$\frac{1}{1-t^n}$	$\sum_{k=0}^{\infty} t^{nk}$	$1+t^n+t^{2n}+t^{3n}+\cdots$
$\frac{1}{1-ct^n}$	$\sum_{k=0}^{\infty} c^k t^{nk}$	$1+ct^n+c^2t^{2n}+c^3t^{3n}+\cdots$
$\frac{1}{(1-t)^n}$	$\sum_{k=0}^{\infty} \binom{n+k-1}{k} t^k$	$1+nt+\binom{n+1}{2}t^2+\binom{n+2}{3}t^3+\cdots$
$\frac{1}{(1-t)^2}$	$\sum_{k=0}^{\infty} (k+1)t^k$	$1+2t+3t^2+4t^3+\cdots$
$(1+t)^c$	$\sum_{k=0}^{\infty} \binom{c}{k} t^k$	$1+ct+\binom{c}{2}t^2+\binom{c}{3}t^3+\cdots$
e^t	$\sum_{k=0}^{\infty} \frac{1}{k!} t^k$	$1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots$

Table 1.2: Table elementary ordinary generating functions

Operation OGF	Squence
$f(t)=\sum_{n\geq 0} a_n t^n$	$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$
$g(t)=\sum_{n\geq 0} b_n t^n$	$b_0, b_1, b_2, b_3, \ldots, b_n, \ldots$
$tf(t)=\sum_{n\geq 1} a_{n-1} t^n$	$a_0, a_1, a_2, a_3, \ldots, a_{n-1}, \ldots$
$\frac{f(t)-a_0}{t}=\sum_{n\geq 0} a_{n+1} t^n$	$a_1, a_2, a_3, \ldots, a_{n+1}, \ldots$
$f'(t)=\sum_{n\geq 0}(n+1)a_{n+1} t^n$	$a_1, 2a_2, \ldots, (n+1)a_{n+1}, \ldots$
$\int_0^t f(t)dt=\sum_{n\geq 1} \frac{a_{n-1}}{n} t^n$	$a_0, \frac{a_1}{2}, \frac{a_2}{3}, \frac{a_3}{4}, \ldots, \frac{a_{n-1}}{n}, \ldots$
$f(\lambda t)=\sum_{n\geq 0} \lambda^n a_n t^n$	$a_0, \lambda a_1, \lambda^2 a_2, \lambda^3 a_3, \ldots, \lambda^n a_n, \ldots$
$f(t)+g(t)=\sum_{n\geq 0}(a_n+b_n)t^n$	$a_0+b_0, a_1+b_1, a_2+b_2, a_3+b_3, \ldots, a_n+b_n, \ldots$
$(1-t)f(t)=a_0+\sum_{n\geq 1}(a_n-a_{n-1})t^n$	$a_0, a_1-a_0, \ldots, a_n-a_{n-1}, \ldots$
$f(t)\cdot g(t)=\sum_{n\geq 0}\left(\sum_{0\leq k\leq n} a_k b_{n-k}\right) t^n$	$a_0b_0, a_1b_0, a_0b_1, \ldots, \sum_{0\leq k\leq n} a_k b_{n-k}, \ldots$

Table 1.3: Operation on ordinary generating functions

1.6 The Stirling numbers

1.6.1 Stirling numbers of the first kind

Definition 1.6.1 *The signed Stirling numbers of the first kind, denoted as $s(n,k)$, are the coefficients of the falling factorial polynomials in the canonical basis t^n, that's to say :*

$$t^{\underline{n}} = \sum_{k=0}^{n} s(n,k)t^k.$$

Definition 1.6.2 *The unsigned Stirling numbers of the first kind, denoted as $\mid s(n,k) \mid$, are the coefficients of the rising factorial polynomials in the canonical basis t^n, that's to say :*

$$t^{\overline{n}} = \sum_{k=0}^{n} |s(n,k)|t^k,$$

we have :

$$|s(n,k)| = (-1)^{n-k}s(n,k).$$

Definition 1.6.3 *The Stirling numbre of the first kind $\mid s(n,k) \mid$ counts the number of permutations of n objects that consist of exactly k disjoint cycles.*

Example 1.11 *Write $t^{\underline{2}}$ as a polynomial*

$$\begin{aligned} t^{\underline{2}} &= t(t-1) \\ &= t^2 - t. \end{aligned}$$

We can say from this example that :
$s(2,0) = 0.$
$s(2,1) = -1.$
$s(2,2) = 1.$

The generating function

The exponential generating function of the signed Stirling numbers of the first kind is given by :

$$\sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!} = \frac{(\ln(1+t))^k}{k!}.$$

The explicit formula

The explicit formula for the signed Stirling numbers of the first kind :

$$\mid s(n,k) \mid = \sum_{0 \leqslant j \leqslant h \leqslant n-k}^{\infty} (-1)^{j+h} \binom{h}{j} \binom{n-1+h}{n-k+h}.$$

The explicit formula for the signed Stirling numbers of the first kind :

$$s(n,k) = \frac{n!}{k!} \sum_{i_1+\cdots+i_k=n} \frac{1}{i_1 \cdots i_k}.$$

The recurrence formula

The Stirling numbers of the first kind satisfy the following recurrence relation :

$$\begin{cases} s(n,0) = s(0,k) = 0, \quad \text{except} \quad s(0,0) = 1, \\ s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \quad (n,k \geqslant 1). \end{cases} \tag{1.23}$$

Table of the Stirling numbres of the first kind

Using the triangular recurrence relation (1.23), we can calculate the Stirling numbers of the first kind $s(n,k)$:

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	−1	1					
3	0	2	−3	1				
4	0	−6	11	−6	1			
5	0	24	−50	35	−10	1		
6	0	−120	274	−225	85	−15	1	
7	0	720	−1764	1624	−735	175	−21	1

Table 1.4: Stirling numbres of the first kind

1.6.2 Stirling numbers of the second kind

Definition 1.6.4 *Stirling numbers of the second kind, denoted as* $\left\{ {n \atop k} \right\}$ *are the coefficients of the polynomial* t^n*, that's to say :*

$$t^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} t^{\underline{k}}.$$

Definition 1.6.5 *The Stirling numbers of the second kind,* $\left\{ {n \atop k} \right\}$ *counts the number of partitions of the set* $\{1, 2, \ldots, n\}$ *into* k *non-empty subsets (*$0 \leq k \leq n$*).*

Example 1.12 *The set* $[4]$ *contain* 7 *partitions into two non-empty subsets* $\left\{ {4 \atop 2} \right\}$ *represent the partitions into* 2 *subsets :*

$$\{\{1\}\{2,3,4\}\}, \quad \{\{2\}\{1,3,4\}\}, \quad \{\{3\}\{1,2,4\}\}, \quad \{\{4\}\{1,2,3\},$$
$$\{1,2\}\{3,4\}\}, \quad \{\{1,3\}\{2,4\}\}, \quad \{\{1,4\}\{2,3\}\}.$$

As a result, $\left\{ {4 \atop 2} \right\} = 7$.

The generating function

The ordinary generating function of Stirling numbers of the second kind is given by :

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} t^n = \frac{1}{(1-t)(1-2t)\cdots(1-kt)} = \prod_{j=1}^{k} \frac{1}{(1-jt)}.$$

The exponential generating function of Stirling numbers of the second kind is given by :

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{t^n}{n!} = \frac{1}{k!}(e^t - 1)^k.$$

The explicit formula

The Stirling numbers of the second kind are given by the following explicit formula :

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

The recurrence formula

The Stirling numbers of the second kind satisfy the following recurrence relation :

$$\begin{cases} \left\{ {n \atop 0} \right\} = \left\{ {0 \atop k} \right\} = 1, \quad \text{except} \quad \left\{ {0 \atop 0} \right\} = 1 \\ \\ \left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}, \ n \geq 0 \end{cases} .$$

Table of Stirling numbers of second kind

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	1					
3	0	1	3	1				
4	0	1	7	6	1			
5	0	1	15	25	10	1		
6	0	1	31	90	65	15	1	
7	0	1	63	301	350	140	20	1

Table 1.5: Stirling numbers of the second kind

2

The Method of Coefficients

The objective of this chapter is to introduce the concept of coefficients. From our standpoint, we can provide a general description of this concept as follows: when faced with the task of proving an identity or calculating an expression (such as finding a closed or asymptotic formula), rather than directly working with the identity or expression itself, we focus on finding the generating functions associated with the quantities involved in the problem. By manipulating these functions, our goal is to derive a consolidated expression, which we achieve by extracting the coefficient of the specific power t^n. This approach yields the solution [10, 11, 18].

2.1 Coefficient extraction

We introduce an infinite number of linear functionals $[t^n] : K \to \mathbb{K}(n = 0, 1, 2, \ldots)$, defined based on their behavior on the monomials t^k :

$$[t^n]t^k = \delta_{n,k}, \tag{2.1}$$

where $\delta_{n,k}$ is the Kronecker delta. From this definition we have :

$$[t^n]f(t) = [t^n]\sum_{k=0}^{\infty} f_k t^k = \sum_{k=0}^{\infty}[t^n]f_k t^k = \sum_{k=0}^{\infty} f_k[t^n]t^k = \sum_{k=0}^{\infty} f_k\delta_{n,k} = f_n. \tag{2.2}$$

The $[t^n]$ functionals play a crucial role in tackling combinatorial problems and analyzing algorithms. Their significance lies in the fact that they allow for the development of the "method of coefficients". While it is possible to establish a set of properties for these functionals and derive the method from them, we will adopt a more direct approach inspired by Egorychev [5]. We will rely on six fundamental properties of the $[t^n]$ functionals, which we will state without presenting their proofs. With the exception of proprty (2.8), these properties naturally follow from the definitions of basic operations involving formal power series. The implications of these rules form the basis of the "method of coefficients". We can state formally the main properties of this operator.

Property 2.1.1 *If $f(t)$ and $g(t)$ are formal power series, then the following statements hold :*

1. ***(Linearity)***
$$[t^n]\left(\alpha f(t)+\beta g(t)\right)=\alpha[t^n]f(t)+\beta[t^n]g(t),\quad \alpha,\beta\in\mathbb{K}. \tag{2.3}$$

2. ***(Shifting)***
$$[t^n]tf(t)=[t^{n-1}]f(t). \tag{2.4}$$

3. ***(Differentiation)***
$$[t^n]f'(t)=(n+1)[t^{n+1}]f(t). \tag{2.5}$$

4. ***(Convolution)***
$$[t^n]f(t)g(t)=\sum_{k=0}^{n}\Big([y^k]f(y)\Big)\,[t^{n-k}]g(t). \tag{2.6}$$

5. ***(Composition)***
$$[t^n]f(g(t))=\sum_{k=0}^{\infty}\Big([y^k]f(y)\Big)\,[t^n]g(t)^k. \tag{2.7}$$

6. ***(Inversion)***
$$[t^n]\bar{f}(t)^k=\frac{k}{n}[t^{n-k}]\left(\frac{t}{f(t)}\right)^n. \tag{2.8}$$

Proof.

