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On Cauchy Numbers and Their Polynomials

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This achievement is the fruit of a collective effort, and I am grateful to everyone who contributed to its realization. This achievement is a shared endeavor, and I am grateful to all who contributed to its realization.ealization.

Dedication

I dedicate this work to my beloved family: my mother, my father, and my siblings, whose unwavering support, encouragement, and patience have been the foundation of my academic journey. I also extend my deepest gratitude to everyone who believed in my abilities and offered guidance and mentorship throughout the preparation of this thesis. To all who contributed, directly or indirectly, to the completion of this work, I offer my sincere thanks and appreciation, as well as to a number of my friends, whom I thank for their wonderful support and encouragement.

Abstract

Cauchy numbers are a special sequence of numbers used in mathematics, especially in analysis and combinatorics. They are defined using a generating function related to the natural logarithm. These numbers appear in formulas for approximations, finite differences, and expansions of functions. They are also linked to other well-known number sequences, such as Bernoulli and Stirling numbers. Because of these connections, Cauchy numbers are useful in studying relationships between discrete and continuous mathematics. Their properties make them helpful in areas like numerical methods and theoretical research.

This thesis generalizes these results by introducing **p-Cauchy numbers** and **Cauchy polynomials of both kinds**, establishing their integral representations, hypergeometric generating series, recurrence relations, as well as their connections with r-Stirling numbers and Bernoulli polynomials. The study highlights the Appell-type nature of these polynomials and opens perspectives toward q-analogues and multivariate generalizations.

Keywords: Generating function, explicit formula, recurrence relation, Stirling, Bernoulli and Bell numbers, Cauchy numbers and polynomials, p-Cauchy numbers.

الملخص

أعداد كوشي هي متتالية عددية خاصة تُستخدم في الرياضيات، لا سيما في التحليل الرياضي والتوافقيات. تُعرَّف هذه الأعداد باستخدام دالة مولِّدة مرتبطة باللوغاريتم الطبيعي. تظهر أعداد كوشي في صيغ التقريبات، والفروق المحدودة، ومفكوكات الدوال. كما ترتبط بمتتاليات عددية معروفة أخرى، مثل أعداد برنولي وستيرلينغ. وبسبب هذه الصلات، تكون أعداد كوشي مفيدة في دراسة العلاقات بين الرياضيات المتقطعة والمتصلة. وتجعلها خصائصها أدوات قيّمة في مجالات كالطرق العددية والبحث النظري.

تُعمِّم هذه الرسالة تلك النتائج من خلال تقديم **أعداد كوشي-**p **ومتعددات حدود كوشي من النوعين**، مع إرساء تمثيلاتها التكاملية، ومتسلسلاتها المولِّدة الهندسية الفائقة، وعلاقاتها الاسترجاعية، بالإضافة إلى صلاتها بأعداد ستيرلينغ-r ومتعددات حدود برنولي. وتسلط الدراسة الضوء على الطبيعة من نمط أبيل لهذه متعددات الحدود، وتفتح آفاقًا نحو المماثلات-q والتعميمات متعددة المتغيرات.

الكلمات المفتاحية: دالة مولِّدة، صيغة صريحة، علاقة استرجاعية، أعداد ستيرلينغ وبرنولي وبيل، أعداد ومتعددات حدود كوشي، أعداد كوشي-p.

Résumé

Les nombres de Cauchy constituent une suite particulière de nombres utilisés en mathématiques, notamment en analyse et en combinatoire. Ils sont définis à l'aide d'une fonction génératrice liée au logarithme népérien. Ces nombres interviennent dans les formules d'approximation, de différences finies et de développements de fonctions. Ils sont également liés à d'autres suites numériques bien connues, telles que les nombres de Bernoulli et de Stirling. De par ces liens, les nombres de Cauchy sont utiles pour étudier les relations entre les mathématiques discrètes et continues. Leurs propriétés les rendent précieux dans des domaines comme les méthodes numériques et la recherche théorique.

Ce mémoire généralise ces résultats en introduisant les **nombres p-Cauchy** et les **polynômes de Cauchy des deux espèces**, en établissant leurs représentations intégrales, leurs séries génératrices hypergéométriques, leurs relations de récurrence, ainsi que leurs liens avec les nombres r-Stirling et les polynômes de Bernoulli. L'étude met en évidence le caractère de type Appell de ces polynômes et ouvre des perspectives vers des q-analogues et des généralisations multivariées.

Mots clés : Fonction génératrice, forme explicite, relation de récurrence, nombres de Stirling, Bernoulli et Bell, nombres et polynômes de Cauchy, nombres p-Cauchy.

Notations

$\mathbb{N}$	The set of natural numbers
$\mathbb{C}$	The set of complex numbers
$:=$	Equality by definition (assignment)
$\delta_{i,j}$	Kronecker symbol equal to 1 if $i = j$ and 0 otherwise
$x^{\overline{n}}$	The product $x(x+1)\cdots(x+n-1)$ for $n \in \mathbb{N}$ and $x^{\overline{0}} = 1$
$x^{\underline{n}}$	The product $x(x-1)\cdots(x-n+1)$ for $n \in \mathbb{N}$ and $x^{\underline{0}} = 1$
${}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix};x\right)$	The hypergeometric function
$\binom{n}{k}$	The binomial coefficient
$s(n,k)$	The signed Stirling number of the first kind
$\left\{ {n \atop k} \right\}$	The Stirling number of the second kind
$\left\{ {n \atop k} \right\}_r$	The r-Stirling number of the second kind
$\delta_{k,r}$	The Kronecker delta
ϕ_n	The Bell number
$\phi_n(x)$	The Bell polynomial
B_n	The Bernoulli number
$B_n(x)$	The Bernoulli polynomial
C_n	The Cauchy number of the first kind
$\widehat{C}_n$	The Cauchy number of the second kind
$C_n(x)$	The Cauchy polynomial of the first kind
$\widehat{C}_n(x)$	The Cauchy polynomial of the second kind
$C_{n,p}$	The p-Cauchy number of the first kind
$\widehat{C}_{n,p}$	The p-Cauchy number of the second kind
$C_{n,p}(x)$	The p-Cauchy polynomial of the first kind
$\widehat{C}_{n,p}(x)$	The p-Cauchy polynomial of the second kind

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Introduction

In this thesis entitled "On Cauchy Numbers and Their Polynomials," we gather several methods used to determine and study these numbers and polynomials, namely explicit formulas, generating functions, and recurrence relations. The objective of this thesis is to study a generalization of Cauchy numbers and polynomials through a parameter p, as well as their connections with classical combinatorial sequences.

This project contains three chapters, where:

Chapter 1: Basic Notions In the first part, we present the elementary objects in enumerative combinatorics, namely: falling factorials, rising factorials, and binomial coefficients. Next, we introduce classical sequences of numbers and polynomials, namely: signed and unsigned Stirling numbers of the first kind, Stirling numbers of the second kind, r-Stirling numbers of the second kind, Bell numbers, Bernoulli numbers, etc. We also recall hypergeometric functions, confluent hypergeometric functions, generating functions, and Riordan arrays.

Chapter 2: Cauchy Numbers and Their Polynomials The second chapter is devoted to the study of classical Cauchy numbers of both kinds. We introduce and investigate some properties of these numbers, including integral representations, generating functions, explicit formulas, and recurrence relations. We also present Cauchy polynomials of the first and second kind, as well as their connections with r-Stirling numbers and Bernoulli polynomials.

Chapter 3: Generalization of Cauchy Numbers and Polynomials The generalization from the second chapter is the subject of the third chapter. We study a new class of Cauchy numbers called p-Cauchy numbers, introduced by M. Rahmani [20]. We also examine the p-Cauchy polynomials of both kinds, which are of Appell type, and establish their fundamental properties, including recurrence relations, explicit formulas, and relations between the two kinds.

Chapter 1

Basic Concepts

In this chapter, we present some basic elements, starting by recalling descending factorials, Stirling numbers, Cauchy numbers, and confluent hypergeometric functions. The material presented here is based on standard references in combinatorics and special functions, which were mentioned in detail in the references [6, 7, 10, 23].

1.1 Falling Factorial

In this section we will discuss the falling factorial, some properties with their proofs, which were mentioned in detail in the references [11, 23].

Definition 1.1.1. *Let $x \in \mathbb{C}$ and $n \in \mathbb{N}$. The Falling Factorial of x of order n, denoted $x^{\underline{n}}$, is defined by:*

$$x^{\underline{n}} = \begin{cases} 1, & \text{if } n = 0 \\ x(x-1)(x-2)\cdots(x-n+1) & \text{if } n > 0. \end{cases}$$

If $x = n$, then $n^{\underline{n}} = n!$ (n factorial).

Remark 1.1.1. *The notation $(x)_n$ [Notation Conventions] is historically known as the Pochhammer symbol, though care must be taken as different authors use it to represent either descending or ascending factorials. In combinatorics and finite difference calculus, the underline notation $x^{\underline{n}}$ (introduced by Knuth) unambiguously denotes descending factorials, which we adopt in this paper [10].*

Example 1.1.1 (First Few Falling Factorials)**.**

$$\begin{aligned} x^{\underline{0}} &= 1, \\ x^{\underline{1}} &= x, \\ x^{\underline{2}} &= x(x-1) = x^2 - x, \\ x^{\underline{3}} &= x(x-1)(x-2) = x^3 - 3x^2 + 2x, \\ x^{\underline{4}} &= x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x. \end{aligned}$$

Property 1.1.1. *Let $x \in \mathbb{R}$ and $m \in \mathbb{Z}$, then*

1.
$$m^{\underline{m}} = m!. \tag{1.1}$$

2.
$$x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}. \tag{1.2}$$

3.
$$x^{\underline{-m}} = \frac{1}{(x+m)^{\underline{m}}}, \quad x \neq -1, -2, -3 \cdots - m. \tag{1.3}$$

1.2 Rising Factorial

In this section we will discuss the rising factorial, some properties with their proofs, which were mentioned in detail in the references [10, 11, 23].

Definition 1.2.1. *Let $x \in \mathbb{C}$ and $n \in \mathbb{N}$. The rising factorial of x of order n, denoted $x^{\overline{n}}$, is defined by*

$$x^{\overline{n}} = \begin{cases} 1 & \text{if } n = 0 \\ x(x+1)(x+2)\cdots(x+n-1) & \text{if } n > 0. \end{cases}$$

Example 1.2.1.

$$\begin{aligned} x^{\overline{0}} &= 1, \\ x^{\overline{1}} &= x, \\ x^{\overline{2}} &= x(x+1) = x^2 + x, \\ x^{\overline{3}} &= x(x+1)(x+2) = x^3 + 3x^2 + 2x, \\ x^{\overline{4}} &= x(x+1)(x+2)(x+3) = x^4 + 6x^3 + 11x^2 + 6x. \end{aligned}$$

Property 1.2.1. *Let $x \in \mathbb{R}$, $y \in \mathbb{C}$ and $m \in \mathbb{Z}$, then*

1.

$$x^{\overline{m}} = (-1)^m (-x)^{\underline{m}}. \tag{1.4}$$

2.

$$x^{\overline{m+n}} = x^{\overline{m}} (x+m)^{\overline{n}}. \tag{1.5}$$

3.

$$x^{\overline{m}} = (x+m-1)^{\underline{m}}. \tag{1.6}$$

4.

$$x^{\overline{n}} = \frac{(x+n-1)!}{(x-1)!}. \tag{1.7}$$

5.

$$x^{\overline{n}} = (-1)^n (1-n-x)^{\overline{n}}. \tag{1.8}$$

6.

$$(x+1)^{\overline{n}} = x^{\overline{n}} \frac{(x+n)}{x}. \tag{1.9}$$

7.

$$x^{\overline{n+1}} = x(x+1)^{\overline{n}}. \tag{1.10}$$

8.

$$(x+n-1)^{\underline{n}} = (-1)^n (-x)^{\overline{n}}. \tag{1.11}$$

9.

$$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}}. \tag{1.12}$$

1.3 Binomial Coefficients

In this section we will discuss the binomial coefficients, some properties with their proofs, which were mentioned in detail in the references [10, 23].

Definition 1.3.1. *Let $x \in \mathbb{C}$ and $n, k \in \mathbb{N}$. The binomial coefficient $\binom{x}{n}$ is defined by*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \tag{1.13}$$

In particular, $\binom{x}{0} = 1$. For positive integers n and k, $\binom{n}{k}$ is the number of ways to choose a k-element subset from an n-element set.

Example 1.3.1. *In the set $[4]$, there are 6 two-element subsets. Moreover, $\binom{4}{2} = \frac{4!}{2!\,2!} = 6$.*

Property 1.3.1. *1. For positive integers n and k,*

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}. \tag{1.14}$$

2.

$$\binom{n}{n-k} = \binom{n}{k}. \tag{1.15}$$

3. For all $1 \leq k \leq n$, binomial coefficients satisfy Pascal's recurrence:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{1.16}$$

with $\binom{n}{0} = \binom{n}{n} = 1$.

4. For $x \in \mathbb{R}$ and $m \in \mathbb{Z}$, then

$$x^{\underline{n}} = n! \binom{x}{n}. \tag{1.17}$$

5.

$$(x+y)^{\underline{m}} = \sum_{k=0}^{m} \binom{m}{k} x^{\underline{k}} y^{\underline{m-k}}. \tag{1.18}$$

From relation (1.16), we can construct Pascal's triangle

$n=0$				1			
$n=1$			1		1		
$n=2$		1		2		1	
$n=3$	1		3		3		1
$n=4$	1	4		6		4	1

Table 1.1: Pascal's Triangle

Theorem 1.3.1 (Newton's Binomial Formula)**.** *Let $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$ Then*

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}. \tag{1.19}$$

Proof. For $n = 0$:

$$(x+y)^0 = 1, \quad \sum_{j=0}^{0} \binom{0}{j} x^j y^{0-j} = 1.$$

For $n = 1$:

$$(x+y)^1 = x+y, \quad \sum_{j=0}^{1} \binom{1}{j} x^j y^{1-j} = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = y + x.$$

So the formula (1.19) is true for $n = 0, 1$. Assume that Newton's Binomial Formula holds for n, then we show it's true for $n+1$

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y) \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} \\
&= x \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} + y \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} \\
&= \sum_{j=0}^{n} \binom{n}{j} x^{j+1} y^{n-j} + \sum_{j=0}^{n} \binom{n}{j} x^j y^{n+1-j} \\
&= \sum_{j=1}^{n+1} \binom{n}{j-1} x^j y^{n+1-j} + \sum_{j=0}^{n} \binom{n}{j} x^j y^{n+1-j} \\
&= \sum_{j=0}^{n+1} \left[\binom{n}{j-1} + \binom{n}{j} \right] x^j y^{n+1-j} \\
&= \sum_{j=0}^{n+1} \binom{n+1}{j} x^j y^{n+1-j}.
\end{aligned}$$

Thus Newton's Binomial Formula is true for $n+1$, so (1.19) is proven. □

1.4 Generating Functions

In this section we will discuss the generating functions, their types, some properties with their proofs, and the operations on them, which were mentioned in detail in the references [11, 23].