1. (Linearity) Using the Definition 1.5.5, we can deduce
$$\begin{aligned}[t^n]\left(\alpha f(t)+\beta g(t)\right)&=[t^n]\left(\alpha\sum_{k=0}^{\infty}f_kt^k+\beta\sum_{k=0}^{\infty}g_kt^k\right)\\&=[t^n]\left(\sum_{k=0}^{\infty}\alpha f_kt^k+\sum_{k=0}^{\infty}\beta g_kt^k\right)\\&=[t^n]\sum_{k=0}^{\infty}\left(\alpha f_k+\beta g_k\right)t^k.\end{aligned}$$
According to the formula (2.2), we obtain
$$\begin{aligned}[t^n]\left(\alpha f(t)+\beta g(t)\right)&=\sum_{k=0}^{\infty}[t^n]\left(\alpha f_k+\beta g_k\right)t^k\\&=\sum_{k=0}^{\infty}\left(\alpha f_k+\beta g_k\right)[t^n]t^k\\&=\sum_{k=0}^{\infty}\left(\alpha f_k+\beta g_k\right)\delta_{n,k}\\&=\alpha f_n+\beta g_n\\&=\alpha[t^n]f(t)+\beta[t^n]g(t).\end{aligned}$$

2. (Shifting) We have
$$f(t)=\sum_{k=0}^{\infty}f_kt^k=f_0+f_1t+f_2t^2+\cdots+f_{n-1}t^{n-1}+f_nt^n+\cdots$$
and
$$tf(t)=\sum_{k=0}^{\infty}f_kt^{k+1}=f_0t+f_1t+f_2t^3+\cdots+f_{n-1}t^n+\cdots$$

then

$$[t^{n-1}]f(t) = f_{n-1}. \tag{2.9}$$

$$[t^{n}]tf(t) = f_{n-1}. \tag{2.10}$$

From (2.9) and (2.10), we conclude

$$[t^{n}]tf(t) = [t^{n-1}]f(t).$$

3. (Differentiation) According to the Definition 1.5.10, we have

$$f'(t) = \sum_{k=0}^{\infty} ka_k \cdot t^{n-1} = f_1 + 2f_2t + 3f_3t^2 + \cdots + nf_nt^{n-1} + (n+1)f_{n+1}t^n + \cdots$$

therefore

$$[t^n]f'(t) = (n+1)f_{n+1} \tag{2.11}$$

and we have

$$(n+1)f(t) = (n+1)f_0 + (n+1)f_1t + \cdots + (n+1)f_nt^n + (n+1)f_{n+1}t^{n+1} + \cdots$$

then

$$[t^{n+1}](n+1)f(t) = (n+1)f_{n+1}. \tag{2.12}$$

From (2.11) and (2.12), we conclude

$$[t^n]f'(t) = (n+1)[t^{n+1}]f(t).$$

4. (Convolution) Let f and g be written in the following form :

$$f(t) = f_0 + f_1t + f_2t^2 + \cdots + f_nt^n + \cdots$$
$$g(t) = g_0 + g_1t + g_2t^2 + \cdots + g_nt^n + \cdots$$

By Definition 1.5.8, we have

$$f(t)g(t) = f_0g(t) + f_1tg(t) + f_2t^2g(t) + \cdots + f_nt^ng(t) + \cdots$$

According (1.19), we deduce that

$$\begin{aligned}[t^n]f(t)g(t) &= \sum_{k=0}^{n} f_kg_{n-k} \\ &= \sum_{k=0}^{n}([y^k]f(y))[t^{n-k}]g(t).\end{aligned}$$

5. (Composition) We have

$$f(t) = f_1t + f_2t^2 + \cdots + f_nt^n + \cdots$$

and by the Definition 1.5.12, we also have

$$f(g(t)) = \sum_{k=0}^{\infty} f_kg(t)^k,$$

then

$$\begin{aligned}f\left(g(t)\right) = {} & f_1(g_0 + g_1t + g_2t^2 + \cdots + g_nt^n + \cdots) \\ & + f_2(g_0 + g_1t + g_2t^2 + \cdots + g_nt^n + \cdots)^2 + \cdots \\ & + f_n(g_0 + g_1t + g_2t^2 + \cdots + g_nt^n + \cdots)^n + \cdots\end{aligned}$$

Now we want to calculate the term $[t^n]f_2(g(t)^2)$ To generalize the operation on $[t^n]f(g(t))$, first we have to calculate $[t^n]g(t)^2$
from convolution (2.6), we have

$$[t^n]g(t)^2 = \sum_{k=0}^{\infty}[t^k]g(t)t^{n-k}g(t),$$

then

$$[t^n]f_2(g(t)^2) = \sum_{k\geq 2} f_2[t^k]g(t)t^{n-k}g(t).$$

Consequently

$$[t^n]f(g(t)) = \sum_{k=0}^{\infty}\Big([y^k]f(y)\Big)\,[t^{n-k}]g(t)^k.$$

6. (Inversion) see [18].

■

Remark 1

- *In (2.6) and (2.7) the indeterminate y is used only to distinguish the action of the functionals on different formal power series.*
- *In (2.7) the composition is only possible when $g(0) = 0$ or $f(t)$ is a polynomial, so the sum is actually finite.*
- *In (2.8), $\bar{f}(t)$ denotes the compositional inverse of $f(t)$, that is, the formal power series such that $f(\bar{f}(t)) = \bar{f}(f(t)) = t$.*
- *It is known that a formal power series $f(t)$ has a compositional inverse if and only if $f(t)$ belongs to K_1.*
- *We state explicitly the principle of identity, a very important rule that is often applied without mention: the formal power series $f(t)$ and $g(t)$ are equal if and only if $[t^n]f(t) = [t^n]g(t)$ holds for $n = 0, 1, 2, \ldots$*
- *The property of shifting can be easily generalized to $[t^n]t^k f(t) = [t^{n-k}]f(t)$ and also to negative powers : $[t^n]/t^k f(t) = [t^{n+k}]f(t)$.*
- *The property of differentiation for $n = -1$ gives $[t^{-1}]f'(t) = 0$, a well-known and important situation.*

Property 2.1.2 *Rule Differentiation (2.5) can be expressed as follows :*

$$[t^n]f(t) = \frac{1}{n}[t^{n-1}]f'(t). \tag{2.13}$$

Proof. Let f be written as

$$f(t) = f_0 + f_1 t + f_2 t^2 + \cdots + f_n t^n + \cdots$$

then

$$[t^n]f(t) = f_n$$

and

$$\frac{1}{n}f'(t) = \frac{1}{n}f_1 + \frac{2}{n}f_2 t + \cdots + f_n t^{n-1} + \frac{n+1}{n}f_{n+1}t^n + \cdots$$

hence

$$\frac{1}{n}[t^{n-1}]f'(t) = f_n$$

and therefore

$$[t^n]f(t) = \frac{1}{n}[t^{n-1}]f'(t).$$

■

Certain power series are used with great frequency in what follows :

- The exponential series :

$$e^t = \exp(t) = \sum_{k=0}^{\infty} \frac{1}{k!}t^k. \tag{2.14}$$

- The logarithm series :

$$\ln \frac{1}{1-t} = \sum_{k=1}^{\infty} \frac{1}{k}t^k. \tag{2.15}$$

- The binomial series :

$$(1+t)^r = \sum_{k=0}^{\infty} \binom{r}{k} t^k. \tag{2.16}$$

Rule composition (2.7) leads to the more general formula (known as Newton's rule)

$$(1+\alpha t)^r = \sum_{n=0}^{\infty} \binom{r}{n} \alpha^n t^n, \tag{2.17}$$

or, equivalently

$$[t^n](1+\alpha t)^r = \binom{r}{n} \alpha^n. \tag{2.18}$$

We remark that when $r = -1$ and by (1.11), we thus obtain

$$[t^n]\frac{1}{1-\alpha t} = \binom{-1}{n}(-\alpha)^n = \binom{1+n-1}{n}\alpha^n = \alpha^n. \tag{2.19}$$

Theoreme 2.1 ***(Partial Sum Theorem).*** *If $f(t)$ belongs to K, then*

$$[t^n]\frac{f(t)}{1-t} = \sum_{k=0}^{n}[t^k]f(t) = f_0 + f_1 + \cdots + f_n. \tag{2.20}$$

Proof. We suppose $g(t) = \dfrac{1}{1-t}$ then $\dfrac{f(t)}{1-t} = f(t)g(t)$.
By applying the property of convolution (2.6), we obtain

$$\begin{aligned}[t^n]f(t)g(t) &= \sum_{k=0}^{n}\Big([y^k]f(y)\Big)\,[t^{n-k}]g(t) \\ &= \sum_{k=0}^{n}\Big([y^k]f(y)\Big)\,[t^{n-k}]\frac{1}{1-t}\end{aligned}$$

and according (2.19), we find

$$[t^{n-k}]\frac{1}{1-t} = 1,$$

then

$$[t^n]f(t)g(t) = \sum_{k=0}^{n}[y^k]f(y).$$

Finally, we get the result

$$[t^n]\frac{f(t)}{1-t} = \sum_{k=0}^{n}[y^k]f(y).$$

■

Example 2.1

By using Theorem 2.1 we can easily find the sum of geometric progression $1+\alpha+\alpha^2+\cdots+\alpha^n$. *Since as observed before :*

$$\alpha^k = [t^k](1-\alpha t)^{-1}. \tag{2.21}$$

We have

$$\sum_{k=0}^{n}\alpha^k = [t^n]\frac{1}{1-t}\frac{1}{1-\alpha t} = [t^n]\frac{1}{\alpha-1}\left(\frac{\alpha}{1-\alpha t}-\frac{1}{1-t}\right) = \frac{\alpha^{n+1}-1}{\alpha-1}. \tag{2.22}$$

From (2.21) and (2.20), we get

$$\sum_{k=0}^{n}\alpha^k = \sum_{k=0}^{n}[t^k]f(t) = \sum_{k=0}^{n}[t^k]\frac{1}{1-\alpha t} = [t^n]\frac{1}{1-t}\frac{1}{1-\alpha t} = [t^n]\frac{1}{\alpha-1}\left(\frac{\alpha}{1-\alpha t}-\frac{1}{1-t}\right). \tag{2.23}$$

Now we want to represent this function $\frac{1}{1-t}\frac{1}{1-\alpha t}$ as a sum of two fractions, by the partial fraction decomposition we find

$$\frac{1}{1-t}\frac{1}{1-\alpha t} = \frac{A}{1-t}+\frac{B}{1-\alpha t}.$$

By unifying fractions, expanding and simplifying, we find that

$$\begin{aligned}\frac{1}{1-t}\frac{1}{1-\alpha t} &= \frac{A-A\alpha t+B-Bt}{(1-t)(1-\alpha t)}\\ &= \frac{A+B-(A\alpha+B)t}{(1-t)(1-\alpha t)}.\end{aligned}$$

Now we will search for the unknowns A and B. Therefore, we need to solve the system of the following two equations :

$$\begin{cases}A+B=1\\ \alpha A+B=0\end{cases} \Leftrightarrow \begin{cases}B=1-A\\ \alpha A+1-A=0\end{cases} \Leftrightarrow \begin{cases}A=\dfrac{-1}{\alpha-1}\\ B=\dfrac{\alpha}{\alpha-1},\end{cases}$$

we further have

$$\frac{1}{1-t}\frac{1}{1-\alpha t} = \frac{\frac{-1}{\alpha-1}}{1-t}+\frac{\frac{\alpha}{\alpha-1}}{1-\alpha t} = \frac{1}{\alpha-1}\left(\frac{\alpha}{1-\alpha t}-\frac{1}{1-t}\right).$$

Now we returne to (2.23)

$$\begin{aligned}[t^n]\frac{1}{1-t}\frac{1}{1-\alpha t} &= [t^n]\frac{1}{\alpha-1}\left(\frac{\alpha}{1-\alpha t}-\frac{1}{1-t}\right)\\ &= [t^n]\frac{1}{\alpha-1}\frac{\alpha}{1-\alpha t}-[t^n]\frac{1}{\alpha-1}\frac{1}{1-t}\\ &= \frac{\alpha}{\alpha-1}[t^n]\frac{1}{1-\alpha t}-\frac{1}{\alpha-1}[t^n]\frac{1}{1-t}.\end{aligned}$$

From (2.19), we have $[t^n]\frac{1}{1-\alpha t} = \alpha^n$ and when $\alpha = 1$, we obtain $[t^n]\frac{1}{1-t} = 1$. Then

$$\frac{\alpha}{\alpha-1}[t^n]\frac{1}{1-\alpha t}-\frac{1}{\alpha-1}[t^n]\frac{1}{1-t} = \frac{\alpha}{\alpha-1}\alpha^n-\frac{1}{\alpha-1} = \frac{\alpha^{n+1}-1}{\alpha-1}.$$

Example 2.2

We are now prepared to demonstrate a well-known result in binary sarching (referenced in [10]). Consider a sorted table T with n keys: $T = (k_1, k_2, \ldots, k_n)$, where $k_1 < k_2 < \ldots < k_n$. Our objective is to find the index j at which $k = k_j$ (successful searching) for a given key k in T. A binary search involves comparing k with the median element k_r, where $r = \lfloor (n+1)/2 \rfloor$ (here, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x). If $k = k_r$, the search is complete. If $k < k_r$, we continue with the subtable $T' = (k_1, \ldots, k_{r-1})$. If $k > k_r$, we proceed with the subtable $T'' = (k_{r+1}, \ldots, k_n)$. Our interest lies in determining the average number b_n of comparisons required to complete the search for any key k in T. There exists only one element that can be found with a single comparison, which is the median element in T. Two elements can be found with two comparisons, representing the median elements in T' and T'. Four elements can be found with three comparisons, and so on, up to a maximum of $1 + \lfloor \log_2 n \rfloor$ comparisons. By summing the number of comparisons corresponding to each of the n keys, we obtain

$$b_n = \frac{B_n}{n} = \frac{1}{n}\left(1 + 2 + 2 + 3 + 3 + 3 + 3 + \cdots + (1 + \lfloor \log_2 n \rfloor)\right).$$

To calculate the sum, we consider the infinite sequence $L = (0, 1, 2, 2, 3, 3, 3, 3, 4, \ldots)$ generated by summing sequences $(0, 1, 1, 1, \ldots)$, $(0, 0, 1, 1, 1, \ldots)$, $(0, 0, 0, 0, 1, 1, \ldots)$, and so on, the k^{th} sequence begins with 2^k zeros. Consequently, the generating function of L can be defined as :

$$L(t) = \frac{t}{1-t} + \frac{t^2}{1-t} + \frac{t^4}{1-t} + \cdots = \frac{t^{2^0}}{1-t} + \frac{t^{2^1}}{1-t} + \frac{t^{2^2}}{1-t} + \cdots = \sum_{k=0}^{\infty} \frac{t^{2^k}}{1-t}.$$

We can determine the total number of comparisons B_n for T using the partial sum Theorem 2.1. To proceed, we follow these steps :

$$\begin{aligned} B_n &= [t^n] \frac{1}{1-t} \sum_{k=0}^{\infty} \frac{t^{2^k}}{1-t} \\ &= [t^n] \sum_{k=0}^{\infty} \frac{t^{2^k}}{(1-t)^2} \\ &= \sum_{k=0}^{\infty} \frac{[t^n] t^{2^k}}{(1-t)^2} \\ &= \sum_{k=0}^{\infty} \frac{[t^{n-2^k}]}{(1-t)^2}. \end{aligned}$$

We have for k greater than $\lfloor \log_2 n \rfloor$ the coefficients are zero, where

$$\sum_{k=0}^{\infty} \frac{[t^{n-2^k}]}{(1-t)^2} = \sum_{k=0}^{\lfloor \log_2 n \rfloor} \frac{[t^{n-2^k}]}{(1-t)^2} + \sum_{k \geqslant \lfloor \log_2 n \rfloor} \frac{[t^{n-2^k}]}{(1-t)^2}$$

and

$$\sum_{k \geqslant \lfloor \log_2 n \rfloor} \frac{[t^{n-2^k}]}{(1-t)^2} = 0,$$

then

$$\sum_{k=0}^{\infty} \frac{[t^{n-2^k}]}{(1-t)^2} = \sum_{k=0}^{\lfloor \log_2 n \rfloor} \frac{[t^{n-2^k}]}{(1-t)^2}.$$

By using (2.19), we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{[t^{n-2^k}]}{(1-t)^2} &= \sum_{k=0}^{\lfloor \log_2 n \rfloor} \binom{2+n-2^k-1}{n-2^k} \\
&= \sum_{k=0}^{\lfloor \log_2 n \rfloor} \binom{n-2^k+1}{n-2^k} \\
&= \sum_{k=0}^{\lfloor \log_2 n \rfloor} \frac{(n-2^k+1)!}{(n-2^k)!} \\
&= \sum_{k=0}^{\lfloor \log_2 n \rfloor} (n-2^k+1) \\
&= \sum_{k=0}^{\lfloor \log_2 n \rfloor} (n+1) - \sum_{k=0}^{\lfloor \log_2 n \rfloor} 2^k \\
&= (n+1)\left(\lfloor \log_2 n + 1 \rfloor\right) - \sum_{k=0}^{\lfloor \log_2 n \rfloor} 2^k.
\end{aligned}$$

From (2.22), we have $\sum_{k=0}^{n} \alpha^k = \frac{\alpha^{n+1}-1}{\alpha-1}$ *so* $\sum_{k=0}^{\lfloor \log_2 n \rfloor} 2^k = \frac{2^{\lfloor \log_2 n \rfloor+1}-1}{2-1} = 2^{\lfloor \log_2 n \rfloor+1} - 1$.
Then

$$B_n = (n+1)\lfloor \log_2 n \rfloor + n - 2^{\lfloor \log_2 n \rfloor+1} + 2.$$

Example 2.3
We will now present a recurrence that emerges during the analysis of quicksort (a fast sorting method for a list of keys), or during the analysis of the internal path length of a binary tree (refer to [10]). In the case of quicksort, the average number C_n *of key comparisons satisfies : we have initial condition* $C_0 = 0$

$$C_{n+1} = n + 2 + \frac{2}{n+1}\sum_{k=0}^{n} C_k \quad or \quad (n+1)C_{n+1} = (n+2)(n+1) + 2\sum_{k=0}^{n} C_k$$

and the generating function of sequence $(n+1)(n+2)$ *is*

$$(n+1)(n+2) = [t^n]\frac{2}{(1-t)^3}, \tag{2.24}$$

by the partial sum Thorem 2.1 we have

$$[t^n]\frac{C(t)}{1-t} = \sum_{k=0}^{n}[t^k]C_k. \tag{2.25}$$

Using the proprty of differentiation (2.5), we can deduce

$$(n+1)C_{n+1} = [t^n]C'(t),$$

as a result, the recurrence can be represented as the corresponding differential equation :

$$[t^n]C'(t) = [t^n]\left(\frac{2}{(1-t)^3} + \frac{2}{1-t}C(t)\right).$$

By suming (2.24) and (2.25), and using (2.3), we find

$$\begin{aligned}
(n+1)(n+2) + 2\sum_{k=0}^{n}[t^k]C_k &= [t^n]\frac{2}{(1-t)^3} + [t^n]\frac{2}{1-t}C(t) \\
&= [t^n]\left(\frac{2}{(1-t)^3} + \frac{2}{1-t}C(t)\right) \\
&= [t^n]C'(t) \\
&= (n+1)C_{n+1},
\end{aligned}$$

then

$$(n+1)C_{n+1} = [t^n]\left(\frac{2}{(1-t)^3} + \frac{2}{1-t}C(t)\right). \tag{2.26}$$

So the solution to this equation is

$$C(t) = \frac{2}{(1-t)^2}\ln\frac{1}{1-t}. \tag{2.27}$$

To derive an explicit formula for C_n, we need to extract the coefficient $[t^n]C(t)$. By using the formula (2.13), we obtain

$$[t^n]\ln\frac{1}{1-t} = \frac{1}{n}[t^{n-1}]\frac{1}{1-t}, \tag{2.28}$$

from (2.15), we deduce that

$$[t^n]\ln\frac{1}{1-t} = \frac{1}{n},$$

then, we get

$$[t^n]\ln\frac{1}{1-t} = \frac{1}{n}[t^{n-1}]\frac{1}{1-t} = \frac{1}{n},$$

for positive n, while the coefficient is 0 for $n = 0$. We can infer from the partial sum theorem that

$$[t^n]\frac{1}{1-t}\ln\frac{1}{1-t} = 0 + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = H_n, \tag{2.29}$$

where H_n is the n^{th} harmonic number.
we have

$$\frac{d}{dt}\left(\ln\frac{1}{1-t}\right) = \frac{1}{1-t}$$

and

$$\frac{d}{dt}\left(\frac{1}{1-t}\right) = \frac{1}{(1-t)^2}.$$

Differentiating this expression, we find by (2.5) that

$$[t^n]\frac{1}{1-t}\ln\frac{1}{1-t} = \frac{1}{n}[t^{n-1}]\left(\frac{1}{(1-t)^2}\ln\frac{1}{1-t} + \frac{1}{(1-t)^2}\right).$$

We suppose $n := n+1$ in the last expression

$$[t^{n+1}]\frac{1}{1-t}\ln\frac{1}{1-t} = \frac{1}{n+1}[t^n]\left(\frac{1}{(1-t)^2}\ln\frac{1}{1-t} + \frac{1}{(1-t)^2}\right).$$

$$(n+1)[t^{n+1}]\frac{1}{1-t}\ln\frac{1}{1-t} = [t^n]\left(\frac{1}{(1-t)^2}\ln\frac{1}{1-t} + \frac{1}{(1-t)^2}\right).$$