Definition 1.4.1. *Generating functions are an algebraic tool that allows reformulating combinatorial problems to transform them into algebraic manipulation problems. In particular, in combinatorics, it often involves determining the number of objects of certain types that are of size n; which gives rise to a sequence $(a_n)_{n\geq 0}$ whose n-th term we seek to determine.*

Ordinary Generating Function

Let n in $\mathbb{N}$ and $(a_n)_{n\geq 0}$ a sequence of complex numbers. The ordinary generating function of the sequence $(a_n)_{n\geq 0}$ is the function

$$A(x) = \sum_{n\geq 0} a_n x^n.$$

Addition Of Ordinary Generating Series

The addition of two Ordinary series defined by

$$\sum_{n\geq 0} a_n x^n + \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0} (a_n + b_n) x^n.$$

Multiplication Of Ordinary Generating Series

Multiplication of two Ordinary series defined by

$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0}\sum_{k=0}^{n} a_k b_{n-k} x^n.$$

Exponential Generating Function

Let n be a natural number and $(a_n)_{n\geq 0}$ a sequence of complex numbers. The exponential generating function of the sequence $(a_n)_{n\geq 0}$ is the function

$$B(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}.$$

Addition Of Exponential Generating Series

The addition of two Exponential series defined by

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} + \sum_{n\geq 0} b_n \frac{x^n}{n!} = \sum_{n\geq 0} (a_n + b_n)\frac{x^n}{n!}.$$

Multiplication Of Exponential Generating Series

Multiplication of two Exponential series defined by

$$\left(\sum_{n\geq 0} a_n \frac{x^n}{n!}\right)\left(\sum_{n\geq 0} b_n \frac{x^n}{n!}\right) = \sum_{n\geq 0}\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \frac{x^n}{n!}.$$

Double Generating Function

Let n, m be natural numbers and $(a_{n,m})_{n,m\geq 0}$ a sequence of complex numbers. The ordinary and exponential double generating functions of the sequence $(a_{n,m})_{n,m\geq 0}$ are given respectively by

$$A(x,y) = \sum_{n\geq 0}\left(\sum_{m\geq 0} a_{n,m} x^m\right) y^n.$$

and

$$B(x,y) = \sum_{n\geq 0}\left(\sum_{m\geq 0} a_{n,m} \frac{x^m}{m!}\right) \frac{y^n}{n!}.$$

as well as the mixed generating function

$$C(x,y) = \sum_{n\geq 0}\left(\sum_{m\geq 0} a_{n,m} \frac{x^m}{m!}\right) y^n.$$

The Laplace Transform

There are the Laplace transform of a function h is defined by

$$\mathcal{L}(h) = \int_0^{+\infty} e^{-st} h(t) dt.$$

The Inverse Laplace Transform

The inverse Laplace transform $\mathcal{L}^{-1}$, defined on the algebra of formal series, maps any ordinary generating function to the corresponding exponential generating function, it defined by

$$\mathcal{L}^{-1}\left(\sum_{n\geq 0} a_n z^n\right) = \sum_{n\geq 0} a_n \frac{z^n}{n!}.$$

Proposition 1.4.1. *Let $m \in \mathbb{Z}$, where $m > 0$*

$$\mathcal{L}^{-1}\left(z^m(1-z)^{-m-1}\right) = e^z \frac{z^m}{m!}.$$

Proof. Let $n, m \in \mathbb{Z}$, where $n, m > 0$ and with change of variables:

$$N = n - m, \quad n = N + m$$

$$\begin{aligned}
\mathcal{L}^{-1}\left(\sum_{n\geq 0} \binom{n}{m} z^n\right) &= \sum_{n\geq 0} \binom{n}{m} \frac{z^n}{n!} \\
&= \sum_{n\geq 0} \frac{n!}{m!(n-m)!} \frac{z^n}{n!} \\
&= \sum_{n\geq m} \frac{z^n}{m!(n-m)!} \\
&= \sum_{N\geq 0} \frac{z^{N+m}}{m!N!} \\
&= \frac{z^m}{m!} \sum_{N\geq 0} \frac{z^N}{N!} \\
&= e^z \frac{z^m}{m!}.
\end{aligned}$$

□

1.5 Hypergeometric Functions

In this section, we will discuss hypergeometric functions, which appear in two distinct forms: integrals and power series. This section is dedicated to presenting definitions of some properties of both types of hypergeometric functions, beginning with the gamma and beta function, which were mentioned in detail in the references [1, 9].

1.6 Gamma Function

Definition 1.6.1. *For all $x \in \mathbb{C}$ such that $\Re(x) > 0$, the Gamma function, denoted Γ, is defined by*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\, dt. \tag{1.20}$$

Integrating by parts yields the immediate consequence, for all $x \in \mathbb{C}$ with $\Re(x) > 0$

$$\Gamma(x+1) = x\,\Gamma(x). \tag{1.21}$$

In particular, since $\Gamma(1) = 1$, we can deduce $\forall n \in \mathbb{N}^*$

$$\Gamma(n) = (n-1)!. \tag{1.22}$$

Property 1.6.1. *Let $x \in \mathbb{R}$ and $m \in \mathbb{Z}$, then*

1.
$$x^{\underline{m}} = \frac{\Gamma(x+1)}{\Gamma(x-m+1)}. \tag{1.23}$$

2.
$$x^{\overline{m}} = \frac{\Gamma(x+m)}{\Gamma(x)}. \tag{1.24}$$

1.6.1 Beta Function

Definition 1.6.2. *For all $x \geq 0$ and $y \geq 0$, the beta function denoted $\beta(x,y)$, is a type of Euler integral defined by*

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{1.25}$$

The Beta function $\beta(x,y)$ is related to the Gamma function by the following relation.

Theorem 1.6.1.
$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.26}$$

Proof. we need the following relation, for $\Re(x) \geq 0$, $\Re(y) \geq 0$
$$\beta(x,y) = \frac{x+y}{y}\beta(x,y+1). \tag{1.27}$$
we use (1.25), we obtain
$$\beta(x,y+1) = \int_0^1 t^{x-1}(1-t)^{y}dt = \beta(x,y) - \beta(x+1,y). \tag{1.28}$$
on the other hand, by integration by parts
$$\beta(x,y+1) = \int_0^1 t^{x-1}(1-t)^{y}dt = \frac{y}{x}\beta(x+1,y). \tag{1.29}$$
From (1.27) and (1.28), we obtain
$$\beta(x,y) = \beta(x,y+1) + \beta(x+1,y) = \frac{x}{y}\beta(x,y+1) + \beta(x,y+1) = \frac{x+y}{y}\beta(x,y+1).$$
From this relation, and by recurrence
$$\beta(x,y) = \frac{(x+y)^{\overline{n}}}{y^{\overline{n}}}\beta(x,y+n).$$
now,
$$\beta(x,y) = \frac{(x+y)^{\overline{n}}n!}{n!y^{\overline{n}}}\int_0^n \left(\frac{t}{n}\right)^{x-1}\left(1-\frac{t}{n}\right)^{y+n-1} d\left(\frac{t}{n}\right) = \frac{(x+y)^{\overline{n}}}{n^x y^{\overline{n}}}\int_0^n t^{x-1}\left(1-\frac{t}{n}\right)^{y+n-1} dt.$$
as $n \to \infty$, we have
$$\int_0^n t^{x-1}\left(1-\frac{t}{n}\right)^{y+n-1} dt \to \int_0^\infty t^{x-1}e^{-t}dt = \Gamma(x).$$
also,
$$\frac{(x+y)^{\overline{n}}}{n^x y^{\overline{n}}} \to \frac{\Gamma(y)}{\Gamma(x+y)}.$$
hence,
$$\beta(x,y) = \frac{\Gamma(y)}{\Gamma(x+y)}\Gamma(x) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.30}$$
□

1.6.2 Hypergeometric Function

In this section we will discuss the hypergeometric function, generating function and some properties, which were mentioned in detail in the reference [1, 23]

Definition 1.6.3. *1. The generalized hypergeometric function ${}_pF_q$ is defined by*

$$ {}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};x\right) := \sum_{n\geq 0} \frac{a_1^{\overline{n}}\cdots a_p^{\overline{n}}}{b_1^{\overline{n}}\cdots b_q^{\overline{n}}}\frac{x^n}{n!}. $$

2. The hypergeometric function refers to the special case $p=2$, $q=1$

$$ {}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};x\right) := \sum_{n\geq 0} \frac{a^{\overline{n}}b^{\overline{n}}}{c^{\overline{n}}}\frac{x^n}{n!}. $$

Example 1.6.1. *For $p=2$, $q=1$, and $a=b=c=1$, the hypergeometric function reduces to the geometric series $1+x+x^2+\ldots$, i.e.:*

$$ {}_2F_1\left(\begin{array}{c} 1,1 \\ 1 \end{array};x\right) := \sum_{n\geq 0} x^n = \frac{1}{1-x}, \quad |x|<1. $$

Theorem 1.6.2. *(Euler's Integral Representation) If $\Re(c)>\Re(b)>0$, then*

$$ {}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt. \tag{1.31} $$

Proof.

$$ \begin{aligned} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\sum_{n\geq 0}\frac{(a)^{\overline{n}}(xt)^n}{n!}dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n\geq 0}\frac{(a)^{\overline{n}}x^n}{n!}\int_0^1 t^{n+b-1}(1-t)^{c-b-1}dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n\geq 0}\frac{(a)^{\overline{n}}x^n}{n!}\mathrm{B}(n+b,c-b) \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n\geq 0}\frac{(a)^{\overline{n}}x^n}{n!}\frac{\Gamma(n+b)\Gamma(c-b)}{\Gamma(n+c)} \\ &= \sum_{n\geq 0}\frac{(a)^{\overline{n}}\cdot(b)^{\overline{n}}}{(c)^{\overline{n}}}\frac{x^n}{n!} \\ &= {}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};x\right). \end{aligned} $$

□

Theorem 1.6.3. *(Pfaff's Transformation)*

$$ {}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};x\right) = (1-x)^{-a}\,{}_2F_1\left(\begin{array}{c} a,c-b \\ c \end{array};\frac{x}{x-1}\right). \tag{1.32} $$

Proof. Using the formula (1.31), we have

$$ \begin{aligned} {}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};x\right) &= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt, \end{aligned} $$

let $t = 1 - s$, then $dt = -ds$, when $t = 0$, $s = 1$; when $t = 1$, $s = 0$

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right) &= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_1^0 (1-s)^{b-1}s^{c-b-1}(1-x(1-s))^{-a}(-ds)\\
&= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 s^{c-b-1}(1-s)^{b-1}(1-x+xs)^{-a}ds\\
&= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 s^{c-b-1}(1-s)^{b-1}\left[(1-x)\left(1+\frac{xs}{1-x}\right)\right]^{-a}ds\\
&= (1-x)^{-a}\frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)}\int_0^1 s^{c-b-1}(1-s)^{b-1}\left(1-\left(-\frac{x}{1-x}\right)s\right)^{-a}ds\\
&= (1-x)^{-a}\,{}_2F_1\left(\begin{matrix}a,c-b\\c\end{matrix};\frac{x}{x-1}\right).
\end{aligned}$$

□

Theorem 1.6.4. *(Euler's Transformation)*

$$ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right) = (1-x)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a,c-b\\c\end{matrix};x\right). \tag{1.33}$$

Proof. Using the formula (1.32)

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right) &= (1-x)^{-a}\,{}_2F_1\left(\begin{matrix}a,c-b\\c\end{matrix};\frac{x}{x-1}\right)\\
&= (1-x)^{-a}\,{}_2F_1\left(\begin{matrix}c-b,a\\c\end{matrix};\frac{x}{x-1}\right)\\
&= (1-x)^{-a}\left(1-\frac{x}{x-1}\right)^{-(c-b)}{}_2F_1\left(\begin{matrix}c-b,c-a\\c\end{matrix};\frac{\frac{x}{x-1}}{\frac{x}{x-1}-1}\right)\\
&= (1-x)^{-a}\left(\frac{x-1-x}{x-1}\right)^{-(c-b)}{}_2F_1\left(\begin{matrix}c-b,c-a\\c\end{matrix};x\right)\\
&= (1-x)^{-a}\,(1-x)^{c-b}\,{}_2F_1\left(\begin{matrix}c-b,c-a\\c\end{matrix};x\right)\\
&= (1-x)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a,c-b\\c\end{matrix};x\right).
\end{aligned}$$

□

Theorem 1.6.5. *(Gauss's Formula) If* $\Re(c-a-b) > 0$, *then*

$$ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};1\right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}. \tag{1.34}$$

Proof. Using the formula (1.31) for $x = 1$, we have

$$\begin{aligned}
{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};1\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}\,dt\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-a-b-1}\,dt\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\mathrm{B}(b,c-a-b)\\
&= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}\\
&= \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.
\end{aligned}$$

□

1.7 Confluent Hypergeometric Function

In this section we will discuss the confluent hypergeometric function, generating function and some properties, which were mentioned in detail in the reference [1, 9]

Definition 1.7.1. *For $c \neq 0$, the confluent hypergeometric function is defined by*

$$ {}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right)=\sum_{n=0}^{\infty}\frac{a^{\overline{n}}x^n}{c^{\overline{n}}n!}, \qquad -\infty<x<+\infty. \tag{1.35} $$

Theorem 1.7.1. *Let the following formula*

$$ \lim_{b\to\infty}{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};\frac{x}{b}\right)={}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right). \tag{1.36} $$

Proof. We have

$$ \lim_{b\to\infty}{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array};\frac{x}{b}\right)=\lim_{b\to\infty}\sum_{n=0}^{\infty}\frac{a^{\overline{n}}b^{\overline{n}}\left(\frac{x}{b}\right)^n}{c^{\overline{n}}n!}=\lim_{b\to\infty}\sum_{n=0}^{\infty}\frac{a^{\overline{n}}\left(b(1+\frac{1}{b})(1+\frac{2}{b})\cdots(1+\frac{n-1}{b})\right)\frac{x^n}{b^n}}{c^{\overline{n}}n!}=\lim_{b\to\infty}\sum_{n=0}^{\infty}\frac{a^{\overline{n}}(1+\frac{1}{b})(1 \tag{1.37} $$

□

Theorem 1.7.2. *(Differential representation) For $c \neq 0$, we have*

$$ \frac{d}{dx}{}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right)=\frac{a}{c}{}_1F_1\left(\begin{array}{c} a+1 \\ c+1 \end{array};x\right). \tag{1.38} $$

Proof.

$$ \begin{aligned} \frac{d}{dx}{}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right)&=\frac{d}{dx}\sum_{n\geq 0}\frac{a^{\overline{n}}x^n}{c^{\overline{n}}n!}\\ &=\sum_{n\geq 1}\frac{a^{\overline{n}}}{c^{\overline{n}}n!}\frac{dx^n}{dx}\\ &=\sum_{n\geq 1}\frac{a^{\overline{n}}x^{n-1}}{c^{\overline{n}}(n-1)!}\\ &=\sum_{n\geq 0}\frac{a^{\overline{n+1}}x^n}{c^{\overline{n+1}}n!}\\ &=\sum_{n\geq 0}\frac{a(a+1)^{\overline{n}}x^n}{c(c+1)^{\overline{n}}n!}\\ &=\frac{a}{c}{}_1F_1\left(\begin{array}{c} a+1 \\ c+1 \end{array};x\right). \end{aligned} $$

In general

$$ \frac{d^k}{dx^k}{}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right)=\frac{a^{\overline{k}}}{c^{\overline{k}}}{}_1F_1\left(\begin{array}{c} a+k \\ c+k \end{array};x\right). $$

(its proof by recurrence) □

Theorem 1.7.3. *(Integral representation) [1] Based on Euler's integral representation for the ${}_1F_1$ hypergeometric function, one might expect that the confluent hypergeometric function satisfies, we have $c > a > 0$*

$$ {}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{c-a-1}dt. \tag{1.39} $$

Proof.