By using linearity rule (2.3), we obtain

$$(n+1)[t^{n+1}]\frac{1}{1-t}\ln\frac{1}{1-t} = [t^n]\frac{1}{(1-t)^2}\ln\frac{1}{1-t} + [t^n]\frac{1}{(1-t)^2}, \tag{2.30}$$

we have

$$[t^n]\frac{1}{(1-t)^2} = (n+1) \tag{2.31}$$

and by (2.29)

$$[t^{n+1}]\frac{1}{1-t}\ln\frac{1}{1-t} = H_{n+1}. \tag{2.32}$$

According to (2.27), we have

$$[t^n]C(t) = [t^n]\frac{2}{(1-t)^2}\ln\frac{1}{1-t},$$

then

$$C_n = [t^n]\frac{2}{(1-t)^2}\ln\frac{1}{1-t}. \tag{2.33}$$

By replacing (2.31), (2.32) and (2.33) in (2.30), we have

$$(n+1)H_{n+1} = \frac{C_n}{2} + (n+1)$$
$$\frac{C_n}{2} = (n+1)H_{n+1} - (n+1),$$

then, the explicit formula for C_n is giving by

$$C_n = 2(n+1)\left(H_{n+1} - 1\right).$$

2.2 Convolution and Composition

We utilized the convolution rule in the proof of (2.1). Another classical application of this rule is in counting binary trees based on the number of their nodes. A binary tree can be either empty or consists of a node (known as the root) with two binary trees appended to it (known as the left and right subtrees). In a binary tree with $n+1$ nodes, the left and right subtrees contain k and $n-k$ nodes, respectively ($k = 0, 1, 2, \ldots, n$). If C_{n+1} represents the number of binary trees with $n+1$ nodes, we can establish the recurrence relation as $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$. If $C(t) = \sum_{k\geq 0} C_k t^k$ denotes the generating function of the sequence, we can observe, according to rule of shifting (2.4), that

$$[t^{n+1}]C(t) = [t^n]\frac{C(t)-C(0)}{t} = [t^n]\frac{C(t)-1}{t},$$

as the empty tree is the only tree with zero nodes, we have

$$[t^n]\frac{C(t)-1}{t} = [t^n]C(t)^2,$$

or

$$C(t) = 1 + tC(t)^2,$$

the previous identity is equal to

$$tC(t)^2 - C(t) + 1 = 0.$$

We have utilized the convolution rule and the principle of identity to transition from a coefficient to a formal power series identity. By solving this functional equation, while considering the initial condition $C(0) = C_0 = 1$, we obtain :

$$C(t) = \frac{1-\sqrt{1-4t}}{2t}.$$

An interesting point is how we know whether to take $+\sqrt{1-4t}$ or $-\sqrt{1-4t}$ here. To see that the $-$ sign must be right, note that $2tC(t)$ must equal the numerator in the above fraction, so this numerator must have zero constant term as a formal series. Since $1+\sqrt{1-4t}$ does not vanish at $t=0$, it is the wrong numerator.
By using (2.17), we have

$$\sqrt{1-4t} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-4t)^n = 1 + \binom{\frac{1}{2}}{1}(-4t) + \binom{\frac{1}{2}}{2}(-4t)^2 + \cdots$$

then

$$C(t) = \frac{1-\sqrt{1-4t}}{2t} = -\frac{1}{2}\left(\binom{\frac{1}{2}}{1}(-4) + \binom{\frac{1}{2}}{2}(-4)^2 t + \cdots\right) = -\frac{1}{2}\left(\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n+1}(-4)^{n+1}t^n\right),$$

therefore

$$\begin{aligned}
C_n &= [t^n]C(t) \\
&= [t^{n+1}]\frac{1-\sqrt{1-4t}}{2} \\
&= -\frac{1}{2}\binom{\frac{1}{2}}{n+1}(-4)^{n+1} \\
&= -\frac{1}{2}\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n)}{(n+1)!}(-4)^{n+1} \\
&= \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{(n+1)!}\frac{2^{2n+2}}{2^{n+2}} \\
&= \frac{(2n)!}{(2^n n!)(n+1)!}.2^n \\
&= \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$

We have therefore proved that binary trees are counted by the well-known Catalan numbers.

Example 2.4
Sometimes, in order to apply the convolution rule, certain manipulations are required. For instance, let's consider the following sum :

$$S_{m,n} = \sum_{k=0}^{n}\binom{k-m-1}{m-2}\binom{n-k+m-1}{m-1}. \tag{2.34}$$

The summation variable k is in the numerator of the binomial coefficient, but it is more convenient to have it in the denominator in order to apply the method of coefficients. This is easily done as follows :

$$\binom{k-m-1}{m-2} = \binom{k-m-1}{k-2m+1} = \binom{-m+1}{k-2m+1}(-1)^{k-2m+1} = [t^k]\frac{t^{2m-1}}{(1-t)^{m-1}}, \tag{2.35}$$

$$\binom{n-k+m-1}{m-1} = \binom{n-k+m-1}{n-k} = \binom{1-m-1}{n-k}(-1)^{n-k} = [t^{n-k}]\frac{1}{(1-t)^m}. \tag{2.36}$$

We obviously have

$$S_{m,n} = [t^n]\frac{t^{2m-1}}{(1-t)^{2m-1}} = [t^{n-2m+1}]\frac{1}{(1-t)^{2m-1}} = \binom{n-1}{2m-2}, \tag{2.37}$$

we want now to detailed the equation (2.35)

$$\binom{k-m-1}{m-2} = \frac{(k-m-1)!}{(k-2m+1)!(m-2)!} = \binom{k-m-1}{k-2m+1}.$$

We use (1.11) to suppose that :

$$\binom{r+n-1}{n} = \binom{k-m-1}{k-2m+1},$$

then

$$r+n-1 = k-m-1 \quad \Rightarrow \quad r = m-1.$$

So

$$\binom{k-m-1}{k-2m+1} = \binom{-m+1}{k-2m+1}(-1)^{k-2m+1} = [t^k]\frac{t^{2m-1}}{(1-t)^{m-1}}. \tag{2.38}$$

Now, we want to do the same with the equation (2.36)

$$\binom{n-k+m-1}{m-1} = \frac{(n-k+m-1)!}{(n-k)!(m-1)!} = \binom{n-k+m-1}{n-k}.$$

By following the same previous steps, we obtain :

$$-r = 1-m-1$$

$$\binom{n-k+m-1}{n-k} = \binom{1-m-1}{n-k}(-1)^{n-k} = [t^{n-k}]\frac{1}{(1-t)^m}, \tag{2.39}$$

from (2.38) and (3.14), we find

$$S_{m,n} = \sum_{k=0}^{n}[t^k]\frac{t^{2m-1}}{(1-t)^{m-1}}[t^{n-k}]\frac{1}{(1-t)^m},$$

from (2.6), we have

$$S_{m,n} = [t^n]\frac{t^{2m-1}}{(1-t)^{m-1}}\frac{1}{(1-t)^m} = [t^n]\frac{t^{2m-1}}{(1-t)^{2m-1}} = [t^{n-2m+1}]\frac{1}{(1-t)^{2m-1}}.$$

By using (2.19), we find that

$$S_{m,n} = \binom{2m-1+n-2m+1-1}{n-2m+1} = \binom{n-1}{n-2m+1} = \binom{n-1}{2m-2}.$$

Example 2.5
In our view, the rule of composition stands out as the most remarkable aspect of the coefficient method. For instance, consider the Grosswald sum (2)

$$\sum_{k=0}^{n-r}(-2)^{-k}\binom{n}{r+k}\binom{n+r+k}{k} = (-1)^{(n-1)/2}2^{r-n}\binom{n}{\frac{n-r}{r}},$$

after the change of variable $k \longmapsto n-r-k$:

$$G_{n,r} = \sum_{k=0}^{n-r}(-2)^{-k}\binom{n}{r+k}\binom{n+r+k}{k} = \sum_{k=0}^{n-r}(-2)^{-n+r+k}\binom{n}{k}\binom{2n-k}{n-r-k}.$$

After the substitution of variables $k \mapsto n-r-k$, we have

$$\begin{aligned}
\binom{n}{r-k} &= \binom{n}{r-n+r+k} \\
&= \binom{n}{2r-n+k} \\
&= \frac{n!}{(2n-2r-k)!(2r-n+k)!} \\
&= \binom{n}{2n-2r-k} \\
&= \binom{n}{2n-2r-n+r-k} \\
&= \binom{n}{n-r-k} \\
&= \binom{n}{k}
\end{aligned}$$

and

$$\binom{n+r+k}{k} = \binom{n+r+n-r-k}{n-r-k} = \binom{2n-k}{n-r-k},$$

then

$$G_{n,r} = \sum_{k=0}^{n-r}(-2)^{-n+r-k}\binom{n}{k}\binom{2n-k}{n-r-k} = (-2)^{-n+r}\sum_{k=0}^{n-r}(-2)^{k}\binom{n}{k}\binom{2n-k}{n-r-k},$$

by using (2.18), we find

$$(-2)^k\binom{n}{k} = \binom{-(-n)}{k}(-2)^k = [u^k]\frac{1}{(1-2u)^{-n}} = [u^k](1-2u)^n \tag{2.40}$$

and according (2.18), we find

$$\binom{2n-k}{n-r-k} = \binom{2n-k}{n-r-k}(1)^{n-r-k} = [v^{n-r-k}](1+v)^{2n-k}, \tag{2.41}$$

from (2.40) and (2.41), we have

$$\begin{aligned} G_{n,r} &= (-2)^{-n+r}\sum_{k=0}^{n-r}[u^k](1-2u)^n[v^{n-r-k}](1+v)^{2n-k} \\ &= (-2)^{-n+r}\sum_{k=0}^{n-r}[u^k](1-2u)^n[v^{n-r}]v^k(1+v)^{2n}(1+v)^{-k} \\ &= (-2)^{-n+r}[v^{n-r}](1+v)^{2n}\sum_{k=0}^{n-r}[u^k](1-2u)^n\frac{v^k}{(1+v)^{-k}}. \end{aligned}$$

Belonging to property (2.7), we find

$$G_{n,r} = (-2)^{-n+r}(1+v)^{2n}[v^{n-r}]\left(1-\frac{2v}{1+v}\right)^n,$$

with $f(u) = (1-2u)^n$ *and* $g(v) = \dfrac{v}{1+n}$ *then*

$$G_{n,r} = (-2)^{-n+r}[v^{n-r}](1+v)^2\left(1-\frac{2v}{1+v}\right)^n.$$

Now we'll calculate the term $(1+v)^2\left(1-\dfrac{2v}{1+v}\right)$

$$\begin{aligned} (1+v)^2\left(1-\frac{2v}{1+v}\right) &= \frac{(1+v)^2(1+v)-2v(1+v)^2}{1+v} \\ &= \frac{v^2+v^3-2v^3+1+v-2v+2v+2v^2-4v^2}{1+v} \\ &= \frac{-v^2-v^3+1+v}{1+v} \\ &= \frac{(-v^2+1)(1+v)}{(1+v)} \\ &= (1-v^2), \end{aligned}$$

then

$$G_{n,r} = (-2)^{-n+r}[v^{n-r}](1-v^2)^n.$$

Now we'll calculate $[v^{n-r}]\dfrac{1}{(1-v^2)^{-n}}$ *on suppuse that* $v^2 = t$, *then*

$$[v^{n-r}]\frac{1}{(1-v^2)^{-n}} = [t^{n-r/2}]\frac{1}{(1-t)^{-n}} = \binom{n}{\frac{n-r}{2}}(-1)^{n-r/2}$$

and therefore

$$G_{n,r} = (-1)^{n-r/2}(-2)^{r-n}\binom{n}{\frac{n-r}{2}} = (-1)^{n-r/2}(-1)^{r-n}(2)^{r-n}\binom{n}{\frac{n-r}{2}}.$$