$$\begin{aligned}\int_0^1 e^{xt}t^{a-1}(1-t)^{c-a-1}dt &= \int_0^1 \sum_{n\geq 0}\frac{(xt)^n}{n!}t^{a-1}(1-t)^{c-a-1}dt\\ &= \sum_{n\geq 0}\frac{x^n}{n!}\int_0^1 t^{a+n-1}(1-t)^{c-a-1}dt\\ &= \sum_{n\geq 0}\frac{x^n}{n!}\beta(a+n,c-a)\\ &= \sum_{n\geq 0}\frac{x^n}{n!}\frac{\Gamma(a+n)\Gamma(c-a)}{\Gamma(c+n)}\\ &= \Gamma(c-a)\sum_{n\geq 0}\frac{x^n}{n!}\frac{a^{\overline{n}}\Gamma(a)}{c^{\overline{n}}\Gamma(c)}\\ &= \frac{\Gamma(c-a)\Gamma(a)}{\Gamma(c)}\sum_{n\geq 0}\frac{a^{\overline{n}}}{c^{\overline{n}}}\frac{x^n}{n!}.\end{aligned}$$

thus

$${}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{c-a-1}dt.$$

□

Theorem 1.7.4. *For $0<a<b$, we have*

$${}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right) = e^x\,{}_1F_1\left(\begin{array}{c} c-a \\ c \end{array};-x\right). \tag{1.40}$$

Proof. For $0<a<b$, we have

$${}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{xt}t^{a-1}(1-t)^{c-a-1}dt = e^x\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{-x(1-t)}t^{a-1}(1-t)^{c-a-1}dt = e^x\frac{\Gamma($$

□

Lemma 1.7.1. *We have*

$${}_1F_1\left(\begin{array}{c} 1 \\ p+2 \end{array};z\right) = \left(1+\frac{z}{p+2}\right){}_1F_1\left(\begin{array}{c} 1 \\ p+3 \end{array};z\right) - \frac{z}{p+3}{}_1F_1\left(\begin{array}{c} 1 \\ p+4 \end{array};z\right). \tag{1.41}$$

Proof. Using the confluent hypergeometric function

$${}_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right) = \sum_{n=0}^{\infty}\frac{a^{\overline{n}}x^n}{c^{\overline{n}}n!}, \qquad -\infty<x<+\infty$$

We get

$$\begin{aligned}&\left(1+\frac{z}{p+2}\right){}_1F_1\left(\begin{array}{c} 1 \\ p+3 \end{array};z\right) - \frac{z}{p+3}{}_1F_1\left(\begin{array}{c} 1 \\ p+4 \end{array};z\right)\\ &= {}_1F_1\left(\begin{array}{c} 1 \\ p+3 \end{array};z\right) + \frac{z}{p+2}{}_1F_1\left(\begin{array}{c} 1 \\ p+3 \end{array};z\right) - \frac{z}{p+3}{}_1F_1\left(\begin{array}{c} 1 \\ p+4 \end{array};z\right)\\ &= \sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^n}{(p+3)^{\overline{n}}n!} + \sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^{n+1}}{(p+2)(p+3)^{\overline{n}}n!} - \sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^{n+1}}{(p+3)(p+4)^{\overline{n}}n!}\end{aligned}$$

and we have

$$(p+3)^{\overline{n}} = (p+3)(p+4)\cdots(p+n+2)$$

$$\frac{(p+2)(p+3)^{\overline{n}}}{p+n+2} = (p+2)^{\overline{n}}$$

and

$$\frac{(p+2)(p+3)(p+4)^{\overline{n}}}{(p+n+2)(p+n+3)} = (p+2)^{\overline{n}}$$

So we get

$$\left(1+\frac{z}{p+2}\right){}_1F_1\left(\begin{array}{c}1\\p+3\end{array};z\right) - \frac{z}{p+3}{}_1F_1\left(\begin{array}{c}1\\p+4\end{array};z\right)$$
$$= \sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^n}{(p+3)^{\overline{n}}n!} + \sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^{n+1}}{(p+2)(p+3)^{\overline{n}}n!} - \sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^{n+1}}{(p+4)^{\overline{n}}n!}$$
$$= \sum_{n=0}^{\infty}\frac{(p+n+2)}{(p+2)}\frac{1^{\overline{n}}z^n}{(p+3)^{\overline{n}}n!} + \sum_{n=0}^{\infty}(p+n+2)\frac{1^{\overline{n}}z^{n+1}}{(p+2)(p+3)^{\overline{n}}n!} - \sum_{n=0}^{\infty}\frac{(p+n+2)(p+n+3)}{p+2}\frac{1^{\overline{n}}z^{n+1}}{(p+4)^{\overline{n}}n!}$$
$$= \sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^n}{(p+2)^{\overline{n}}n!} + \sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^{n+1}}{(p+2)^{\overline{n}}n!} - \sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^{n+1}}{(p+2)^{\overline{n}}n!}$$
$$= \sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^n}{(p+2)^{\overline{n}}n!} + \frac{(p+n+1)n}{n}\sum_{n=0}^{\infty}\frac{1^{\overline{n-1}}z^n}{(p+2)^{\overline{n}}(n-1)!} - \frac{(p+n+1)n}{n}\sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^{n+1}}{(p+2)^{\overline{n}}n!}$$
$$= \sum_{n=0}^{\infty}\frac{1^{\overline{n}}z^n}{(p+2)^{\overline{n}}n!} = {}_1F_1\left(\begin{array}{c}1\\p+2\end{array};z\right)$$

□

1.7.1 Recurrence Relations

1.

$$(c-a-1){}_1F_1\left(\begin{array}{c}a\\c\end{array};x\right) + a{}_1F_1\left(\begin{array}{c}a+1\\c\end{array};x\right) = (c-1){}_1F_1\left(\begin{array}{c}a\\c-1\end{array};x\right). \tag{1.42}$$

2.

$${}_1F_1\left(\begin{array}{c}a\\c\end{array};x\right) - c{}_1F_1\left(\begin{array}{c}a-1\\c\end{array};x\right) = x{}_1F_1\left(\begin{array}{c}a\\c+1\end{array};x\right). \tag{1.43}$$

Proof. 1. The proof of (1.42)

$$(c-a-1){}_1F_1\left(\begin{array}{c}a\\c\end{array};x\right) + a{}_1F_1\left(\begin{array}{c}a+1\\c\end{array};x\right) = (c-a-1)\sum_{n\geq 0}\frac{a^{\overline{n}}x^n}{c^{\overline{n}}n!} + a\sum_{n\geq 0}\frac{(a+1)^{\overline{n}}x^n}{c^{\overline{n}}n!}.$$

We have $(a+1)^{\overline{n}} = \frac{a^{\overline{n}}(a+n)}{a}$, and $c^{\overline{n}} = \frac{(c-1)^{\overline{n}}(c-1+n)}{c-1}$

$$\begin{aligned}(c-a-1){}_1F_1\left(\begin{array}{c}a\\c\end{array};x\right) + a{}_1F_1\left(\begin{array}{c}a+1\\c\end{array};x\right) &= (c-a-1)\sum_{n\geq 0}\frac{a^{\overline{n}}}{(c-1)^{\overline{n}}}\frac{c-1}{c-1+n}\frac{x^n}{n!}\\ &\quad + a\sum_{n\geq 0}\frac{a^{\overline{n}}(a+n)}{a}\frac{1}{(c-1)^{\overline{n}}}\frac{c-1}{c-1+n}\frac{x^n}{n!}\\ &= \sum_{n\geq 0}\frac{a^{\overline{n}}}{(c-1)^{\overline{n}}}\frac{c-1}{c-1+n}(c-a-1+a+n)\frac{x^n}{n!}\\ &= (c-1)\sum_{n\geq 0}\frac{a^{\overline{n}}(c+n-1)}{(c-1)^{\overline{n}}(c-1+n)}\frac{x^n}{n!}\\ &= (c-1)\sum_{n\geq 0}\frac{a^{\overline{n}}x^n}{(c-1)^{\overline{n}}n!}\\ &= (c-1){}_1F_1\left(\begin{array}{c}a\\c-1\end{array};x\right).\end{aligned}$$

2. The proof of (1.43)

$$\alpha = c_1F_1\left(\begin{array}{c} a \\ c \end{array};x\right) - c_1F_1\left(\begin{array}{c} a-1 \\ c \end{array};x\right) = c\sum_{n\geq 0}\frac{a^{\overline{n}}x^n}{c^{\overline{n}}n!} - c\sum_{n\geq 0}\frac{(a-1)^{\overline{n}}x^n}{c^{\overline{n}}n!}.$$

We have

$$c^{\overline{n}} = \frac{(c+1)^{\overline{n}}c}{c+n}.$$

So,

$$\begin{aligned}\alpha &= c\sum_{n\geq 0}\frac{a^{\overline{n}}}{(c+1)^{\overline{n}}}\frac{c+n}{c}\frac{x^n}{n!} - c\sum_{n\geq 0}\frac{(a-1)^{\overline{n}}}{(c+1)^{\overline{n}}}\frac{c+n}{c}\frac{x^n}{n!} \\ &= \sum_{n\geq 0}\frac{a^{\overline{n}}(c+n)}{(c+1)^{\overline{n}}}\frac{x^n}{n!} - \sum_{n\geq 0}\frac{(a-1)^{\overline{n}}(c+n)}{(c+1)^{\overline{n}}}\frac{x^n}{n!}.\end{aligned}$$

Note that $(a-1)^{\overline{n}} = \frac{a^{\overline{n}}}{a+n-1}$, we have

$$\begin{aligned}\alpha &= \sum_{n\geq 0}\frac{a^{\overline{n}}(c+n)}{(c+1)^{\overline{n}}}\frac{x^n}{n!} - \sum_{n\geq 0}\frac{a^{\overline{n}}}{a+n-1}\frac{(c+n)}{(c+1)^{\overline{n}}}\frac{x^n}{n!} \\ &= \sum_{n\geq 0}\frac{a^{\overline{n}}(c+n)}{(c+1)^{\overline{n}}}\left(1-\frac{1}{a+n-1}\right)\frac{x^n}{n!} = \sum_{n\geq 0}\frac{a^{\overline{n}}(c+n)}{(c+1)^{\overline{n}}}\frac{n}{a+n-1}\frac{x^n}{n!} \\ &= \sum_{n\geq 1}\frac{a^{\overline{n}}}{(c+1)^{\overline{n}}}\frac{c+n}{a+n-1}\frac{x^n}{(n-1)!} = x\sum_{n\geq 1}\frac{a^{\overline{n}}}{(c+1)^{\overline{n}}}\frac{c+n}{a+n-1}\frac{x^{n-1}}{(n-1)!} \\ &= x\sum_{n\geq 0}\frac{a^{\overline{n+1}}}{(c+1)^{\overline{n+1}}}\frac{c+n+1}{a+n}\frac{x^n}{n!} = x\sum_{n\geq 0}\frac{a(a+1)^{\overline{n}}}{(c+1)(c+2)^{\overline{n}}}\frac{c+n+1}{a+n}\frac{x^n}{n!} \\ &= x\sum_{n\geq 0}\frac{a^{\overline{n}}(a+n)}{(c+1)^{\overline{n}}(c+n+1)}\frac{c+n+1}{a+n}\frac{x^n}{n!} = x\sum_{n\geq 0}\frac{a^{\overline{n}}x^n}{(c+1)^{\overline{n}}n!} = x_1F_1\left(\begin{array}{c} a \\ c+1 \end{array};x\right).\end{aligned}$$

□

1.8 Classical Sequences

In this section, we will study the Stirling numbers that appear in many combinatorics problems; there are two named types: (signed and unsigned Stirling numbers of the first kind, and Stirling numbers of the second kind), which were mentioned in detail in the references [7, 10, 11].

1.8.1 Stirling Numbers Of The First Kind

Stirling Numbers Of The First Kind Unsigned

Definition 1.8.1. *Stirling numbers of the first kind (unsigned), denoted* $\left[\begin{array}{c} n \\ k \end{array}\right]$, *are defined as the number of permutations of the set* $[n]$ *having exactly* k *cycles. They appear as coefficients of the polynomials* $z^{\overline{n}}$ *written in the canonical basis* z^n

$$z^{\overline{n}} = \sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]z^k. \tag{1.44}$$

Remark 1.8.1. *We have*

$$\left[\begin{array}{c} n \\ k \end{array}\right] = (-1)^{n-k}s(n,k).$$

Generating Function

The generating function of Stirling Numbers of the First Kind unsigned given by

$$\sum_{n\geq k} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{z^n}{n!} = \frac{(-\ln(1-z))^k}{k!}.$$

Explicit Formula

The Explicit Formula of Stirling Numbers of the First Kind unsigned given by

$$\left[\begin{array}{c} n \\ k \end{array} \right] = \frac{n!}{k!} \sum_{i_1+i_2+\cdots+i_k=n} \frac{(-1)^{k-n}}{i_1 \cdots i_k}.$$

Recurrence Relation

Let $n, k \in \mathbb{N}$, the recurrence relation is given by

$$\left[\begin{array}{c} n+1 \\ k \end{array} \right] = n \left[\begin{array}{c} n \\ k \end{array} \right] + \left[\begin{array}{c} n \\ k-1 \end{array} \right].$$

with $\left[\begin{array}{c} 0 \\ 0 \end{array} \right] = 1, \quad \left[\begin{array}{c} 1 \\ 0 \end{array} \right] = 0, \quad \left[\begin{array}{c} n \\ n \end{array} \right] = 1, \quad$ *and* $\quad \left[\begin{array}{c} 0 \\ k \end{array} \right] = 0,$

Table 1.2: First values for unsigned Stirling numbers of the first kind

$n\backslash k$ 9	0	1	2	3	4	5	6	7	8
0	1								
1	0	1							
2	0	1	1						
3	0	2	3	1					
4	0	6	11	6	1				
5	0	24	50	35	10	1			
6	0	120	274	225	85	15	1		
7	0	720	1764	1624	735	175	21	1	
8	0	5040	13068	13132	6769	1960	322	28	1
9	0	40320	109584	118124	67284	22449	4536	546	36
1									

Stirling Numbers Of The First Kind Signed

Definition 1.8.2. *The signed Stirling numbers of the first kind, denoted $s(n,k)$, are the coefficients of the development of the falling factorial $z^{\underline{n}}$*

$$z^{\underline{n}} = \sum_{k=0}^{n} s(n,k) z^k. \tag{1.45}$$

Generating Function

Theorem 1.8.1.

$$\sum_{n,k\geq 0} s(n,k) \frac{z^n}{n!} x^k = (1+z)^x. \tag{1.46}$$

$$\sum_{n\geq 0} s(n,k) \frac{z^n}{n!} = \frac{(\ln(1+z))^k}{k!}. \tag{1.47}$$

Proof. Using Newton's binomial to prove the formula (1.46)

$$\begin{aligned}(1+z)^x &= \sum_{n\geq 0}\binom{x}{n}z^n \\ &= \sum_{n\geq 0} x^{\underline{n}}\frac{z^n}{n!} \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} s(n,k)x^k\right)\frac{z^n}{n!} \\ &= \sum_{n,k\geq 0} s(n,k)\frac{z^n}{n!}x^k.\end{aligned}$$

□

Proof. Using the formula (1.45) to prove (1.47)

$$\begin{aligned}\sum_{n,k\geq 0} s(n,k)\frac{z^n}{n!}x^k &= (1+z)^x \\ &= e^{x\ln(1+z)} \\ &= \sum_{k\geq 0}\frac{x^k(\ln(1+z))^k}{k!}.\end{aligned}$$

□

Explicit Formula

$$s(n,k) = \frac{n!}{k!}\sum_{i_1+i_2+\cdots+i_k=n}\frac{1}{i_1\cdots i_k}. \tag{1.48}$$

Recurrence Relation

For $n,k\geq 1$,

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k),$$

With $s(n,0) = s(0,k) = 0$ *except* $s(0,0) = 1$.

Table 1.3: First values for signed Stirling numbers of the first kind

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	-1	1							
3	0	2	-3	1						
4	0	-6	11	-6	1					
5	0	24	-50	35	-10	1				
6	0	-120	274	-225	85	-15	1			
7	0	720	-1764	1624	-735	175	-21	1		
8	0	-5040	13068	-13132	6769	-1960	322	-28	1	
9	0	40320	-109584	118124	-67284	22449	-4536	546	-36	1

1.8.2 Stirling Numbers Of The Second Kind

Definition 1.8.3. *Stirling numbers of the second kind, denoted $\left\{ {n \atop k} \right\}$, count the number of partitions of a set with n elements into k blocks; in other words, into k pairwise disjoint nonempty subsets of $[n]$. They appear as coefficients of the monomial x^n in the basis $z^{\underline{k}}$, that is:*

$$z^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} z^{\underline{k}}. \tag{1.49}$$

Generating Function

The generating function of the Stirling number of the second kind is given by

1. *The exponential generating function of the Stirling numbers of second kind*

$$\sum_{n \geq k} \left\{ {n \atop k} \right\} \frac{z^n}{n!} = \frac{(e^z - 1)^k}{k!}, \qquad k \geq 0. \tag{1.50}$$

2. *The ordinary generating function for Stirling number is given by*

$$\sum_{n \geq k} \left\{ {n \atop k} \right\} z^n = \frac{z^k}{(1-z)(1-2z)\dots(1-kz)}. \tag{1.51}$$

3. *Double generating function*

$$\sum_{n,k \geq 0} \left\{ {n \atop k} \right\} \frac{x^n}{n!} y^k = e^{y(e^x - 1)}. \tag{1.52}$$

Explicit Formula

Theorem 1.8.2. *The explicit formula of the Stirling numbers of second kind given by*

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-j)^n. \tag{1.53}$$

Proof. Using the exponential generating function (1.50), we have

$$\sum_{n \geq k} \left\{ {n \atop k} \right\} \frac{z^n}{n!} = \frac{(e^z - 1)^k}{k!}.$$

Then using Newton's binomial to get

$$\begin{aligned}
\frac{(e^z - 1)^k}{k!} &= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{(k-j)z} \\
&= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \sum_{n \geq 0} (k-j)^n \frac{z^n}{n!} \\
&= \sum_{n \geq 0} \left(\frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-j)^n \right) \frac{z^n}{n!} \\
&= \sum_{n \geq k} \left\{ {n \atop k} \right\} \frac{z^n}{n!}.
\end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-j)^n.$$

□

Recurrence Relation

For $n \geq 1$ and $k \geq 1$, the Stirling numbers of the second kind satisfy the following recurrence relation

$$\left\{ n \atop k \right\} = \left\{ n-1 \atop k-1 \right\} + k \left\{ n-1 \atop k \right\}. \tag{1.54}$$

With

$$\left\{ n \atop 0 \right\} = \left\{ 0 \atop k \right\} = 0, \qquad \left\{ 0 \atop 0 \right\} = 1$$

The first values of $\left\{ n \atop k \right\}$ are:

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

Table 1.4: First values of Stirling numbers of the second kind $\left\{ n \atop k \right\}$

1.9 r-Stirling Numbers

In this section, we will study one of the generalizations of Stirling numbers that appears in many combinatoric problems; namely r-Stirling Numbers, and also there are two kinds of these Numbers: r-Stirling numbers of the first and second kind [4, 11].

1.9.1 r-Stirling Numbers Of The First Kind

r-Stirling Numbers Of The First Kind (Unsigned)

Definition 1.9.1. *The (unsigned) r-Stirling numbers of the first kind, denoted $\left[\begin{matrix} n+r \\ k+r \end{matrix} \right]_r$, count the number of partitions of set n into k non-empty subsets such that the first r elements are in distinct subsets with $r < k < n$.*

$$(z+r)^{\underline{n}} = \sum_{k=0}^{n} \left[\begin{matrix} n+r \\ k+r \end{matrix} \right]_r z^n. \tag{1.55}$$

Generating Function

The exponential generating function of the first kind r-Stirling number is given by :

$$\sum_{n \geq k} \left[\begin{matrix} n+r \\ k+r \end{matrix} \right]_r \frac{z^n}{n!} = \frac{(-\ln(1-z))^k}{k!(1-z)^r}. \tag{1.56}$$

r-Stirling Numbers Of The First Kind (Signed)

Definition 1.9.2. *The (signed) r-Stirling numbers of the first kind, denoted $s(n+r,k+r)$, count the number of partitions of set n into k non-empty subsets such that the first r elements are in distinct subsets with $r<k<n$.*

$$(z+r)^{\underline{n}}=\sum_{k=0}^{n}s(n+r,k+r)z^n. \tag{1.57}$$

Generating Function

The exponential generating function of the first kind (signed) r-Stirling number is given by :

$$\sum_{n\geq k}s(n+r,k+r)\frac{z^n}{n!}=\frac{(\ln(1+z))^k}{k!(1+z)^r}. \tag{1.58}$$

Explicit Formula

The explicit formula of these numbers given by

Theorem 1.9.1.

$$s(n+r,k+r)=\sum_{j=0}^{n}\binom{n}{j}s(j,k)(-1)^{n-j}r^{\overline{n-j}}. \tag{1.59}$$

Proof. Using the generating function to prove the explicit formula

$$\begin{aligned}\frac{(\ln(1+z))^k}{k!(1+z)^r}&=\frac{(\ln(1+z))^k}{k!}(1+z)^{-r}\\&=\left(\sum_{n\geq 0}s(n,k)\frac{z^n}{n!}\right)\left(\sum_{n\geq 0}(-1)^n r^{\overline{n}}\frac{z^n}{n!}\right)\\&=\sum_{n\geq 0}\left(\sum_{j=0}^{n}\binom{n}{j}s(j,k)(-1)^{n-j}r^{\overline{n-j}}\right)\frac{z^n}{n!}\\&=\sum_{n\geq k}s(n+r,k+r)\frac{z^n}{n!}.\end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find

$$s(n+r,k+r)=\sum_{j=0}^{n}\binom{n}{j}s(j,k)(-1)^{n-j}r^{\overline{n-j}}.$$

□

1.9.2 r-Stirling Numbers Of The Second Kind

Definition 1.9.3. *The r-Stirling numbers of the second kind, denoted $\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r$, count the number of partitions of set n into k non-empty subsets such that the first r elements are in distinct subsets.*

$$(z+r)^n=\sum_{k=0}^{n}\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r z^{\underline{n}}.$$

Generating Function And Explicit Formula

The exponential generating function of the second kind r-Stirling number is given by

$$\sum_{n\geq k}\left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r \frac{z^n}{n!} = \frac{1}{k!}e^{rz}(e^z-1)^k. \tag{1.60}$$

The ordinary generating functions of the second kind r-Stirling number is given by

$$\sum_{n\geq k}\left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r z^n = \frac{z^k}{(1-rz)(1-(r+1)z)\cdots(1-(k+r)z)}; \qquad k\geq r\geq 0. \tag{1.61}$$

The double generating function given by

$$\sum_{n,k\geq 0}\left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r \frac{z^n}{n!}y^k = e^{y(e^z-1)+rz}. \tag{1.62}$$

Theorem 1.9.2. *The explicit formula given by*

$$\left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(r+j)^n. \tag{1.63}$$

Proof. Using the generating function to prove the explicit formula

$$\begin{aligned}
\frac{1}{k!}e^{rz}(e^z-1)^k &= \frac{e^{rz}}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}e^{jz} \\
&= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}e^{z(r+j)} \\
&= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{n\geq 0}\frac{z^n(r+j)^n}{n!} \\
&= \sum_{n\geq 0}\left(\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(r+j)^n\right)\frac{z^n}{n!} \\
&= \sum_{n\geq k}\left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r \frac{z^n}{n!}.
\end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find

$$\left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(r+j)^n.$$

□

Recurrence Relation

These numbers verify the following recurrence relation for $k\geq 0$ and $n\geq r$

$$\left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r = \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\}_r + \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\}_r.$$

With $\left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r = 0 \qquad if \quad n<r$

$$\left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r = \delta_{k,r} \qquad if \quad n=r$$

where $\delta_{k,r}$ is the Kronecker delta.

0	1							
1	2	1						
2	4	5	1					
3	8	19	9	1				
4	16	65	55	14	1			
5	32	211	285	215	20	1		
6	64	655	1351	910	245	27	1	
7	128	2059	6069	5901	2380	434	35	1

Table 1.5: First values for 2-Stirling numbers of the second kind

1.10 Bernoulli Numbers And Polynomials

In this section, we will study the Bernoulli numbers and their polynomials, their properties, generating function, explicit formula and recurrence relation, which were mentioned in detail in the references [2, 10, 23].

1.10.1 Bernoulli Numbers

Definition 1.10.1. *Bernoulli numbers, denoted by B_n, form a sequence of rational numbers that can be defined by the following recurrence relation*

$$\begin{cases} B_0 = 1 \\ B_n = -\frac{1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k. \end{cases}$$

Generating Function

The generating function of the Bernoulli numbers is defined by

$$\sum_{n\geq 0} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1}; \qquad |z| < 2\pi.$$

Explicit Formula

Theorem 1.10.1. *The explicit formula for Bernoulli numbers is defined by*

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} \left\{ n \atop k \right\}. \tag{1.64}$$

Proof. Using the exponential generating function we have

$$\sum_{n\geq 0} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1}; \qquad |z| < 2\pi.$$

and:

$$\sum_{k=0}^{n} \left\{ n \atop k \right\} \frac{z^n}{n!} = \frac{(e^z - 1)^k}{k!}.$$

assume $y = e^z - 1$, we have :

$$\begin{aligned} z = \ln(e^z) &= \ln((e^z - 1) + 1) \\ &= \ln(y + 1) \\ &= \sum_{n\geq 1} (-1)^{n-1} \frac{y^n}{n} \\ &= \sum_{n\geq 1} (-1)^{n-1} \frac{(e^z - 1)^n}{n}. \end{aligned}$$

assume $N = n-1$, $n = N+1$.

$$z = \sum_{N\geq 0} (-1)^N \frac{(e^z-1)^{N+1}}{N+1}.$$

we have

$$\begin{aligned}
\sum_{n\geq 0} B_n \frac{z^n}{n!} &= \frac{z}{e^z-1} \\
&= \frac{1}{e^z-1} \sum_{k\geq 0} (-1)^k \frac{(e^z-1)^{k+1}}{k+1} \\
&= \sum_{k\geq 0} (-1)^k \frac{(e^z-1)^k}{k+1} \\
&= \sum_{k\geq 0} (-1)^k \frac{k!}{k+1} \frac{(e^z-1)^k}{k!} \\
&= \sum_{k\geq 0} (-1)^k \frac{k!}{k+1} \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{z^n}{n!} \\
&= \sum_{n\geq 0} \left(\sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} \left\{ {n \atop k} \right\} \right) \frac{z^n}{n!}.
\end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} \left\{ {n \atop k} \right\}.$$

□

Recurrence Relation

The recurrence relation of Bernoulli numbers is given by

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = n+1, \qquad n \geq 0$$

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	1/2	1/6	0	-1/30	0	1/42	0	-1/30	0	5/66

Table 1.6: The first values of Bernoulli numbers with $B_1 = \frac{1}{2}$.

1.10.2 Bernoulli Polynomials

Definition 1.10.2. *For $n \in \mathbb{N}$, the Bernoulli polynomials denoted $B_n(x)$, are defined by generating function and explicit formula, given by :*

Generating Function

Theorem 1.10.2. *The exponential generating function of Bernoulli polynomials is defined by*

$$\sum_{n\geq 0} B_n(x) \frac{z^n}{n!} = \frac{ze^{zx}}{e^z-1}; \qquad |z| < 2\pi. \tag{1.65}$$

Explicit Formula

Theorem 1.10.3. *The explicit formula for Bernoulli polynomials is defined by*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{1.66}$$

Proof. Using the exponential generating function we have

$$\sum_{n\geq 0} B_n(x)\frac{z^n}{n!} = \frac{ze^{zx}}{e^z-1}; \qquad |z| < 2\pi.$$

we have

$$\begin{aligned}
\frac{ze^{zx}}{e^z-1} &= \left(\sum_{n\geq 0} B_n \frac{z^n}{n!}\right)\left(\sum_{n\geq 0} \frac{(zx)^n}{n!}\right) \\
&= \left(\sum_{n\geq 0} B_n \frac{z^n}{n!}\right)\left(\sum_{n\geq 0} \frac{x^n z^n}{n!}\right) \\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \frac{B_k}{k!}\frac{x^{n-k}}{(n-k)!}\right) z^n \\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} B_k x^{n-k}\right) \frac{z^n}{n!} \\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}\right) \frac{z^n}{n!}.
\end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

□

Properties *The sequence of Bernoulli polynomials denoted by $(B_n(x))_{n\in\mathbb{N}}$, is defined such that its first term $B_0(x) = 1$, and the subsequent terms are determined recursively according to certain formulas involving Bernoulli numbers and derivatives, for $(n \geq 1)$*

1.

$$\int_0^1 B_n(x)dx = 0. \tag{1.67}$$

2.

$$B_n'(x) = nB_{n-1}(x). \tag{1.68}$$

Proof. 1. The proof of (1.67), we have

$$\begin{aligned}
\int_0^1 \sum_{n\geq 0} B_n(x)\frac{z^n}{n!}dx &= \int_0^1 \frac{ze^{zx}}{e^z-1}dx \\
&= \frac{z}{e^z-1}\int_0^1 e^{zx}dx \\
&= \frac{z}{e^z-1}\left[\frac{e^{zx}}{z}\right]_0^1 \\
&= \frac{z}{e^z-1}\frac{e^z-1}{z} = 1.
\end{aligned}$$

This gives the identity $\sum_{n\geq 0}\int_0^1 B_n(x)dx\frac{z^n}{n!}=1$. On the other hand, we have

$$\begin{aligned}\int_0^1\left(\sum_{n\geq 0}B_n(x)\frac{z^n}{n!}\right)dx&=\int_0^1 B_0(x)dx+\sum_{n\geq 1}\left(\int_0^1 B_n(x)dx\right)\frac{z^n}{n!}\\&=1+0=1.\end{aligned}$$

Thus $\int_0^1 B_n(x)dx=0$ for $n\geq 1$.

2. From (1.68), we have

$$\begin{aligned}\frac{d}{dx}\left(\sum_{n\geq 0}B_n(x)\frac{z^n}{n!}\right)&=\frac{d}{dx}\frac{ze^{zx}}{e^z-1}\\&=\frac{z^2e^{zx}}{e^z-1}\\&=z\sum_{n\geq 0}B_n(x)\frac{z^n}{n!}\\&=\sum_{n\geq 0}B_n(x)\frac{z^{n+1}}{n!}\\&=\sum_{n\geq 1}B_{n-1}(x)\frac{z^n}{(n-1)!}.\end{aligned}$$

On the other hand

$$\frac{d}{dx}\left(\sum_{n\geq 0}B_n(x)\frac{z^n}{n!}\right)=\sum_{n\geq 0}\left(\frac{dB_n(x)}{dx}\right)\frac{z^n}{n!}=\sum_{n\geq 0}B_n'(x)\frac{z^n}{n!}=\sum_{n\geq 1}\frac{B_n'(x)}{n}\frac{z^n}{(n-1)!}.$$

So,

$$\sum_{n\geq 1}B_{n-1}(x)\frac{z^n}{(n-1)!}=\sum_{n\geq 1}\frac{B_n'(x)}{n}\frac{z^n}{(n-1)!}.$$

By identification of the coefficients of $\frac{z^n}{(n-1)!}$

$$\frac{B_n'(x)}{n}=B_{n-1}(x)\quad\Rightarrow\quad B_n'(x)=nB_{n-1}(x).$$

□

The First Bernoulli Polynomials Values

$$\begin{aligned}B_0(x)&=1,\\B_1(x)&=x-\frac{1}{2},\\B_2(x)&=x^2-x+\frac{1}{6},\\B_3(x)&=x^3-\frac{3}{2}x^2+\frac{1}{2}x,\\B_4(x)&=x^4-2x^3+x^2-\frac{1}{30},\\B_5(x)&=x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x.\end{aligned}$$

1.11 Bell Numbers And Polynomials

In this section, we will study the Bell numbers and their polynomials, their properties, generating function, explicit formula and recurrence relation, which were mentioned in detail in the references [2, 3, 6, 7].