The result is

$$G_{n,r} = (-1)^{n-r/2}(-2)^{r-n}\binom{n}{\frac{n-r}{2}},$$

which is 0 *when* $n-r$ *is odd.*

We now adopt a notational convention. When we wish to denote the substitution of a formal power series $g(t)$ for the indeterminate y in $f(y)$ (i.e., to form the composition $f(g(t))$), we write

$$f(g(t)) = [f(y) \mid y = g(t)]\,,$$

which is a variant of the common notation $f(y)\mid_{y=g(t)}$. The latter notation becomes awkward when $g(t)$ is a complicated expression. The following general result deals with the so-called Euler (or binomial) transformation and involves sums containing binomial coefficients :

Theorem 2.2 ***(Euler Transformation).*** *If* $f(t)$ *belongs to* K, *then*

$$\sum_{k=0}^{n}\binom{n}{k}f_k = \sum_{k=0}^{n}\binom{n}{k}[t^k]f(t) = [t^n]\frac{1}{1-t}f\left(\frac{t}{1-t}\right). \tag{2.42}$$

Proof. From the property of binomial coefficients the symmetry (1.5), it follows that

$$\begin{aligned}\binom{n}{k} &= \binom{n}{n-k}\\ &= \binom{-(-n)}{n-k}\end{aligned}$$

and from the property of binomial coefficients (1.11)

$$\begin{aligned}\binom{n}{k} &= \binom{-n+n-k-1}{n-k}(-1)^{n-k}\\ &= \binom{-k-1}{n-k}(-1)^{n-k},\end{aligned}$$

which is the coefficient of t^{n-k} in $(1-t)^{-k-1}$. We then observe that the sum in the theorem's statement can be extended to infinity, giving :

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}f_k &= \sum_{k=0}^{n}\binom{-k-1}{n-k}(-1)^{n-k}f_k\\ &= \sum_{k=0}^{n}[t^{n-k}](1-t)^{-k-1}[y^k]f(y)\\ &= (1-t)^{-1}\sum_{k=0}^{n}[t^n]t^k(1-t)^{-k}[y^k]f(y)\\ &= \frac{1}{1-t}\sum_{k=0}^{n}\Big([y^k]f(y)\Big)\,[t^n]\left(\frac{t}{1-t}\right)^k,\end{aligned}$$

from the composition rule (2.7), we get

$$\sum_{k=0}^{n}\binom{n}{k}f_k = [t^n]\frac{1}{1-t}f\left(\frac{t}{1-t}\right).$$

■

Example 2.6

This same theorem allows us to perform a tow-line procedure to obtain the closed form of S_m, the sum reported by Greene and Knuth in [10] and quoted in the introduction.

$$S_m = \sum_{k=0}^{m}\binom{m}{k}\left(-\frac{1}{2}\right)^k\binom{2k}{k}.$$

It is well known that

$$[t^n]\frac{1}{\sqrt{1-4t}} = \binom{2n}{n},$$
$$[t^n]\frac{1}{\sqrt{1+2t}} = \left(-\frac{1}{2}\right)^n\binom{2n}{n}. \tag{2.43}$$

Hence we have

$$\sum_{k=0}^{m}\binom{m}{k}\left(-\frac{1}{2}\right)^k\binom{2k}{k} = \sum_{k=0}^{m}[t^k]\binom{m}{k}\frac{1}{\sqrt{1+2t}},$$

by using Theorem 2.2, we have

$$\begin{aligned}
\sum_{k=0}^{m}[t^k]\binom{m}{k}\frac{1}{\sqrt{1+2t}} &= [t^m]\frac{1}{1-t}\frac{1}{\sqrt{1+\dfrac{2t}{1-t}}}\\
&= [t^m]\frac{1}{1-t}\frac{1}{\sqrt{\dfrac{1-t+2t}{1-t}}}\\
&= [t^m]\frac{1}{1-t}\frac{\sqrt{1-t}}{\sqrt{1+t}}\\
&= [t^m]\frac{1}{\sqrt{1-t}\sqrt{1+t}}\\
&= [t^m]\frac{1}{\sqrt{1-t^2}}.
\end{aligned}$$

This quantity is clearly 0 when m is odd and equals $\binom{m}{m/2}/2^m$ when m is even.

3

Riordan Arrays and Combinatorial Sums

An interesting approach for evaluating combinatorial sums has emerged through the concept of Riordan arrays. These arrays serve as a generalization of two previously discussed theorems: the partial sum theorem 2.1 and Euler's transformation 2.2 [11, 18].

3.1 Combinatorial Sums

A Riordan array consists of a pair of formal power series, denoted as $D = (d(t), h(t))$. If both $d(t)$ and $h(t)$ belong to the field K_0. then the Riordan array is called proper. Throughout our discussion, we assume all Riordan arrays to be proper.

Definition 3.1.1 *A Riordan array D can be identified with the infinite, lower triangular array (or traingle) $(d_{n,k})_{n,k\geq 0}$ defined by*

$$d_{n,k} = [t^n]d(t)(th(t))^k. \tag{3.1}$$

The sequence of functions represent the column generating functions of the triangl $(d_k(t))$ is defined recursively as follows :

$$\begin{cases} d_0(t) = d(t), \\ d_k(t) = d_{k-1}(t)th(t) = d(t)(th(t))^k \quad (k = 1, 2, \ldots). \end{cases}$$

One well-known example of a Riordan array is the Pascal triangle, where both $d(t)$ and $h(t)$ are equal to $\dfrac{1}{1-t}$. This is confirmed by equation (3.1).
By using (2.4), (2.19) and (1.5), we find

$$\begin{aligned} [t^n]\frac{1}{1-t}\left(\frac{t}{1-t}\right)^k &= [t^{n-k}]\,(1-t)^{-k-1} \\ &= \binom{-k-1}{n-k}(-1)^{n-k} \\ &= \binom{k+1+n-k-1}{n-k} \\ &= \binom{n}{n-k} \\ &= \binom{n}{k}. \end{aligned}$$

The key algebraic property of Riordan arrays is their closure under the usual row-by-column product operation. When two Riordan arrays, denoted as $(d(t), h(t))$ and $(a(t), b(t))$, are multiplied together, the resulting product is also a Riordan array. This property can be proven by considering the product of the two arrays and examining its generic element, which is given by $\sum_j d_{n,j} f_{j,k}$. Here, $d_{n,j}$ represents

the generic element of $(d(t),h(t))$, and $f_{j,k}$ represents the generic element of $(a(t),b(t))$.

From (3.1) and (2.7), we have

$$\begin{aligned}\sum_{j=0}^{\infty} d_{n,j}f_{j,k} &= \sum_{j=0}^{\infty}[t^n]d(t)(th(t))^j[y^j]a(y)(yb(y))^k\\ &= [t^n]d(t)\sum_{j=0}^{\infty}(th(t))^j[y^j]a(y)(yb(y))^k\\ &= [t^n]d(t)a(th(t))(th(t)b(th(t)))^k.\end{aligned}$$

We conclude that

$$(d(t),h(t))\cdot(a(t),b(t)) = (d(t)a(th(t)),h(t)b(th(t))). \tag{3.2}$$

We suppose that $(d(t),h(t))$ is a proper Riordan array. In view of formula (3.2), we can look for a proper Riordan array $(a(t),b(t))$ such that $(d(t),h(t))\cdot(a(t),b(t)) = (1,1)$. If this is the case, we should have $d(t)a(th(t)) = 1$ and $h(t)b(th(t)) = 1$. Setting $y = th(t) \Rightarrow t = yh(t)^{-1}$ we require that

$$\begin{cases} a(y) = \dfrac{1}{d(t)} = d(t)^{-1}\\ b(y) = \dfrac{1}{h(t)} = h(t)^{-1}, \end{cases}$$

hence

$$a(y) = \left[d(t)^{-1} \quad | \quad t = yh(t)^{-1}\right], \qquad b(y) = \left[h(t)^{-1} \quad | \quad t = yh(t)^{-1}\right].$$

Theorem 3.1 *If $D = (d(t),h(t))$ is a Riordan array and $f(t)$ is the generating function of the sequence $(f_k)_{k\geq 0}$, then*

$$\sum_{k=0}^{n} d_{n,k}f_k = [t^n]d(t)f(th(t)). \tag{3.3}$$

Proof. By using (3.1), (2.2) and (2.7), we have

$$\begin{aligned}\sum_{k=0}^{n} d_{n,k}f_k &= \sum_{k=0}^{n}[t^n]d(t)(th(t))^k f_k\\ &= \sum_{k=0}^{n} d(t)[t^n](th(t))^k[t^k]f(t)\\ &= d(t)\sum_{k=0}^{n}[t^k]f(t)[t^n](th(t))^k\\ &= [t^n]d(t)f(th(t)).\end{aligned}$$

■

In the case of the Pascal triangle 3.1 reduces to the Euler transformation 2.2. More general formulas are obtained if we consider simple binomail coefficiebts (i.e.binomail coeffficients of the form $\binom{n+ak}{m+bk}$, where a and b are parameters and k is a non-negative integer varaible). If we consider n a variable and m parameter, and vice versa, we get two diffrent infinite arrays $(d_{n,k})$ and $(\widehat{d}_{m,k})$ whose elements depend on the parameters a,b,m and a,b,n, respectively. In either case if certain conditions on a and b hold, we have Riordan arrays and therefore we can applay formela (3.3) to determine the values of many sums, in this way we arrive at the following formulas :

Theoreme 3.2

$$\sum_{k=0}^{n}\binom{n+ak}{m+bk}f_k=[t^n]\frac{t^m}{(1-t)^{m+1}}f\left(\frac{t^{b-a}}{(1-t)^b}\right)\quad(b>a),\tag{3.4}$$

$$\sum_{k=0}^{n}\binom{n+ak}{m+bk}f_k=[t^m](1+t)^n f(t^{-b}(1+t)^a)\quad(b<0).\tag{3.5}$$

Proof.