1.11.1 Bell Numbers

Definition 1.11.1. *For $n \geq 1$, the Bell numbers, denoted ϕ_n, are the total number of partitions of an n-element set. They are defined as the sum of Stirling numbers of the second kind*

$$\phi_n = \sum_{k=0}^{n} \left\{ n \atop k \right\}. \tag{1.69}$$

Example 1.11.1. *For $n = 4$ we have $\phi_4 = \left\{ 4 \atop 1 \right\} + \left\{ 4 \atop 2 \right\} + \left\{ 4 \atop 3 \right\} + \left\{ 4 \atop 4 \right\} = 1 + 7 + 6 + 1 = 15$.*

Table 1.7: The first values of Bell numbers

n	1	2	3	4	5	6	7	8	9	10	11	12
ϕ_n	1	2	5	15	52	203	877	4140	21147	115975	678570	4213597

Generating Function

Theorem 1.11.1. *The exponential generating functions of Bell numbers are given by*

$$\sum_{n \geq 0} \phi_n \frac{z^n}{n!} = \exp\left(e^z - 1\right). \tag{1.70}$$

Proof. Using the exponential generating (1.70), we have

$$\begin{aligned}
\exp\left(e^z - 1\right) &= \sum_{k \geq 0} \frac{(e^z - 1)^k}{k!} \\
&= \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j e^{z(k-j)} \\
&= \sum_{k \geq 0} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \sum_{n \geq 0} \frac{z^n}{n!} (k-j)^n \\
&= \sum_{n \geq 0} \left(\frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^j (k-j)^n \right) \frac{z^n}{n!} \\
&= \sum_{n \geq 0} \left\{ n \atop k \right\} \frac{z^n}{n!} \\
&= \sum_{n \geq 0} \phi_n \frac{z^n}{n!}.
\end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find

$$\phi_n = \sum_{k=0}^{n} \left\{ n \atop k \right\}.$$

□

Proposition 1.11.1. *We have the following Dobiński formula, for $n \geq 0$,*

$$\phi_n = \frac{1}{e}\sum_{k\geq 0}\frac{k^n}{k!}. \tag{1.71}$$

Proof. Let f a function be defined on $\mathbb{R}$ by: $f(z) = e^{e^z-1}$. According to formula (1.69) since the function f is derivable on this interval, we have

$$\begin{aligned}
f'(z) &= e^z e^{e^z-1} \\
&= \left(\sum_{n\geq 0}\frac{z^n}{n!}\right)\left(\sum_{n\geq 0}\phi_n\frac{z^n}{n!}\right) \\
&= \left(\sum_{n\geq 0}\frac{z^n}{n!}\right)\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}\phi_k\right)\frac{z^n}{n!} \\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\phi_k\right)\frac{z^n}{n!} \\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{\phi_k}{k!(n-k)!}\right)z^n = \sum_{n\geq 0}\sum_{k=0}^{n}\frac{z^{n-k}}{(n-k)!}\frac{z^k}{k!}\phi_k \\
&= \sum_{k=0}^{\infty}\phi_k\frac{z^k}{k!}\sum_{n\geq k}\frac{z^{n-k}}{(n-k)!} = f(z)e^z.
\end{aligned}$$

So we have $f(z) = e^{-1}e^{e^z}$, then for $z \in \mathbb{C}$:

$$e^{e^z} = \sum_{n\geq 0}\frac{(e^z)^n}{n!} = \sum_{n\geq 0}\frac{e^{nz}}{n!} = \sum_{n\geq 0}\sum_{k\geq 0}\frac{(nz)^k}{k!n!} = \sum_{k\geq 0}\frac{z^k}{k!}\sum_{n\geq 0}\frac{n^k}{n!}.$$

So,

$$e^{e^z-1} = \sum_{k\geq 0}\frac{z^k}{k!}\frac{1}{e}\sum_{n\geq 0}\frac{n^k}{n!}.$$

Secondly we have

$$\sum_{k\geq 0}\phi_k\frac{z^k}{k!} = e^{e^z-1}.$$

So,

$$\sum_{k\geq 0}\phi_k\frac{z^k}{k!} = \sum_{k\geq 0}\frac{z^k}{k!}\frac{1}{e}\sum_{n\geq 0}\frac{n^k}{n!}.$$

Equating the coefficients of $\frac{z^k}{k!}$, we get the result. □

Recurrence Relation

$$\phi_0 = 1, \quad \phi_{n+1} = \sum_{k=0}^{n}\binom{n}{k}\phi_k. \tag{1.72}$$

1.11.2 Bell Polynomials

Definition 1.11.2. *For $n \in \mathbb{N}$, the Bell polynomials $\phi_n(x)$ are defined by the explicit formula*

$$\phi_n(x) = \sum_{k=0}^{n}\left\{{n \atop k}\right\}x^k. \tag{1.73}$$

Generating Function

The exponential generating function of Bell polynomials is given by

$$\sum_{n\geq 0}\phi_n(x)\frac{z^n}{n!}=e^{x(e^z-1)}. \tag{1.74}$$

Explicit Formula

Theorem 1.11.2. *The explicit formula of Bell polynomials defined by*

$$\phi_n(x)=\sum_{k=0}^{n}\left\{{n \atop k}\right\}x^k. \tag{1.75}$$

Proof. Using the exponential generating function, we have

$$\sum_{n\geq 0}\phi_n(x)\frac{z^n}{n!}=e^{x(e^z-1)}.$$

We have

$$\begin{aligned}
e^{x(e^z-1)} &= \sum_{k\geq 0}x^k\frac{(e^z-1)^k}{k!}\\
&= \sum_{k\geq 0}\frac{x^k}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{z(k-j)}\\
&= \sum_{k\geq 0}\frac{x^k}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\sum_{n\geq 0}\frac{z^n}{n!}(k-j)^n\\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{x^k}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j(k-j)^n\right)\frac{z^n}{n!}\\
&= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\left\{{n \atop k}\right\}x^k\right)\frac{z^n}{n!}\\
&= \sum_{n\geq 0}\phi_n(x)\frac{z^n}{n!}.
\end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find

$$\phi_n(x)=\sum_{k=0}^{n}\left\{{n \atop k}\right\}x^k.$$

□

Dobiński's Formula For Polynomials

$$\phi_n(x)=e^{-x}\sum_{k\geq 0}\frac{k^n}{k!}x^k. \tag{1.76}$$

Proof. Let f a function be defined on $\mathbb{R}$ by: $f(z)=e^{x(e^z-1)}$. According to formula (1.74) since the

function f is derivable on this interval, we have

$$\begin{aligned}
f'(z) &= xe^z e^{x(e^z-1)} \\
&= x\left(\sum_{n\geq 0}\frac{z^n}{n!}\right)\left(\sum_{n\geq 0}\phi_n(x)\frac{z^n}{n!}\right) \\
&= x\left(\sum_{n\geq 0}\frac{z^n}{n!}\right)\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}\phi_k(x)\right)\frac{z^n}{n!} \\
&= x\sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\phi_k(x)\right)\frac{z^n}{n!} \\
&= x\sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{\phi_k(x)}{k!(n-k)!}\right)z^n = x\sum_{n\geq 0}\sum_{k=0}^{n}\frac{z^{n-k}}{(n-k)!}\frac{z^k}{k!}\phi_k(x) \\
&= x\sum_{k=0}^{\infty}\phi_k(x)\frac{z^k}{k!}\sum_{n\geq k}\frac{z^{n-k}}{(n-k)!} = xf(z)e^z.
\end{aligned}$$

so we have $f(z) = e^{-x}e^{xe^z}$. Then for $z \in \mathbb{C}$,

$$e^{xe^z} = \sum_{n\geq 0}\frac{x^n e^{nz}}{n!} = \sum_{n\geq 0}x^n\sum_{k\geq 0}\frac{(nz)^k}{k!n!} = \sum_{k\geq 0}\frac{z^k}{k!}\sum_{n\geq 0}\frac{x^n n^k}{n!}.$$

so

$$e^{x(e^z-1)} = \sum_{k\geq 0}\frac{z^k}{k!}\frac{1}{e^x}\sum_{n\geq 0}\frac{x^n n^k}{n!}.$$

Second we have:

$$\sum_{k\geq 0}\phi_k(x)\frac{z^k}{k!} = e^{x(e^z-1)}.$$

so

$$\sum_{k\geq 0}\phi_k(x)\frac{z^k}{k!} = \sum_{k\geq 0}\frac{z^k}{k!}\frac{1}{e^x}\sum_{n\geq 0}\frac{x^n n^k}{n!}.$$

By identification of coefficients $\frac{z^n}{n!}$, we get the result. □

Recurrence Relation

$$\phi_0(x) = 1, \quad \phi_{n+1}(x) = x\sum_{k=0}^{n}\binom{n}{k}\phi_k(x). \tag{1.77}$$

First Bell Polynomials

n	$\phi_n(x)$
0	1
1	x
2	$x + x^2$
3	$x + 3x^2 + x^3$
4	$x + 7x^2 + 6x^3 + x^4$

Table 1.8: Bell polynomials $\phi_n(x)$

Chapter 2

Cauchy Numbers And Their Polynomials

Cauchy numbers, also known as Bernoulli numbers of the second kind, are a sequence of numbers that appear in various contexts in mathematical analysis and combinatorics. They are defined through their generating function and have deep connections with Stirling numbers, harmonic numbers, and other combinatorial objects, which were mentioned in detail in the references [7, 12, 18, 20, 25].

2.1 The Cauchy Numbers Of The First Kind

Definition 2.1.1. *The Cauchy numbers of the first kind are defined by the definite integrals*

$$C_n = \int_0^1 z(z-1)(z-2)\cdots(z-n+1)dz = \int_0^1 z^{\underline{n}}dz. \tag{2.1}$$

Property 2.1.1. *Note that we can rewrite the definition of the Cauchy numbers in a shorter form if we take the binomial coefficients into account*

$$C_n = n!\int_0^1 \binom{z}{n}dz. \tag{2.2}$$

Proof. We have

$$\binom{z}{n} = \frac{z^{\underline{n}}}{n!}.$$

$$\begin{aligned} C_n &= \int_0^1 z(z-1)(z-2)\cdots(z-n+1)dz \\ &= \int_0^1 z^{\underline{n}}dz \\ &= \int_0^1 n!\binom{z}{n}dz \\ &= n!\int_0^1 \binom{z}{n}dz. \end{aligned}$$

□

Generating Function And Explicit Formula

Theorem 2.1.1. *The generating function of the Cauchy numbers of first kind given by :*

$$\sum_{n=0}^{\infty} C_n \frac{z^n}{n!} = \frac{z}{\ln(1+z)}. \tag{2.3}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} C_n \frac{z^n}{n!} &= \sum_{n=0}^{\infty} \int_0^1 \binom{z}{n} dz\, z^n \\ &= \int_0^1 \sum_{n=0}^{\infty} \binom{z}{n} z^n\, dz \\ &= \int_0^1 (1+z)^x dz \\ &= \frac{z}{\ln(1+z)}.\end{aligned}$$

□

Theorem 2.1.2. *These numbers are connected to the Stirling numbers of the first kind, and are defined by*

$$C_n = \sum_{k=0}^{n} s(n,k) \frac{1}{k+1}. \tag{2.4}$$

Proof. We have

$$z^{\underline{n}} = \sum_{k=0}^{n} s(n,k) z^k.$$

So,

$$\begin{aligned}C_n &= \int_0^1 z^{\underline{n}} dz \\ &= \sum_{k=0}^{n} s(n,k) \int_0^1 z^k dz \\ &= \sum_{k=0}^{n} s(n,k) \frac{1}{k+1}.\end{aligned}$$

□

n	0	1	2	3	4	5	6	7
C_n	1	1/2	-1/6	1/4	-19/30	9/4	-863/84	1375/24

Table 2.1: Cauchy numbers of the first kind

Recurrence Relation

For the Cauchy numbers of the first kind [17], we have

$$C_n = n! \sum_{k=0}^{n-1} \frac{C_k}{k!} \frac{(-1)^{n-k+1}}{n-k+1}, \quad (n \geq 1).$$

Theorem 2.1.3. *The following identity [17], defined by*

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} C_k = \frac{1}{1+n}.$$

Remark 2.1.1. *There is a relation between Cauchy numbers of the first kind and the Bernoulli numbers, defined by*

$$b_n = \frac{C_n}{n!}.$$

They are also called the Bernoulli numbers of the first kind b_n.

2.2 The Cauchy Numbers Of The Second Kind

Definition 2.2.1. *The Cauchy numbers of the second kind are defined by the definite integrals*

$$\widehat{C_n} = \int_0^1 z(z+1)(z+2)\cdots(z+n-1)dz = \int_0^1 z^{\overline{n}}dz. \tag{2.5}$$

They are also called Norlund numbers denoted by $B_n^{(n)}$ when

$$\widehat{C}_n = B_n^{(n)}.$$

Property 2.2.1. *Note that we can rewrite the definition of the Cauchy numbers in a shorter form if we take the binomial coefficients into account*

$$\widehat{C_n} = (-1)^n n! \int_0^1 \binom{-z}{n} dz. \tag{2.6}$$

Proof. We have

$$\binom{z}{n} = \frac{z^{\underline{n}}}{n!}.$$

So

$$\begin{aligned}
\widehat{C_n} &= \int_0^1 z(z+1)(z+2)\cdots(z+n-1)dz \\
&= \int_0^1 z^{\overline{n}}dz \\
&= \int_0^1 (-1)^n(-z)^{\underline{n}}dz \\
&= \int_0^1 (-1)^n n!\binom{-z}{n}dz \\
&= n!(-1)^n \int_0^1 \binom{-z}{n}dz.
\end{aligned}$$

□

Property 2.2.2. *Note that we can rewrite the definition of the Cauchy numbers in a shorter form if we take the binomial coefficients into account*

$$\widehat{C_n} = n! \int_0^1 \binom{z+n-1}{n} dz, \tag{2.7}$$

Proof. We have

$$\binom{z}{n} = \frac{z^{\underline{n}}}{n!}.$$

$$\begin{aligned}
n!\int_0^1 \binom{z+n-1}{n}dz &= n!\int_0^1 \frac{(z+n-1)^{\underline{n}}}{n!}dz \\
&= \int_0^1 (z+n-1)(z+n-2)\cdots(z+n-1-n+1)dz \\
&= \int_0^1 z(z+1)\cdots(z+n-1)dz \\
&= \int_0^1 z^{\overline{n}}dz \\
&= \widehat{C_n}
\end{aligned}$$

□

Generating Function And Explicit Formula

Theorem 2.2.1.

$$\sum_{n=0}^{\infty} \widehat{C_n} \frac{z^n}{n!} = \frac{z}{(1-z)\ln(1-z)}. \tag{2.8}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{C_n} \frac{z^n}{n!} &= \sum_{n=0}^{\infty} (-1)^n \int_0^1 \binom{-z}{n} dz\, z^n \\
&= \int_0^1 \sum_{n=0}^{\infty} \binom{-z}{n} (-1)^n z^n \, dz \\
&= \int_0^1 \sum_{n=0}^{\infty} \binom{-z}{n} (-z)^n \, dz \\
&= \int_0^1 (1-z)^{-x} dz \\
&= \frac{z}{(1-z)\ln(1-z)}.
\end{aligned}$$

□

Theorem 2.2.2. *These numbers are connected to the unsigned Stirling numbers of the first kind [24], and are defined by*

$$\widehat{C_n} = \int_0^1 z^{\overline{n}} dz = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{k+1}. \tag{2.9}$$

Proof. We have

$$z^{\overline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} z^k.$$

So,

$$\begin{aligned}
\widehat{C_n} &= \int_0^1 z^{\overline{n}} dz \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \int_0^1 z^k dz \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{k+1}.
\end{aligned}$$

□

n	0	1	2	3	4	5	6	7
$\widehat{C_n}$	1	-1/2	5/6	-9/4	251/30	-475/12	19087/84	-36799/24

Table 2.2: Cauchy numbers of the second kind

Recurrence Relation

For the Cauchy numbers of the second kind [17], we have

$$\widehat{C_n} = n! \sum_{k=0}^{n-1} \frac{C_k}{k!} \frac{(-1)^{n-k}}{(n-k)(n-k+1)}, \quad (n \geq 1).$$

Theorem 2.2.3. *The following identity [11], defined by*

$$\sum_{k=0}^{n} \left\{ n \atop k \right\} \widehat{C_k} = \frac{(-1)^n}{n+1}.$$

2.3 Cauchy Polynomials

In this section, we will study the Cauchy polynomials, which were mentioned in detail in the references [5, 12, 22].