1/ Proof (3.4) by using (2.2), (1.11), (2.4) and (2.7) respectively, then

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n+ak}{m+bk}f_k&=\sum_{k=0}^{n}[t^k]f(t)\binom{n+ak}{m+bk}\\
&=\sum_{k=0}^{n}[t^k]f(t)\binom{n+ak}{n-m+k(a-b)}\\
&=\sum_{k=0}^{n}[t^k]f(t)\binom{-n-ak+n-m+ak-bk-1}{n-m+k(a-b)}(-1)^{n-m+k(a-b)}\\
&=\sum_{k=0}^{n}[t^k]f(t)\binom{-(m+bk+1)}{n-m+k(a-b)}(-1)^{n-m+k(a-b)}\\
&=\sum_{k=0}^{n}[t^k]f(t)[t^{n-m+k(a-b)}]\frac{1}{(1-t)^{m+1+bk}}\\
&=\sum_{k=0}^{n}[t^k]f(t)[t^n]\frac{t^m t^{k(b-a)}}{(1-t)^{m+1}(1-t)^{bk}}\\
&=\sum_{k=0}^{n}[t^k]f(t)[t^n]\frac{t^m}{(1-t)^{m+1}}\left(\frac{t^{b-a}}{(1-t)^b}\right)^k\\
&=[t^n]\frac{t^m}{(1-t)^{m+1}}f\left(\frac{t^{b-a}}{(1-t)^b}\right)\quad(b>a).
\end{aligned}$$

2/ Proof (3.5) using the same steps

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n+ak}{m+bk}f_k&=\sum_{k=0}^{n}[t^k]f(t)\binom{n+ak}{m+bk}\\
&=\sum_{k=0}^{n}[t^k]f(t)\binom{n+ak}{m+bk}(1)^{m+bk}\\
&=\sum_{k=0}^{n}[t^k]f(t)[t^m](t^{-b}(1+t)^a)^k(1+t)^n\\
&=[t^m](1+t)^n f(t^{-b}(1+t)^a)\quad(b<0).
\end{aligned}$$

■

Example 3.1

As a simple example, we consider the following sum, in which we applay (3.5), (2.18), (2.4), (2.15)

(2.7), (2.13) and (2.3) respectivly, with n = 0, m = n, a = 1, and b = -1 :

$$\begin{aligned}
\sum_{k=n/2}^{n}\binom{k}{n-k}\frac{1}{k} &= \sum_{k=n/2}^{n}\frac{1}{k}\binom{k}{n-k}(1)^{n-k}\\
&= \sum_{k=n/2}^{n}\frac{1}{k}[t^{n-k}](1+t)^k\\
&= \sum_{k=n/2}^{n}[t^n]\frac{t^k}{k}(1+t)^k\\
&= \sum_{k=n/2}^{n}[t^k]\ln\left(\frac{1}{1-t}\right)[t^n]t^k(1+t)^k\\
&= \sum_{k=n/2}^{n}[t^k]\ln\left(\frac{1}{1-t}\right)[t^n](t(1+t))^k\\
&= [t^n]\left[\ln\frac{1}{1-y} \quad | \quad y=t(1+t)\right]\\
&= [t^n]\ln\frac{1}{1-t-t^2}\\
&= \frac{1}{n}[t^{n-1}]\frac{1+2t}{1-t-t^2}\\
&= \frac{1}{n}[t^n]\frac{t+2t^2}{1-t-t^2}\\
&= \frac{1}{n}\left([t^n]\frac{t}{1-t-t^2}+[t^{n-1}]\frac{2t}{1-t-t^2}\right)\\
&= \frac{1}{n}(F_n+2F_{n-1})\\
&= \frac{F_n+F_{n-1}+F_{n-1}}{n}\\
&= \frac{F_{n+1}+F_{n-1}}{n},
\end{aligned}$$

where $F_n = [t^n]t/(1-t-t^2)$ is the n^{th} Fibonachi number.

Example 3.2
For a recent application of formula (3.5) in proving the identity :

$$(x+m+1)\sum_{k=0}^{m}(-1)^k\binom{x+y+k}{m-k}\binom{y+2k}{k}-\sum_{k=0}^{m}\binom{x+k}{m-k}(-4)^k=(x-m)\binom{x}{m}.$$

We have

$$d_{n,k}=[t^n]d(t)(th(t))^k$$

and by using (2.4), we find

$$\begin{aligned}
d_{n,k} &= \binom{r+ak}{n+bk}\\
&= [t^{n+bk}](1+t)^{r+ak}\\
&= [t^n]t^{-bk}(1+t)^{r+ak}\\
&= [t^n](1+t)^r\left(\frac{t^b}{(1+t)^a}\right)^k,
\end{aligned}$$

we have

$$D = (d(t), h(t)),$$

then

$$D = \left((1+t)^r, \frac{(1+t)^a}{t^{b+1}}\right).$$

For all sequances f_k having $f(t)$ as generating function we find :

$$\sum_{k=0}^{n} \binom{r+ak}{n+bk} f_k = [t^n](1+t)^r f(t^{-b}(1+t)^a) \qquad (b<0),$$

$$f_k = \binom{y+2k}{k}(-1)^k = [t^k](1-t)^{y+2k} = [t^k](1-t)^y(1-t)^{2k},$$

we have from (2.8)

$$[t^n]\overline{f}(t)^k = \frac{k}{n}[t^{n-k}]\left(\frac{t}{f(t)}\right)^n.$$

By using the Lagrange Inversion theorem , we have

$$[t^n]F(t)\Phi(t)^n = [t^n]\left[\frac{F(w)}{1-t\Phi'(w)} \quad | \quad w = t\Phi(w)\right],$$

then

$$[t^k](1-t)^y((1-y)^2)^k = [t^k]\left[\frac{(1-y)^y}{1+2t(1-w)} \quad | \quad w = t(1-w)^2\right]$$

and

$$w = t(1-w)^2 \quad \Rightarrow \quad w = tw^2 + t - 2tw,$$

$$tw^2 + t - 2tw = 0,$$

therefore

$$w_1 = \frac{1+2t-\sqrt{1+4t}}{2t},$$
$$w_2 = \frac{1+2t+\sqrt{1+4t}}{2t},$$

then

$$[t^k](1-t)^y((1-t)^2)^k = [t^k]\frac{\left(1-\dfrac{1+2t-\sqrt{1+4t}}{2t}\right)^y}{1+2t\left(1-\dfrac{1+2t-\sqrt{1+4t}}{2t}\right)} = [t^k]\frac{\left(\sqrt{1+4t}-1\right)^y}{(2t)^y\sqrt{1+4t}}$$

$$\begin{aligned}\sum_{k=0}^{m}(-1)^k\binom{x+y+k}{m-k}\binom{y+2k}{k} &= \sum_{k=0}^{m}[t^n](x+y)^{x+y}\left(t(1+t)\right)^k[t^k](1-t)^y(1-t)^{2k} \\ &= \sum_{k=0}^{m}[t^k](1+t)^{x+y}\frac{\left(\sqrt{1+4t}-1\right)^y}{(2t)^y\sqrt{1+4t}}[t^m]\left(t(1+t)\right)^k.\end{aligned}$$

We have $g(t) = t(1+t)$, $f(t) = (1+t)^{x+y}\dfrac{\left(\sqrt{1+4t}-1\right)^y}{(2t)^y\sqrt{1+4t}}$,

then, from (2.7), we find

$$
\begin{aligned}
[t^m]f(g(t)) &= (1+t)^{x+y}\frac{\left(\sqrt{1+4t+4t^2}-1\right)^y}{(2t(1+t))^y\sqrt{1+4t+4t^2}} \\
\sum_{k=0}^{m}\binom{x+y+k}{m-k}f_k &= \sum_{k=0}^{m}\binom{x+y+k}{m-k}\binom{y+2k}{k}(-1)^k \\
&= [t^m](1+t)^{x+y}f(t(1+t)) \\
&= [t^m](1+t)^{x+y}\frac{\left(\sqrt{1+4t(1+t)}-1\right)^y}{(2t(1+t))^y\sqrt{1+4t(1+t)}} \\
&= [t^m](1+t)^x(1+t)^y\frac{(2y)^y}{(2t)^y(1+t)^y(1+2t)},
\end{aligned}
$$

therefore

$$
\sum_{k=0}^{m}(-1)^k\binom{x+y+k}{m-k}\binom{y+2k}{k} = [t^m]\frac{(1+t)^x}{1+2t}.
$$

Now we'll calculate $\displaystyle\sum_{k=0}^{m}\binom{x+k}{m-k}(-4)^k$

we have

$$
(-4)^k = \binom{1+k-1}{k}(-4)^k = \binom{-1}{k}(4)^k = [t^k]\frac{1}{1+4t} = [t^k]f(t)
$$

and

$$
\binom{x+k}{m-k} = [t^m](1+t)^x f\left(t(1+t)\right),
$$

then

$$
\sum_{k=0}^{m}\binom{x+k}{m-k}(-4)^k = [t^m](1+t)^x\frac{1}{1+4t+4t^2} = [t^m]\frac{(1+t)^x}{(1+2t)^2}.
$$

Finally, we get

$$
\begin{aligned}
&(x+m+1)\sum_{k=0}^{m}(-1)^k\binom{x+y+k}{m-k}\binom{y+2k}{k} - \sum_{k=0}^{m}\binom{x+k}{m-k}(-4)^k \\
&= (x+m+1)[t^m]\frac{(1+t)^x}{1+2t} - [t^m]\frac{(1+t)^x}{(1+2t)^2} \\
&= (x+1)[t^m]\frac{(1+t)^x}{1+2t} + m[t^m]\frac{(1+t)^x}{1+2t} - [t^m]\frac{(1+t)^x}{(1+2t)^2} \\
&= (x+1)[t^m]\frac{(1+t)^x}{1+2t} + [t^{m-1}]\frac{dt\dfrac{(1+t)^x}{1+2t}}{dt} - [t^m]\frac{(1+t)^x}{(1+2t)^2} \\
&= (x+1)[t^m]\frac{(1+t)^x}{1+2t} + [t^m]t\left(\frac{x(1+t)^{x-1}(1+2t)-2(1+t)^x}{(1+2t)^2}\right) - [t^m]\frac{(1+t)^x}{(1+2t)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= (x+1)[t^m]\frac{(1+t)^x}{1+2t} + x[t^m]t\frac{(1+t)^{x-1}}{1+2t} - [t^m]2t\frac{(1+t)^x}{(1+2t)^2} - [t^m]\frac{(1+t)^x}{(1+2t)^2}\\
&= (x+1)[t^m]\frac{(1+t)^x}{1+2t} + x[t^m]t\frac{(1+t)^{x-1}}{1+2t} - [t^m]\frac{(1+t)^x}{(1+2t)}\\
&= x[t^m]\frac{(1+t)^x}{1+2t} + x[t^m]t\frac{(1+t)^{x-1}}{1+2t}\\
&= x[t^m]\frac{(1+t)^x}{1+2t}\left(1+\frac{t}{1+t}\right)\\
&= x[t^m]\frac{(1+t)^x}{1+2t}\frac{1+2t}{1+t}\\
&= x[t^m](1+t)^{x-1}\\
&= x\binom{x-1}{m}\\
&= x\frac{(x-1)!}{(x-1-m)!m!}\\
&= \frac{x!}{(x-1-m)!m!}\\
&= (x-m)\frac{x!}{(x-m)(x-1-m)!m!}\\
&= (x-m)\frac{x!}{(x-m)!m!}\\
&= (x-m)\binom{x}{m}.
\end{aligned}
$$