2.3.1 Cauchy polynomials Of The First Kind

Definition 2.3.1. *In 2007 Cheon, Hwang and Lee introduced the Cauchy polynomials of the first kind by the definite integral*

$$C_n(x) = n! \int_0^1 \binom{z-x}{n} dz, \tag{2.10}$$

and when we take $x = 0$ we get $C_n(0) = C_n$ the classical Cauchy numbers of the first kind.

Remark 2.3.1. *The polynomials $b_n(x) = \frac{C_n(x)}{n!}$ are sometimes called Bernoulli polynomials of the second kind, it means*

$$\sum_{k=0}^{n} b_n(x) z^n = \frac{z}{\ln(1+z)(1+z)^x}.$$

Generating Function

Theorem 2.3.1. *The generating function of the first kind is defined by*

$$\sum_{n\geq 0} C_n(x) \frac{z^n}{n!} = \frac{z}{\ln(1+z)(1+z)^x}. \tag{2.11}$$

Proof.

$$\begin{aligned}
\sum_{n\geq 0} C_n(x) \frac{z^n}{n!} &= \sum_{n\geq 0} \left(\int_0^1 \binom{z-x}{n} dz \right) z^n \\
&= \int_0^1 \sum_{n\geq 0} \binom{z-x}{n} z^n \, dz \\
&= \int_0^1 (1+z)^{z-x} dz \\
&= \frac{z}{\ln(1+z)(1+z)^x}.
\end{aligned}$$

□

Proposition 2.3.1. *Komatsu proved that the Cauchy polynomials of the first kind satisfy*

$$\sum_{k=0}^{n} \left\{ n \atop k \right\} C_k(x) = \sum_{i=0}^{n} \binom{n}{i} \frac{(-x)^i}{n-i+1}. \tag{2.12}$$

Explicit Formula

For $n, m \in \mathbb{N}$ from (2.12) we have the explicit formula defined by

$$C_n(x) = \sum_{k=0}^{n} \left[n \atop k \right] (-1)^{n-k} \sum_{i=0}^{n} \binom{n}{i} \frac{(-x)^i}{n-i+1}.$$

Recurrence Relation

The recurrence relation is given by:

$$\sum_{k=0}^{n}\binom{n}{k}C_k(x)C_{n-k}(y) = -n(x+y+n-2)C_{n-1}(x+y) - (n-1)C_n(x+y), \qquad n \geq 0 \tag{2.13}$$

n	$C_n(x)$
0	1
1	$\frac{1}{2} - x$
2	$-\frac{1}{6} + x + x^2$
3	$\frac{1}{4} - \frac{1}{2}x + \frac{3}{2}x^2 + x^3$
4	$\frac{-19}{30} + x - x^2 + 2x^3 + x^4$
5	$\frac{9}{4} - \frac{19}{6}x + \frac{5}{2}x^2 - \frac{5}{3}x^3 + \frac{5}{2}x^4 + x^5$

Table 2.3: Cauchy polynomials of the first kind $C_n(x)$ for $n = 0, 1, \ldots, 5$

2.3.2 Cauchy Polynomials Of The Second Kind

Definition 2.3.2. *In [16] defined the Cauchy polynomials of the Second kind by the definite integral*

$$\widehat{C_n}(x) = n!\int_0^1 \binom{z+x}{n}dz, \tag{2.14}$$

And when we take $x = 0$ we get $\widehat{C_n}(0) = \widehat{C_n}$ are the classical Cauchy numbers of the second kind.

Generating Function

Theorem 2.3.2. *The generating function of the second kind is defined by*

$$\sum_{n\geq 0}\widehat{C_n}(x)\frac{z^n}{n!} = \frac{z(1+z)^x}{(1+z)\ln(1+z)}. \tag{2.15}$$

Proof.

$$\begin{aligned}
\sum_{n\geq 0}\widehat{C_n}(x)\frac{z^n}{n!} &= \sum_{n\geq 0}\left(n!\int_0^1\binom{z+x}{n}dz\right)z^n \\
&= \int_0^1 \sum_{n\geq 0}\binom{z+x}{n}.z^n\,dz \\
&= \int_0^1 \sum_{n\geq 0}\binom{z+x}{n}z^n\,dz \\
&= \int_0^1 (1+z)^{z+x}dz \\
&= \frac{z(1+z)^x}{(1+z)\ln(1+z)}.
\end{aligned}$$

□

Explicit Formula

For $n, m \in \mathbb{N}$ from (2.12) we have the explicit formula defined by

$$\widehat{C_n}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} \sum_{i=0}^{n} \binom{n}{i} \frac{x^i}{n-i+1}.$$

Proposition 2.3.2. *We have*

$$\beta_n = \sum_{i=0}^{n} \binom{n}{i} \frac{x^i}{n-i+1}.$$

So,

$$\widehat{C_n}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} \beta_k.$$

Recurrence Relation

The recurrence relation is given by

$$\widehat{C_n}(x) = \frac{1}{n+1} \left((2n-1)\widehat{C}_{n-1}(x) + (x+n-1)\widehat{C}_{n-2}(x) \right).$$

Example 2.3.1. *For example with $\widehat{C_0}(x) = 1$ and $\widehat{C_1}(x) = -\frac{1}{2} + x$.*

n	$\widehat{C_n}(x)$
0	1
1	$-\frac{1}{2} + x$
2	$\frac{5}{6} - x + x^2$
3	$-\frac{9}{4} + \frac{5}{2}x - \frac{3}{2}x^2 + x^3$
4	$\frac{251}{30} - 9x + 5x^2 - 2x^3 + x^4$
5	$-\frac{475}{12} + \frac{251}{6}x - \frac{45}{2}x^2 + \frac{25}{3}x^3 - \frac{5}{2}x^4 + x^5$

Table 2.4: Cauchy polynomials of the second kind $\widehat{C_n}(x)$ for $n = 0, 1, \ldots, 5$

Proposition 2.3.3. *We have the relation $\widehat{C_n}(x) = C_n(-x)$*

Proof. We have

$$\widehat{C_n}(x) = n! \int_0^1 \binom{z+x}{n} dz,$$

So

$$\begin{aligned} C_n(-x) &= n! \int_0^1 \binom{z-(-x)}{n} dz \\ &= n! \int_0^1 \binom{z+x}{n} dz \\ &= \widehat{C_n}(x). \end{aligned}$$

□

2.4 Relation Between The Cauchy Numbers And r-Stirling Numbers

In this section, we will discuss the relation between the Cauchy numbers of both kinds and r-Stirling numbers, which were mentioned in detail in the reference [16].

2.4.1 Relation With The Cauchy Numbers Of First Kind

For an integer r, Cauchy polynomials of the first kind $C_n(r)$ can be expressed in terms of the r-Stirling numbers of the first kind $\left[\begin{matrix} n+r \\ k+r \end{matrix} \right]_r$.

Theorem 2.4.1. *For nonnegative integers n and r, we have*

$$C_n(r) = \sum_{k=0}^{n} \left[\begin{matrix} n+r \\ k+r \end{matrix} \right]_r \frac{(-1)^{n-k}}{k+1} \tag{2.16}$$

Proof. We put

$$G_{n,r}(x) = \sum_{k=0}^{n} \left[\begin{matrix} n+r \\ k+r \end{matrix} \right]_r x^k. \tag{2.17}$$

Using (1.56) the generating function of the r-Stirling numbers of the first kind we have

$$\begin{aligned} \sum_{n\geq 0} G_{n,r}(x) \frac{z^n}{n!} &= \sum_{n\geq 0} \left(\sum_{k=0}^{n} \left[\begin{matrix} n+r \\ k+r \end{matrix} \right]_r x^k \right) \frac{z^n}{n!} \\ &= \sum_{k=0}^{n} x^k \sum_{n\geq 0} \left[\begin{matrix} n+r \\ k+r \end{matrix} \right]_r \frac{z^n}{n!} \\ &= \sum_{k=0}^{n} x^k \frac{(-\ln(1-z))^k}{k!(1-z)^r} \\ &= \frac{1}{(1-z)^r} (-\ln(1-z))^x \\ &= \frac{1}{(1-z)^r (1-z)^x}. \end{aligned}$$

By (2.11), and by integrating with respect to x from $[-1, 0]$ we find

$$\begin{aligned} \int_0^{-1} \frac{1}{(1-z)^r (1-z)^x} \, dz &= \frac{z}{(1-z)^r \ln(1-z)} \\ &= -\sum_{n\geq 0} C_n(r)(-1)^n \frac{z^n}{n!}. \end{aligned}$$

And also we integrating (2.17), with respect to x from $[-1,0]$ on both sides, we have

$$\int_0^{-1}\sum_{n\geq 0}G_{n,r}(x)\frac{z^n}{n!}\,dx=\sum_{n\geq 0}\left(\sum_{k=0}^{n}\left[\begin{array}{c} n+r \\ k+r \end{array}\right]_r\frac{z^n}{n!}\right)\int_0^{-1}x^k dx$$
$$=\sum_{n\geq 0}\left(\sum_{k=0}^{n}\left[\begin{array}{c} n+r \\ k+r \end{array}\right]_r\frac{(-1)^{k+1}}{k+1}\frac{z^n}{n!}\right).$$

Comparing the coefficients of both sides, we get the identity (2.16) □

Remark 2.4.1. *If $r=0$, the identity (2.16) is reduced to*

$$C_n=\sum_{k=0}^{n}\frac{s(n,k)}{k+1}.$$

This gives the Cauchy numbers of the second kind (see [7], Ch. VII, [14], Theorem 1, [15], p.1908).

Generating Function

The generating function of these kind of numbers is defined by

$$\sum_{n\geq 0}C_n(r)\frac{z^n}{n!}=\frac{z}{\ln(1+z)(1+z)^r}.$$

2.4.2 Relation With The Cauchy Numbers Of Second Kind

For an integer r[7, 16], Cauchy polynomials of the second kind $\widehat{C_n}(r)$ can also be expressed in terms of the r-Stirling numbers of the first kind $\left[\begin{array}{c} n+r \\ k+r \end{array}\right]_r$.

Theorem 2.4.2. *For nonnegative integers n and r, we have*

$$\widehat{C_n}(-r)=(-1)^n\sum_{k=0}^{n}\left[\begin{array}{c} n+r \\ k+r \end{array}\right]_r\frac{1}{k+1}. \tag{2.18}$$

Proof. From (1.56) the generating function of the r-Stirling numbers of the first kind, we have

$$\sum_{n\geq 0}G_{n,r}(-x)=\frac{1}{(1-z)^r}(-\ln(1-z))^x$$
$$=\frac{(1-z)^x}{(1-z)^r}.$$

By integrating with respect to x from $[-1,0]$ on both sides, we have

$$\int_0^{-1}\sum_{n\geq 0}G_{n,r}(-x)\frac{z^n}{n!}dx=\sum_{n\geq 0}\sum_{k=0}^{n}\left[\begin{array}{c} n+r \\ k+r \end{array}\right]_r\frac{z^n}{n!}\int_0^{-1}(-x)^k dx$$
$$=-\sum_{n\geq 0}\sum_{k=0}^{n}\left[\begin{array}{c} n+r \\ k+r \end{array}\right]_r\frac{1}{k+1}\frac{z^n}{n!}.$$

And by (2.15) we have

$$\int_0^{-1}\frac{(1-z)^x}{(1-z)^r}dx=\frac{z(1-z)^r}{(1-z)\ln(1-z)}$$
$$=-\sum_{n\geq 0}\widehat{C_n}(r)(-1)^n\frac{z^n}{n!}$$

Comparing the coefficients of both sides, we get the identity (2.18)

$$\widehat{C_n}(-r) = (-1)^n \sum_{k=0}^{n} \left[\begin{array}{c} n+r \\ k+r \end{array} \right]_r \frac{1}{k+1}.$$

□

Remark 2.4.2. *If $r = 0$, the identity (2.6) is reduced to*

$$\widehat{C_n}(0) = (-1)^n \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right] \frac{1}{k+1}.$$

This gives the Cauchy numbers of the second kind (see [7], Ch. VII, [14], Theorem 4, [15]).

Generating Function

The generating function of these kind of numbers is defined by

$$\sum_{n\geq 0} \widehat{C_n}(-r) \frac{z^n}{n!} = \frac{z(1-z)^r}{(1-z)\ln(1-z)}.$$

Chapter 3

Generalization Of Cauchy Numbers And Polynomials

In this section, we introduce one of the generalizations of Cauchy Numbers is called the p-Cauchy numbers and polynomials, which generalize the classical Cauchy numbers through a parameter p. We examine their fundamental properties, including recurrence relations and explicit representations [13, 20].

3.1 The p-Cauchy Numbers

3.1.1 The p-Cauchy Numbers Of The First Kind

Definition 3.1.1. *We define the p-Cauchy numbers of the first kind denoted $C_{n,p}(n \geq 0)$, for every integer $p \geq -1$ by*

$$f_p(z) = \sum_{n\geq 0} C_{n,p}\frac{z^n}{n!} = {}_1F_1\left(\begin{array}{c} 1 \\ p+2 \end{array}; \ln(1+z)\right). \tag{3.1}$$

where ${}_1F_1\left(\begin{array}{c} a \\ b \end{array}; z\right)$ denotes the Kummer confluent hypergeometric function [1].

Remark 3.1.1. *For $p = -1$, we have*

$$\sum_{n\geq 0} C_{n,-1}\frac{z^n}{n!} = {}_1F_1\left(\begin{array}{c} 1 \\ 1 \end{array}; \ln(1+z)\right) = 1+z.$$

It is because for $p = -1$ we find

$$\begin{aligned}\sum_{n\geq 0} C_{n,-1}\frac{z^n}{n!} &= {}_1F_1\left(\begin{array}{c} 1 \\ 1 \end{array}; \ln(1+z)\right) \\ &= \sum_{k\geq 0}\frac{1^{\overline{n}}}{1^{\overline{n}}}\frac{(\ln(1+z))^k}{k!} \\ &= \exp(\ln(1+z)) \\ &= 1+z.\end{aligned}$$

and for $p = 0$, we find C_n the classical Cauchy numbers of the first kind.

Explicit Formula

Theorem 3.1.1. *For $p \geq -1$, these numbers can be written explicitly as*

$$C_{n,p} = \sum_{k=0}^{n} s(n,k)\left(\begin{array}{c} k+p+1 \\ k \end{array}\right)^{-1}. \tag{3.2}$$

Proof. Using (1.47) the generating function of Stirling numbers of the first kind (signed), we have

$$\begin{aligned}\sum_{n\geq 0}\sum_{k=0}^{n}\left(\begin{array}{c} k+p+1 \\ k \end{array}\right)^{-1} s(n,k)\frac{z^n}{n!} &= \sum_{k\geq 0}\left(\begin{array}{c} k+p+1 \\ k \end{array}\right)^{-1}\sum_{n\geq k} s(n,k)\frac{z^n}{n!} \\ &= \sum_{k\geq 0}\frac{(p+1)!k!}{(k+p+1)!}\frac{(\ln(1+z))^k}{k!} \\ &= \sum_{k\geq 0}\frac{1^{\overline{k}}}{(p+2)^{\overline{k}}}\frac{(\ln(1+z))^k}{k!} \\ &= {}_1F_1\left(\begin{array}{c} 1 \\ p+2 \end{array};\ln(1+z)\right) \\ &= \sum_{n\geq 0} C_{n,p}\frac{z^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides, we arrive at the result. □

Remark 3.1.2. *If we take $p = 0$ in (3.2) we find the classical Cauchy numbers of the first kind C_n.*

The First Few Exponential Generating Functions For $C_{n,p}$

For $p = 1, 2, 3$ we have

$$\begin{aligned}\sum_{n\geq 0} C_{n,1}\frac{z^n}{n!} &= \frac{-2(\ln(1+z)-z)}{(\ln(1+z))^2}. \\ \sum_{n\geq 0} C_{n,2}\frac{z^n}{n!} &= \frac{-3((\ln(1+z))^2+2\ln(1+z)-2z)}{(\ln(1+z))^3}. \\ \sum_{n\geq 0} C_{n,3}\frac{z^n}{n!} &= \frac{-4((\ln(1+z))^3+3(\ln(1+z))^2+6\ln(1+z)-6z)}{(\ln(1+z))^4}.\end{aligned}$$

Example 3.1.1. *For $p = 1$ we have*

$$\begin{aligned}\sum_{n\geq 0} C_{n,1}\frac{z^n}{n!} &= \sum_{k\geq 0}\frac{(1+1)!}{(k+1+1)!}(\ln(1+z))^k \\ &= \sum_{k\geq 0}\frac{2}{(k+2)!}(\ln(1+z))^k.\end{aligned}$$

With $t = \ln(1+z)$ we find

$$\begin{aligned}\sum_{n\geq 0} C_{n,1}\frac{z^n}{n!} &= 2\sum_{k\geq 0}\frac{t^k}{(k+2)!} \\ &= \frac{2}{t^2}\sum_{k\geq 0}\frac{t^{k+2}}{(k+2)!} \\ &= \frac{2}{t^2}\sum_{k\geq 2}\frac{t^k}{k!} \\ &= \frac{2}{t^2}(e^t-1-t).\end{aligned}$$

So we get

$$\begin{aligned}\sum_{n\geq 0} C_{n,1}\frac{z^n}{n!} &= \frac{2(e^{\ln(1+z)}-1-\ln(1+z))}{(\ln(1+z))^2} \\ &= \frac{-2(\ln(1+z)-z)}{(\ln(1+z))^2}.\end{aligned}$$

Corollary 3.1.1. *For $r \geq 0$ we have*

$$\sum_{k=0}^{n} \left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r C_{r+k,p} = \sum_{k=0}^{n} s(r,k) \left(\begin{array}{c} n+k+p+1 \\ n+k \end{array} \right)^{-1},$$

where $\left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r$ *is r-Stirling numbers of the second kind [4]. In particular for $m = 0$ we find*

$$\begin{aligned} \sum_{k=0}^{n} \left\{ n \atop k \right\} C_{k,p} &= s(0,0) \left(\begin{array}{c} n+p+1 \\ n \end{array} \right)^{-1} \\ &= \frac{n!(p+1)!}{(n+p+1)!}. \end{aligned}$$

Proof. We have in [19][Theorem 1, 2 and Corollary 1] the Stirling transform

$$\sum_{k=0}^{n} \left\{ \begin{array}{c} n+r \\ k+r \end{array} \right\}_r a_{r+k} = \sum_{k=0}^{r} s(r,k) b_{n+k}.$$

If we take $a_{r+k} = C_{r+k,p}$, we get $b_{n+k} = \left(\begin{array}{c} n+k+p+1 \\ n+k \end{array} \right)^{-1}$. □

Theorem 3.1.2. *we have*

$$f_p(z) = (p+1) \int_0^1 (1+z)^t (1-t)^p dt. \tag{3.3}$$

Proof. Let the integral representation of the confluent hypergeometric function [1] defined by (1.31) so

$$\begin{aligned} f_p(z) = {}_1F_1 \left(\begin{array}{c} 1 \\ p+2 \end{array} ; \ln(1+z) \right) &= \frac{\Gamma(p+2)}{\Gamma(p+1)\Gamma(1)} \int_0^1 e^{t\ln(1+z)} t^0 (1-t)^{p+2-1-1} dt \\ &= \frac{(p+1)\Gamma(p+1)}{\Gamma(p+1)} \int_0^1 e^{\ln(1+z)^t} (1-t)^p dt \\ &= (p+1) \int_0^1 (1+z)^t (1-t)^p dt. \end{aligned}$$

□

Theorem 3.1.3. *The p-Cauchy numbers of the first kind satisfy the following three-term recurrence relation*

$$C_{n+1,p} = (1-n) C_{n,p} - \frac{p+1}{p+2} C_{n,p+1}, \tag{3.4}$$

with $n = 0$ we get $C_{0,p} = 1$.

Proof. by differentiation of (3.3) with respect to z; we obtain

$$\begin{aligned} (1+z)\frac{d}{dz} f_p(z) &= (p+1) \int_0^1 t(1+z)^t (1-t)^p dt \\ &= -(p+1) \int_0^1 (1-t-1)(1+z)^t (1-t)^p dt \\ &= -(p+1) \int_0^1 (1+z)^t (1-t)^{p+1} dt + (p+1) \int_0^1 (1+z)^t (1-t)^p dt \\ &= -\frac{p+1}{p+2} f_{p+1}(z) + f_p(z). \end{aligned}$$

or equivalently

$$(1+z)\sum_{n\geq 0}\frac{d}{dz}C_{n,p}\frac{z^n}{n!}=-\frac{p+1}{p+2}\sum_{n\geq 0}C_{n,p+1}\frac{z^n}{n!}+\sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}$$

$$(1+z)\sum_{n\geq 0}C_{n,p}n\frac{z^{n-1}}{n!}=-\frac{p+1}{p+2}\sum_{n\geq 0}C_{n,p+1}\frac{z^n}{n!}+\sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}.$$

by change of variables $n-1=N$, and $n=N+1$ we find

$$(1+z)\sum_{n\geq 0}C_{n+1,p}\frac{z^n}{n!}=-\frac{p+1}{p+2}\sum_{n\geq 0}C_{n,p+1}\frac{z^n}{n!}+\sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}$$

$$\sum_{n\geq 0}C_{n+1,p}\frac{z^n}{n!}+\sum_{n\geq 0}C_{n+1,p}\frac{z^{n+1}}{n!}=-\frac{p+1}{p+2}\sum_{n\geq 0}C_{n,p+1}\frac{z^n}{n!}+\sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}.$$

by change of variables $n+1=N$, and $n=N-1$ we find

$$\sum_{n\geq 0}C_{n+1,p}\frac{z^n}{n!}+n\sum_{n\geq 0}C_{n,p}\frac{z^n}{n(n-1)!}=-\frac{p+1}{p+2}\sum_{n\geq 0}C_{n,p+1}\frac{z^n}{n!}+\sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}$$

$$\sum_{n\geq 0}C_{n+1,p}\frac{z^n}{n!}+(n-1)\sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}=-\frac{p+1}{p+2}\sum_{n\geq 0}C_{n,p+1}\frac{z^n}{n!}$$

$$\sum_{n\geq 0}C_{n+1,p}\frac{z^n}{n!}=\sum_{n\geq 0}\left[(1-n)C_{n,p}-\frac{p+1}{p+2}C_{n,p+1}\right]\frac{z^n}{n!}.$$

By identification we find (3.4). □

List Of The p-Cauchy Numbers Of The First Kind $C_{n,p}$

For $0\leq n\leq 6$ *and* $0\leq p\leq 3$.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & \cdots & C_{0,p}\\
\\
1/2 & 1/3 & 1/4 & 1/5 & \cdots & C_{1,p}\\
-1/6 & -1/6 & -3/20 & -2/15 & \cdots & C_{2,p}\\
1/4 & 4/15 & 1/4 & 8/35 & \cdots & C_{3,p}\\
-19/30 & -7/10 & -47/70 & -131/210 & \cdots & C_{4,p}\\
9/4 & 107/42 & 139/56 & 1496/630 & \cdots & C_{5,p}\\
-863/84 & -995/84 & -9809/840 & -2323/210 & \cdots & C_{6,p}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
\\
C_{n,0} & C_{n,1} & C_{n,2} & C_{n,3} & C_{n,p} &
\end{pmatrix}$$

Theorem 3.1.4. *The p-Cauchy numbers of the first kind satisfy the following recurrence relation*

$$C_{n+1,p+1}=(1-n)C_{n,p}-\left(\frac{n}{p+2}+1\right)C_{n,p+1}+\frac{n+1}{p+3}C_{n,p+2}.$$

with the initial sequence $C_{0,p}=1$, *and the final sequence* $C_{n,0}=C_n$.

Proof. By (1.41) with $z:=\ln(1+z)$, we have

$${}_1F_1\left(\begin{array}{c}1\\ p+2\end{array};z\right)=\left(1+\frac{z}{p+2}\right){}_1F_1\left(\begin{array}{c}1\\ p+3\end{array};z\right)-\frac{z}{p+3}{}_1F_1\left(\begin{array}{c}1\\ p+4\end{array};z\right)$$

$$\sum_{n\geq 0} C_{n,p}\frac{z^n}{n!} = \left(1+\frac{z}{p+2}\right)\sum_{n\geq 0} C_{n,p+1}\frac{z^n}{n!} - \frac{z}{p+3}\sum_{n\geq 0} C_{n,p+2}\frac{z^n}{n!}$$

$$\sum_{n\geq 0} C_{n,p}\frac{z^n}{n!} = \sum_{n\geq 0} C_{n,p+1}\frac{z^n}{n!} + \frac{1}{p+2}\sum_{n\geq 0} C_{n,p+1}\frac{z^{n+1}}{n!} - \frac{1}{p+3}\sum_{n\geq 0} C_{n,p+2}\frac{z^{n+1}}{n!}.$$

by change of variables $n+1=N$, so $n=N-1$, we have

$$\sum_{n\geq 0} C_{n,p}\frac{z^n}{n!} = \sum_{n\geq 0} C_{n,p+1}\frac{z^n}{n!} + \frac{1}{p+2}\sum_{n\geq 0} nC_{n-1,p+1}\frac{z^n}{n!} - \frac{1}{p+3}\sum_{n\geq 0} nC_{n-1,p+2}\frac{z^n}{n!}$$

$$\sum_{n\geq 0} C_{n,p}\frac{z^n}{n!} = \sum_{n\geq 0}\left(C_{n,p+1} + \frac{1}{p+2}nC_{n-1,p+1} - \frac{1}{p+3}nC_{n-1,p+2}\right)\frac{z^n}{n!}.$$

equating the coefficients of $\frac{z^n}{n!}$ and using (3.4), we get the result.

$$C_{n+1,p+1} = (1-n)C_{n,p} - \left(\frac{n}{p+2}+1\right)C_{n,p+1} + \frac{n+1}{p+3}C_{n,p+2}.$$

□

Lemma 3.1.1. *We have*

$${}_1F_1\left(\begin{array}{c} p+1 \\ p+2 \end{array};\pm\ln(1+z)\right) = (\pm 1)^p(p+1)\left((1+z)\frac{d}{dz}\right)^p {}_1F_1\left(\begin{array}{c} 1 \\ 2 \end{array};\pm\ln(1+z)\right). \tag{3.5}$$

Proof. The verification of (3.5) follows by induction on p. It follows from induction hypothesis that

$$\pm\frac{p+1}{p+2}{}_1F_1\left(\begin{array}{c} p+2 \\ p+3 \end{array};\pm\ln(1+z)\right) = (1+z)\frac{d}{dz}\left[{}_1F_1\left(\begin{array}{c} p+1 \\ p+2 \end{array};\pm\ln(1+z)\right)\right]$$

$$= (\pm 1)^p(p+1)\left((1+z)\frac{d}{dz}\right)^{p+1} {}_1F_1\left(\begin{array}{c} 1 \\ 2 \end{array};\pm\ln(1+z)\right).$$

□

Theorem 3.1.5. *The exponential generating function for p-Cauchy numbers of the first kind is given by*

$$\sum_{n\geq 0} C_{n,p}\frac{z^n}{n!} = (-1)^p(p+1)(1+z)\left((1+z)\frac{d}{dz}\right)^p\left(\frac{z}{(1+z)\ln(1+z)}\right).$$

Proof. According to the Kummer transformation [1], we can write

$$\sum_{n\geq 0} C_{n,p}\frac{z^n}{n!} = {}_1F_1\left(\begin{array}{c} 1 \\ p+2 \end{array};\ln(1+z)\right)$$

$$= (1+z)\,{}_1F_1\left(\begin{array}{c} p+1 \\ p+2 \end{array};-\ln(1+z)\right),$$

and by (3.5) we get

$$\sum_{n\geq 0} C_{n,p}\frac{z^n}{n!} = (-1)^p(p+1)(1+z)\left((1+z)\frac{d}{dz}\right)^p\left(\frac{z}{(1+z)\ln(1+z)}\right).$$

□

Theorem 3.1.6. *The exponential generating function for p-Cauchy numbers of the first kind is given by*

$${}_1F_1\left(\begin{array}{c} 1 \\ p+2 \end{array};\ln(1+z)\right) = \frac{(p+1)!z}{(\ln(1+z))^{p+1}} - \sum_{k=1}^{p}\frac{(p+1)^{\underline{k}}}{(\ln(1+z))^k}.$$

Proof. Using integration by parts in the formula (3.3) with $u=(1-t)^p$, $dv=(1+z)^t dt$, we get

$$\begin{aligned}
f_p(z) &= (p+1)\left[\left[\frac{(1-t)^p(1+z)^t}{\ln(1+z)}\right]_0^1 - \int_0^1 (-p(1-t)^{p-1})\frac{(1+z)^t}{\ln(1+z)}dt\right] \\
&= (p+1)\left[-\frac{1}{\ln(1+z)} + \frac{p}{\ln(1+z)}\int_0^1 (1+z)^t(1-t)^{p-1}dt\right] \\
&= (p+1)\left(\frac{f_{p-1}(z)}{\ln(1+z)} - \frac{1}{\ln(1+z)}\right) \\
&= \frac{p+1}{\ln(1+z)}f_{p-1}(z) - \frac{p+1}{\ln(1+z)}.
\end{aligned}$$

we have

$$f_{p-1}(z) = \frac{p!z}{(\ln(1+z))^p} - \sum_{k=1}^{p-1}\frac{p^{\underline{k}}}{(\ln(1+z))^k}.$$

so,

$$\begin{aligned}
f_p(z) &= \frac{p+1}{\ln(1+z)}f_{p-1}(z) - \frac{p+1}{\ln(1+z)} \\
&= \frac{p+1}{\ln(1+z)}\left(\frac{p!z}{(\ln(1+z))^p} - \sum_{k=1}^{p-1}\frac{p^{\underline{k}}}{(\ln(1+z))^k}\right) - \frac{p+1}{\ln(1+z)} \\
&= \frac{(p+1)!z}{(\ln(1+z))^{p+1}} - \sum_{k=1}^{p-1}\frac{(p+1)p^{\underline{k}}}{(\ln(1+z))^{k+1}} - \frac{p+1}{\ln(1+z)}.
\end{aligned}$$

we have $(p+1)p^{\underline{k}} = (p+1)^{\underline{k+1}}$; by change of variables $K=k+1$ we find

$$\begin{aligned}
f_p(z) &= \frac{(p+1)!z}{(\ln(1+z))^{p+1}} - \sum_{K=2}^{p}\frac{(p+1)^{\underline{K}}}{(\ln(1+z))^K} - \frac{(p+1)^{\underline{1}}}{\ln(1+z)} \\
&= \frac{(p+1)!z}{(\ln(1+z))^{p+1}} - \sum_{k=1}^{p}\frac{(p+1)^{\underline{k}}}{(\ln(1+z))^k}.
\end{aligned}$$

□

3.1.2 The p-Cauchy Numbers Of The Second Kind

Definition 3.1.2. *We define the p-Cauchy Numbers of the second kind denoted $\widehat{C}_{n,p}(n\geq 0)$, for every integer $p\geq -1$ by*

$$\sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!} = {}_1F_1\left(\begin{matrix}1\\ p+2\end{matrix};-\ln(1+z)\right). \tag{3.6}$$

Here ${}_1F_1\left(\begin{matrix}a\\ b\end{matrix};z\right)$ denotes the Kummer confluent hypergeometric function.