Example 3.3

These examples of combinatorial sums derived from formulas (3.4) and (3.5), very important examples of Riordan arrays are given by the Stirling numbers of both kinds. If we set

$$s_{n,k} = \frac{k!}{n!}s(n,k), \tag{3.6}$$

$$S_{n,k} = \frac{k!}{n!}\left\{{n \atop k}\right\}, \tag{3.7}$$

$$(s_{n,k}) = \left(1, \frac{1}{t}\ln\frac{1}{1-t}\right), \tag{3.8}$$

$$(S_{n,k}) = \left(1, \frac{e^t-1}{t}\right). \tag{3.9}$$

Here we limit ourselves to demonstrating the orthogonality property of the two arrays : by using (3.6), (3.7), (3.8), (3.9), (3.1), (2.7) and (2.1), we have

$$
\begin{aligned}
\sum_{k=0}^{n} s(n,k)\left\{{k \atop m}\right\}(-1)^{n-k} &= \sum_{k=0}^{n}\frac{n!}{n!}\frac{m!}{m!}\frac{k!}{k!}s(n,k)\left\{{k \atop m}\right\}(-1)^n(-1)^k\\
&= (-1)^n\frac{n!}{m!}\sum_{k=0}^{n}\frac{k!}{n!}s(n,k)\frac{m!}{k!}\left\{{k \atop m}\right\}(-1)^k\\
&= (-1)^n\frac{n!}{m!}\sum_{k=0}^{n}(s_{n,k})(S_{n,k})(-1)^k\\
&= (-1)^n\frac{n!}{m!}\sum_{k=0}^{n}[t^n]\left(\ln\frac{1}{1-t}\right)^k[y^k](e^t-1)^m(-1)^k\\
&= (-1)^n\frac{n!}{m!}\sum_{k=0}^{n}[t^n](e^t-1)^m[y^k]\left(-\ln\frac{1}{1-t}\right)^k
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^n \frac{n!}{m!}[t^n]\left[(e^y-1)^m \mid y = \ln\left(\frac{1}{1-t}\right)\right] \\
&= (-1)^n \frac{n!}{m!}[t^n](e^{\ln(1-t)}-1)^m \\
&= (-1)^n \frac{n!}{m!}[t^n](1-t-1)^m \\
&= (-1)^n \frac{n!}{m!}[t^n](-t)^m \\
&= (-1)^{n+m} \frac{n!}{m!} \\
&= \delta_{n,m}.
\end{aligned}
$$

Example 3.4
Another orthogonality property is proved in an analogous way. Namely, the reader is invited to prove the following identity :
By using the same steps

$$
\begin{aligned}
\sum_{k=0}^{n} s(n,k)\left\{ {k \atop m} \right\} &= \frac{n!}{m!}\sum_{k=0}^{n}\frac{k!}{n!}s(n,k)\frac{m!}{k!}\left\{ {k \atop m} \right\} \\
&= \frac{n!}{m!}\sum_{k=0}^{n}[t^n]\left(\ln\frac{1}{1-t}\right)^k [y^k](e^t-1)^m \\
&= \frac{n!}{m!}[t^n]\left(e^{\ln(1-t)}-1\right)^m \\
&= \frac{n!}{m!}[t^n]\left(\frac{t}{1-t}\right)^m \\
&= \frac{n!}{m!}[t^{n-m}]\frac{1}{(1-t)^m} \\
&= \frac{n!}{m!}\binom{-m}{n-m}(-1)^{n-m} \\
&= \frac{n!}{m!}\binom{n-1}{m-1}.
\end{aligned}
$$

Example 3.5
Finally, we consider the following sum, whose generating function is immediately found by the method of coefficients :

$$
\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k} = [t^n]\frac{1}{1-t}\left[\frac{1}{\sqrt{1-4t}} \mid y = \frac{t}{1-t}\right] = [t^n]\frac{1}{\sqrt{(1-t)(1-5t)}}.
$$

We have

$$
\binom{2k}{k} = [t^k]\frac{1}{\sqrt{1-4t}},
$$

and by using Theorem 2.2 in which $f_k = [t^k]\dfrac{1}{\sqrt{1-4t}}$, *we find*

$$
\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k} &= \sum_{k=0}^{n}\binom{n}{k}[t^k]\frac{1}{\sqrt{1-4t}} \\
&= [t^n]\frac{1}{1-t}\left[\frac{1}{\sqrt{1-4t}} \mid y = \frac{t}{1-t}\right]
\end{aligned}
$$

$$= [t^n]\frac{1}{1-t}\frac{1}{\sqrt{1-\dfrac{4t}{1-t}}}$$
$$= [t^n]\frac{1}{1-t}\frac{1}{\dfrac{\sqrt{1-5t}}{\sqrt{1-t}}}$$
$$= [t^n]\frac{1}{\sqrt{(1-t)(1-5t)}}.$$

3.2 Inversion

As already mentioned, the compositional inverse $\bar{f}(t)$ of a formal power series $f(t)$ is defined by the relation $f(\bar{f}(t)) = \bar{f}(f(t)) = t$. The form of $\bar{f}(t)$ is best determined by the Lagrange inversion formula, which is our rule (2.8). There exist many proofs of the Lagrange inversion formula in terms of formal Laurent series, for which the reader is referred to [11] The method of coefficients can be used to prove the Lagrange inversion formula, via the following steps :

1. Consider the proper Riordan array $F = (1, f(t)/t)$ whose columns are the successive powers of $f(t)$.
2. Observe that the inverse Riordan array F^{-1} is just $(1, \bar{f}(t)/t)$, whose columns are the successive powers of $\bar{f}(t)$.
3. Verify directly that the array D defined by
$$d_{n,k} = \frac{k}{n}[t^{n-k}]\left(\frac{t}{f(t)}\right)^n,$$
is also the inverse of F.
4. Conclude that $D = F^{-1}$ and thereby deduce that formula (2.8) is correct for every n and every k.

There is another way to apply the Lagrange inversion formula. Suppose that we have a functional equation $\omega = t\phi(\omega)$, where $\phi(t)$ belongs to K_0, and that we wish to find the formal power series $\omega = \omega(t)$ satisfying this functional equation. Clearly $\omega(t)$ lies in K_1 and if we set $f(y) = y/\phi(y)$, we also have $f(t)$ in K_1. However, the functional equation can be written $f(\omega(t)) = t$, which shows that $\omega(t)$ is the compositional inverse of $f(t)$. We therefore know that $\omega(t)$ is uniquely determined, and we learn from the Lagrange inversion formula that

$$[t^n]\omega(t) = \frac{1}{n}[t^{n-1}]\left(\frac{t}{f(t)}\right)^n = \frac{1}{n}[t^{n-1}]\phi(t), \tag{3.10}$$

we have

$$f(\omega(t)) = t \quad \Rightarrow \bar{f}(f(\omega(t))) = \bar{f}(t) \Rightarrow \omega(t) = \bar{f}(t)$$

and we have

$$f(t) = \frac{t}{\phi(t)} \quad \Rightarrow \phi(t) = \frac{t}{f(t)},$$

then

$$\frac{1}{n}[t^{n-1}]\left(\frac{t}{f(t)}\right)^n = \frac{1}{n}[t^{n-1}]\left(\phi(t)\right)^n.$$

The Lagrange inversion formula can also give us the coefficients of the powers $\omega(t)^k$, but we can obtain an even more general result. Let $F(t)$ be a member of K and consider the composition $F(\omega(t))$, where

$\omega = \omega(t)$ is, as before, the solution to the functional equation $\omega = t\phi(\omega)$ for given $\phi(\omega)$ in K_0. For the coefficient of t^n in $F(\omega(t))$ we have by using (2.2), (3.10), (2.4) and (1.21)

$$\begin{aligned}
[t^n]F(\omega(t)) &= [t^n]\sum_{k=0}^{\infty} F_k\omega(t)^k \\
&= \sum_{k=0}^{\infty} F_k[t^n]\omega(t)^k \\
&= \sum_{k=0}^{\infty} F_k\frac{k}{n}[t^{n-k}]\left(\phi(t)\right)^n \\
&= \sum_{k=0}^{\infty} F_k\frac{k}{n}[t^{n-1}]t^{k-1}\phi(t)^n \\
&= \frac{1}{n}[t^{n-1}]\phi(t)^n\sum_{k=0}^{\infty} kF_kt^{k-1} \\
&= \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n,
\end{aligned}$$

because $\displaystyle\sum_{k=0}^{\infty} kF_kt^{k-1} = F'(t)$.

Theorem 3.3 *If $F(t)$ and $\phi(t)$ belong to K, then*

$$[t^n]F(t)\phi(t)^n = [t^n]\left[\frac{F(\omega)}{1-t\phi'(\omega)} \mid \omega = t\phi(\omega)\right]. \tag{3.11}$$

Proof. We first observe that

$$\begin{aligned}
[t^n]F(t)\phi(t)^n &= \frac{n}{n}[t^{n-1}]\frac{F(t)}{t}\phi(t)^n \\
&= \frac{n}{n}[t^{n-1}]\left(\sum_{k=0}^{\infty} kF_kt^{k-1}\right)\frac{\phi(t)^n}{t} \\
&= n\sum_{k=0}^{\infty} F_k\frac{k}{n}[t^{n-(-k)}]\phi(t)^n \\
&= n\sum_{k=0}^{\infty} F_k[t^n]\omega(t)^{k-1} \\
&= n[t^n]\sum_{k=0}^{\infty}\frac{F_k\omega(t)^k}{\omega(t)} \\
&= n[t^n]\int\frac{F(\omega(t))}{\omega(t)}d\omega(t),
\end{aligned}$$

we suppose $\omega(t) = y$

$$\begin{aligned}
[t^n]F(t)\phi(t)^n &= n[t^n]\left[\int\frac{F(y)}{y}dy \mid y = \omega(t)\right] \\
&= \frac{n}{n}[t^{n-1}]\frac{d}{dt}\left[\int\frac{F(y)}{y}dy \mid y = \omega(t)\right],
\end{aligned}$$

by using (2.8)

$$[t^n]F(t)\phi(t)^n = [t^{n-1}]\frac{d}{dt}\int\frac{F(\omega)}{\omega}d\omega$$

$$[t^n]F(t)\phi(t)^n = [t^{n-1}]\left[\frac{F(\omega)}{\omega} \mid \omega = t\phi(\omega)\right]\frac{d\omega}{dt} \tag{3.12}$$

and we have

$$\begin{aligned}\frac{d\omega}{dt} &= \frac{dt\phi(\omega)}{dt}\\ \frac{d\omega}{dt} &= \phi(\omega) + t\omega'(t)\phi'(\omega)\\ \frac{d\omega}{dt} &= \phi(\omega) + t\frac{d\omega}{dt}\phi'(\omega),\end{aligned}$$

by moving $t\frac{d\omega}{dt}\phi'(\omega)$ to the other side, we find

$$\begin{aligned}\frac{d\omega}{dt} - t\frac{d\omega}{dt}\phi'(\omega) &= \phi(\omega)\\ \frac{d\omega}{dt}(1 - t\phi'(\omega)) &= \phi(\omega),\end{aligned}$$

then

$$\frac{d\omega}{dt} = \frac{\phi(\omega)}{1 - t\phi'(\omega)}, \tag{3.13}$$

by replacing (3.13) in (3.12), we find

$$\begin{aligned}[t^n]F(t)\phi(t)^n &= [t^{n-1}]\left[\frac{F(\omega)}{\omega}\frac{\phi(\omega)}{1 - t\phi'(\omega)} \mid \omega = t\phi(\omega)\right]\\ &= [t^{n-1}]\frac{1}{t}\left[\frac{F(\omega)}{1 - t\phi'(\omega)} \mid \omega = t\phi(\omega)\right],\end{aligned}$$

where $\frac{\phi(\omega)}{\omega} = \frac{1}{t}$,
then, from proprty (2.4) :

$$[t^n]F(t)\phi(t)^n = [t^n]\left[\frac{F(\omega)}{1 - t\phi'(\omega)} \mid \omega = t\phi(\omega)\right].$$

■

Example 3.6
The name "diagonalization" stems from the fact that if we imagine the coefficients of $F(t)\phi(t)^n$ as constituting row n in an infinite matrix, then the $[t^n]F(t)\phi(t)^n$ are just the elements in the main diagonal of this array. One of its simplest applications is determining the generating function of the central binomial coefficients. Since plainly

$$\binom{2n}{n} = [t^n](1+t)^{2n}.$$

Theorem 3.3 can be applied with $F(t) = 1$ and $\phi(t) = (1+t)^{2n}$. In this case, the function $\omega = \omega(t)$ is easily determined by solving the functional equation $\omega = t(1+\omega)^{2n}$. Expanding, we find that

$$t\omega^2 - (1 - 2t)\omega + t = 0,$$

by solving this equation, we find :

$$\omega = \omega(t) = \frac{1 - 2t \pm \sqrt{1 - 4t}}{2t}.$$

Since $\omega = \omega(t)$ should belong to K_1, we must eliminate the solution with the + sign. Consequently, using $\mathcal{G}$ to signify the generating function of a sequence, we have the following well-known result :

$$\begin{aligned}
\mathcal{G}\left\{\binom{2n}{n}\right\} &= \left[\frac{F(\omega)}{1-t\phi'(\omega)} \mid \omega = \frac{1-2t-\sqrt{1-4t}}{2t}\right] \\
&= \left[\frac{1}{1-2t(1+\omega)} \mid \omega = \frac{1-2t-\sqrt{1-4t}}{2t}\right] \\
&= \frac{1}{1-2t\left(1+\dfrac{1-2t-\sqrt{1-4t}}{2t}\right)} \\
&= \frac{1}{1-2t\left(\dfrac{2t+1-2t-\sqrt{1+4t}}{2t}\right)} \\
&= \frac{1}{\dfrac{2t-2t+2t\sqrt{1+4t}}{2t}} \\
&= \frac{1}{\sqrt{1+4t}}.
\end{aligned}$$

Example 3.7
As a more complex example, we prove the famous Abel identity :

$$S_n = \sum_{k=0}^{n} \binom{n}{k} a(a+k)^{k-1}(b+n-k)^{n-k} = (a+b+n)^n.$$

by using Deninition 1.2.1, we can write

$$\begin{aligned}
S_n &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a(a+k)^{k-1}(b+n-k)^{n-k} \\
&= n! \sum_{k=0}^{n} \frac{a(a+k)^{k-1}}{k!} \cdot \frac{(b+n-k)^{n-k}}{(n-k)!} \\
&= n! \sum_{k=0}^{n} a(a+k) \frac{a(a+k)^{k}}{k!} \cdot \frac{(b+n-k)^{n-k}}{(n-k)!},
\end{aligned}$$

then from the exponential series (2.14), we get :

$$e^{(b+k)t} = \sum_{k=0}^{\infty} \frac{(b+k)^k}{k!},$$

therefore

$$\frac{(b+k)^k}{k!} = [t^k]e^{(b+k)t} = [t^k]e^{bt}e^{kt} = [t^k]e^{bt}(e^t)^k.$$

Then $F(t) = e^{bt}$ and $\phi(t) = e^t$

$$\mathcal{G}\left\{\frac{(b+k)^k}{k!}\right\} = \left[\frac{e^{b\omega}}{1-te^{\omega}} \mid \omega = te^{\omega}\right] = \left[\frac{e^{b\omega}}{1-\omega} \mid \omega = te^{\omega}\right], \tag{3.14}$$

we have

$$\frac{a(a+k)^{k-1}}{k!} = \frac{(b+k)^k}{k!} - \frac{(b+k)^{k-1}}{(k-1)!} - \cdots$$

then from the exponential series (2.14), we get :

$$\frac{(b+k)^k}{k!} = [t^k]e^{(b+k)t} \quad \text{and} \quad \frac{(b+k)^{k-1}}{(k-1)!} = [t^{k-1}]e^{(b+k)t},$$

then

$$\begin{aligned}\frac{(b+k)^k}{k!} - \frac{(b+k)^{k-1}}{(k-1)!} &= [t^k]e^{(b+k)t} - [t^{k-1}]e^{(b+k)t} \\ &= [t^k]e^{(b+k)t} - [t^k]te^{(b+k)t},\end{aligned}$$

by using (3.1) and Thereom 3.3, we find :

$$\begin{aligned}\frac{(b+k)^k}{k!} - \frac{(b+k)^{k-1}}{(k-1)!} &= [t^k](1-t)e^{(b+k)t} \\ &= [t^k](1-t)e^{at}(e^t)^k,\end{aligned}$$

$F(t) = (1-t)e^{at}$ *and* $\phi(t) = e^t$, *then*

$$\mathcal{G}\left\{\frac{a(a+k)^{k-1}}{(k-1)!}\right\} = \left[\frac{(1-\omega)e^{a\omega}}{1-te^{\omega}} \mid \omega = te^{\omega}\right] = \left[e^{a\omega} \mid \omega = te^{\omega}\right], \tag{3.15}$$

by using (3.14), we find

$$\begin{aligned}S_n &= [t^n]\sum_{k=0}^{\infty}\frac{a(a+k)^k}{k!}\frac{(a+n-k)^{n-k}}{(n-k)!} \\ &= [t^n]\mathcal{G}\left\{\frac{a(a+k)^{k-1}}{(k-1)!}\right\}\mathcal{G}\left\{\frac{(b+k)^k}{k!}\right\} \\ &= [t^n]\left[\frac{e^{b\omega}}{1-\omega}e^{a\omega} \mid \omega = te^{\omega}\right] \\ &= [t^n]\left[\frac{e^{(a+b)\omega}}{1-\omega} \mid \omega = te^{\omega}\right] \\ &= \frac{(a+b+n)^n}{n!},\end{aligned}$$

after applying the diagonalization rule backwards (since $1-\omega = 1-te^{\omega}$*,as already seen in (3.15)). A systematic approach of this type has been used in [11] to prove a number of identities related to Abel's and Gould's formulas.*

Example 3.8
We conclude this thesis by returning to Riordan arrays in order to illustrate how that concept, together with the technique of diagonalization, allows one to solve combinatorial inversions. Let(a_n) *be a sequence defined by an infinite set of equations*

$$a_n = \sum_{k=0}^{n} f_{n,k}b_k,$$

where the sequence (b_n) *is unknown. The problem of inversion is to find the values of* b_n *in terms of the* a_n*, that is, to express* b_n *in the form*

$$b_n = \sum_{k=0}^{n} g_{n,k}a_k.$$

As an illustrative example, we choose inversion (2.4.2) in Riordan's book [11]. Although it is not particularly important, it does provide an application of Riordan arrays that we have not considered in our previous examples. We consider

$$a_n = \sum_{k=0}^{n} \left(\binom{n}{k} - (c-1)\binom{n}{k-1} \right) b_{n-ck}$$

Let $a(t)$ and $b(t)$ be the generating functions of the two sequences. The relevant Riordan array R is given by $b_{n-ck} = [t^n]b(t)(t^c)^k$, whence $R = (b(t), t^{c-1})$. The generating function of the other term is determined by

$$\begin{aligned} f_k &= \binom{n}{k} - (c-1)\binom{n}{k-1} \\ &= [t^k](1+t)^n - (c-1)[t^{k-1}](1+t)^n \\ &= [t^k](1+t)^n - (c-1)[t^k]t(1+t)^n \\ &= [t^k](1+t)^n \left(1 - t(c-1)\right), \end{aligned}$$

by using (3.3), we get

$$a_n = [t^n]b(t)f(t \cdot t^{c-1}) = [t^n]b(t)(1+t^c)^n(1-t^c(c-1)),$$

then $F(t) = b(t)(1-t^c(c-1))$, and $\phi(t) = 1+t^c$,
therefore

$$\begin{aligned} a(t) &= \left[\frac{b(\omega)(1-\omega^c(c-1)}{1-tc\omega^{c-1}} \mid \omega = t(1-\omega^c) \Rightarrow t = \frac{\omega}{1+\omega^c} \right] \\ a\left(\frac{\omega}{1+\omega^c}\right) &= \frac{b(\omega)(1-\omega^c(c-1)}{1-\dfrac{c\omega^c}{1+\omega^c}} \\ &= \frac{b(\omega)(1-\omega^c(c-1))}{\dfrac{1-\omega^c(c-1)}{1+\omega^c}} \\ &= b(\omega)(1+\omega^c), \end{aligned}$$

so

$$b(\omega) = \frac{1}{1+\omega^c} a\left(\frac{\omega}{1+\omega^c}\right) = d(\omega)a\left(\omega h(\omega)\right), \tag{3.16}$$

then

$$B = (d(\omega), h(\omega)) = \left(\frac{1}{1+\omega^c}, \frac{1}{1+\omega^c}\right).$$

This, however, is the transformation related to the Riordan array

$$\left(\frac{1}{1+\omega^c}, \frac{1}{1+\omega^c}\right),$$

the generic element of which is

$$d_{n,k} = [\omega^n]\frac{1}{1+\omega^c} \cdot \frac{\omega^k}{(1+\omega^c)^k},$$

by using (2.2), (2.4), (2.19) and (3.1), we find

$$\begin{aligned} d_{n,k} &= [\omega^{n-k}]\frac{1}{(1+\omega^c)^{k+1}} \\ &= [\omega^{n-k}](1+\omega^c)^{-k-1} \\ &= [\omega^{n-k}]\sum_{j=0}^{n}\binom{-k-1}{j}\omega^{cj} \\ &= [\omega^{n-k}]\sum_{j=0}^{n}\binom{k+1+j-1}{j}(-1)^j\omega^{cj} \\ &= \sum_{j=0}^{n}[\omega^{n-k-cj}]\binom{k+j}{j}(-1)^j \\ &= \binom{n-k-cj+k+j}{j}(-1)^j \\ &= \binom{n-j(c-1)}{j}(-1)^j. \end{aligned}$$

Since the sum reduces to the case $n-k=cj$. If we now write k instead of j and pass to the coefficients, equation (3.16) becomes

$$b_n = \sum_{k=0}^{n}\binom{n-k(c-1)}{k}(-1)^k a_{n-ck}.$$

Conclusion

Conclusion

In this thesis, we studied the Method of Coefficient, and we provided some practical examples for it. Then we observed the relation between the Riordan arrays and this method, where we were able to apply it in Riordan arrays calculations.

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