Explicit Formula

Theorem 3.1.7. *These numbers can be written explicitly as for $p\geq -1$*

$$\widehat{C}_{n,p} = \sum_{k=0}^{n}(-1)^k s(n,k)\left(\begin{matrix}k+p+1\\ k\end{matrix}\right)^{-1}. \tag{3.7}$$

Proof. Using (1.47) the generating function of Stirling numbers of the first kind (signed), we have

$$\begin{aligned}
\sum_{n\geq 0}\sum_{k=0}^{n}(-1)^k \left(\begin{array}{c} k+p+1 \\ k \end{array} \right)^{-1} s(n,k)\frac{z^n}{n!} &= \sum_{k\geq 0}\left(\begin{array}{c} k+p+1 \\ k \end{array} \right)^{-1}\sum_{n\geq k}(-1)^k s(n,k)\frac{z^n}{n!} \\
&= \sum_{k\geq 0}\frac{(p+1)!k!}{(k+p+1)!}(-1)^k\frac{(\ln(1+z))^k}{k!} \\
&= \sum_{k\geq 0}\frac{(p+1)!k!}{(k+p+1)!}\frac{(-\ln(1+z))^k}{k!} \\
&= \sum_{k\geq 0}\frac{1^{\overline{k}}}{(p+2)^{\overline{k}}}\frac{(-\ln(1+z))^k}{k!} \\
&= {}_1F_1\left(\begin{array}{c} 1 \\ p+2 \end{array} ; -\ln(1+z)\right) \\
&= \sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides, we arrive at the result. □

Remark 3.1.3. *If we take $p=0$ in (3.7) we find the classical Cauchy numbers of the second kind $\widehat{C}_n$.*

The First Few Exponential Generating Functions For $\widehat{C}_{n,p}$

For $p=1,2$ we have

$$\begin{aligned}
\sum_{n\geq 0}\widehat{C}_{n,1}\frac{z^n}{n!} &= \frac{-2((1+z)\ln(1+z)-z)}{(1+z)(\ln(1+z))^2} \\
\sum_{n\geq 0}\widehat{C}_{n,2}\frac{z^n}{n!} &= \frac{-3(((1+z)\ln(1+z))^2-2(1+z)\ln(1+z)+2z)}{(1+z)(\ln(1+z))^3}.
\end{aligned}$$

Example 3.1.2. *For $p=1$ we have*

$$\sum_{n\geq 0}\widehat{C}_{n,1}\frac{z^n}{n!} = \sum_{k\geq 0}\frac{2}{(k+2)!}(-\ln(1+z))^k.$$

For $t=\ln(1+z)$ we get

$$\sum_{k\geq 0}\frac{2}{(k+2)!}(-t)^k = \frac{2}{(-t)^2}\sum_{k\geq 0}\frac{(-t)^{k+2}}{(k+2)!} = \frac{2}{t^2}\sum_{k\geq 2}\frac{(-t)^k}{k!} = \frac{2}{t^2}(e^{-t}-1+t).$$

we have $e^{-t}=\frac{1}{1+z}$ so,

$$\sum_{n\geq 0}\widehat{C}_{n,1}\frac{z^n}{n!} = \frac{-2((1+z)\ln(1+z)-z)}{(1+z)(\ln(1+z))^2}.$$

Theorem 3.1.8. *We have integral representation given by*

$$\widehat{f}_p(z) = (p+1)\int_0^1 (1+z)^{-t}(1-t)^p dt. \tag{3.8}$$

Proof.

$$\begin{aligned}
\widehat{f}_p(z) &= {}_1F_1\left(\begin{array}{c}1\\ p+2\end{array};-\ln(1+z)\right)\\
&= \frac{\Gamma(p+2)}{\Gamma(p+1)\Gamma(1)}\int_0^1 e^{t(-\ln(1+z))}t^0(1-t)^{p+2-1-1}dt\\
&= \frac{(p+1)\Gamma(p+1)}{\Gamma(p+1)}\int_0^1 e^{-t\ln(1+z)}(1-t)^p dt\\
&= (p+1)\int_0^1 (1+z)^{-t}(1-t)^p dt.
\end{aligned}$$

□

Theorem 3.1.9. *The p-Cauchy numbers of the second kind satisfy the following three-term recurrence relation*

$$\widehat{C}_{n+1,p} = \frac{p+1}{p+2}\widehat{C}_{n,p+1} - (n+1)\widehat{C}_{n,p}. \tag{3.9}$$

and for $n = 0$ we get

$$\widehat{C}_{0,p} = 1.$$

Proof. By differentiation of (3.8) with respect to z; we obtain

$$\begin{aligned}
(1+z)\frac{d}{dz}\widehat{f}_p(z) &= (p+1)\int_0^1 (-t)(1+z)^{-t}(1-t)^p dt\\
&= (p+1)\int_0^1 (1-t-1)(1+z)^{-t}(1-t)^p dt\\
&= (p+1)\int_0^1 (1+z)^t(1-t)^{p+1}dt - (p+1)\int_0^1 (1+z)^{-t}(1-t)^p dt\\
&= \frac{p+1}{p+2}\widehat{f}_{p+1}(z) - \widehat{f}_p(z).
\end{aligned}$$

or equivalently

$$\begin{aligned}
(1+z)\sum_{n\geq 0}\frac{d}{dz}\widehat{C}_{n,p}\frac{z^n}{n!} &= (1+z)\sum_{n\geq 0}\widehat{C}_{n,p}n\frac{z^{n-1}}{n!}\\
&= \frac{p+1}{p+2}\sum_{n\geq 0}\widehat{C}_{n,p+1}\frac{z^n}{n!} - \sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!}.
\end{aligned}$$

by change of variables $n-1 = N$ and $n = N+1$ we find

$$(1+z)\sum_{n\geq 0}\widehat{C}_{n+1,p}\frac{z^n}{n!} = \frac{p+1}{p+2}\sum_{n\geq 0}\widehat{C}_{n,p+1}\frac{z^n}{n!} - \sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!}$$

$$\sum_{n\geq 0}\widehat{C}_{n+1,p}\frac{z^n}{n!} + \sum_{n\geq 0}\widehat{C}_{n+1,p}\frac{z^{n+1}}{n!} = \frac{p+1}{p+2}\sum_{n\geq 0}\widehat{C}_{n,p+1}\frac{z^n}{n!} - \sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!}.$$

by change of variables $n+1 = N$ and $n = N-1$ we find

$$\sum_{n\geq 0}\widehat{C}_{n+1,p}\frac{z^n}{n!} + n\sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n(n-1)!} = \frac{p+1}{p+2}\sum_{n\geq 0}\widehat{C}_{n,p+1}\frac{z^n}{n!} - \sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!}$$

$$\sum_{n\geq 0}\widehat{C}_{n+1,p}\frac{z^n}{n!} + (n+1)\sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!} = \frac{p+1}{p+2}\sum_{n\geq 0}\widehat{C}_{n,p+1}\frac{z^n}{n!}$$

$$\sum_{n\geq 0}\widehat{C}_{n+1,p}\frac{z^n}{n!} = \sum_{n\geq 0}\left[\frac{p+1}{p+2}\widehat{C}_{n,p+1} - (n+1)\widehat{C}_{n,p}\right]\frac{z^n}{n!}.$$

By identification we find (3.9). □

List Of The p-Cauchy Numbers Of The Second Kind $\widehat{C}_{n,p}$

For $0 \leq n \leq 6$ and $0 \leq p \leq 3$.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & \widehat{C}_{0,p} \\ -1/2 & -1/3 & -1/4 & -1/5 & \cdots & \widehat{C}_{1,p} \\ 1/6 & 1/6 & 3/20 & 2/15 & \cdots & \widehat{C}_{2,p} \\ -1/4 & -4/15 & -1/4 & -8/35 & \cdots & \widehat{C}_{3,p} \\ 19/30 & 7/10 & 47/70 & 131/210 & \cdots & \widehat{C}_{4,p} \\ -9/4 & -107/42 & -139/56 & -1496/630 & \cdots & \widehat{C}_{5,p} \\ 863/84 & 995/84 & 9809/840 & 2323/210 & \cdots & \widehat{C}_{6,p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ \widehat{C}_{n,0} & \widehat{C}_{n,1} & \widehat{C}_{n,2} & \widehat{C}_{n,3} & \widehat{C}_{n,p} & \end{pmatrix}$$

Theorem 3.1.10.

$$\sum_{n\geq 0} \widehat{C}_{n,p} \frac{z^n}{n!} = \frac{p+1}{1+z} \left((1+z)\frac{d}{dz} \right)^p \left(\frac{z}{\ln(1+z)} \right).$$

$$= \frac{(-1)^p (p+1)! z}{(1+z)(\ln(1+z))^{p+1}} - \sum_{k=1}^{p} \frac{(-1)^k (p+1)_k}{(\ln(1+z))^k}.$$

Proof. As in the previous proof of (3.8) by $u = (1-t)^p$, $dv = (1+z)^{-t}dt$, we get

$$\begin{aligned} \widehat{f}_p(z) &= (p+1)\left[\left[\frac{(1-t)^p (1+z)^{-t}}{\ln(1+z)} \right]_0^1 - \int_0^1 (p(1-t)^{p-1}) \frac{(1+z)^{-t}}{\ln(1+z)} dt \right] \\ &= (p+1)\left[\frac{1}{\ln(1+z)} - \frac{p}{\ln(1+z)} \int_0^1 (1+z)^{-t}(1-t)^{p-1} dt \right] \\ &= (p+1)\left(\frac{1}{\ln(1+z)} - \frac{\widehat{f}_{p-1}(z)}{\ln(1+z)} \right) \\ &= \frac{p+1}{\ln(1+z)} - \frac{p+1}{\ln(1+z)} \widehat{f}_{p-1}(z). \end{aligned}$$

We find finally

$$\sum_{n\geq 0} \widehat{C}_{n,p} \frac{z^n}{n!} = \frac{(-1)^p (p+1)! z}{(1+z)(\ln(1+z))^{p+1}} - \sum_{k=1}^{p} \frac{(-1)^k (p+1)_k}{(\ln(1+z))^k}.$$

□

3.2 Relation Between The First And Second Kind Of p-Cauchy Numbers

We have a relation between both kinds of Cauchy numbers [20], given by

$$C_n = \widehat{C}_n + n\widehat{C}_{n-1}.$$

So we get the following theorem for p-Cauchy numbers

Theorem 3.2.1. *For nonnegative integers n and p, we have*

$$C_{n,p} = (-1)^p (p+1) \sum_{k=0}^{p} \binom{p}{k} \sum_{j=0}^{k+1} \binom{k+1}{j} \binom{n}{j} j! \widehat{C}_{n-j+k}. \tag{3.10}$$

Proof. We have

$$\sum_{n\geq 0} C_{n,p}\frac{z^n}{n!} = (-1)^p(p+1)\left((1+z)\frac{d}{dz}\right)^p \sum_{n\geq 0}\widehat{C}_n\frac{z^n}{n!},$$

and

$$\left((1+z)\frac{d}{dz}\right)^p = \sum_{k=0}^{p}\binom{p}{k}(1+z)^k\frac{d^k}{dz^k}.$$

So we get

$$\begin{aligned}
\sum_{n\geq 0} C_{n,p}\frac{z^n}{n!} &= (-1)^p(p+1)\sum_{k=0}^{p}\binom{p}{k}(1+z)^k\frac{d^k}{dz^k}\left(\sum_{n\geq 0}\widehat{C}_n\frac{z^n}{n!}\right)\\
&= (-1)^p(p+1)\sum_{k=0}^{p}\binom{p}{k}(1+z)^k\left(\sum_{n\geq 0}\widehat{C}_{n+k}\frac{z^n}{n!}\right)\\
&= (-1)^p(p+1)\sum_{k=0}^{p}\binom{p}{k}\sum_{j=0}^{k}\binom{k}{j}z^j\left(\sum_{n\geq 0}\widehat{C}_{n+k}\frac{z^n}{n!}\right)\\
&= (-1)^p(p+1)\sum_{k=0}^{p}\binom{p}{k}\sum_{j=0}^{k}\binom{k}{j}\left(\sum_{m\geq j}\widehat{C}_{m-j+k}\frac{z^m}{(m-j)!}\right)\\
&= (-1)^p(p+1)\sum_{k=0}^{p}\binom{p}{k}\sum_{j=0}^{k}\binom{k}{j}j!\left(\sum_{m\geq 0}\binom{m}{j}\widehat{C}_{m-j+k}\frac{z^m}{m!}\right).
\end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find (3.10).

□

Theorem 3.2.2. *For nonnegative integers n and p, we have*

$$\widehat{C}_{n,p} + n\widehat{C}_{n-1,p} = (p+1)\sum_{k=0}^{p}\binom{p}{k}\sum_{j=0}^{k}\binom{k}{j}\binom{n}{j}j!C_{n-j+k}.$$

Proof. we have

$$\begin{aligned}
\sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!} &= \frac{1}{1+z}\sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}\\
(1+z)\sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!} &= \sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}\\
\sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!} + \sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^{n+1}}{n!} &= \sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}\\
\sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!} + \sum_{n\geq 0}\widehat{C}_{n-1,p}\frac{z^n}{(n-1)!} &= \sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}\\
\sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!} + \sum_{n\geq 0}n\widehat{C}_{n-1,p}\frac{z^n}{n!} &= \sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}\\
\sum_{n\geq 0}\left(\widehat{C}_{n,p} + n\widehat{C}_{n-1,p}\right)\frac{z^n}{n!} &= \sum_{n\geq 0}C_{n,p}\frac{z^n}{n!}.
\end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find

$$\widehat{C}_{n,p} + n\widehat{C}_{n-1,p} = C_{n,p}.$$

□

3.3 The p-Cauchy Polynomials

We now introduce the p-Cauchy polynomials of both kinds. These are Appell polynomials, a family that has attracted attention from many perspectives [8, 20, 21]. As a reminder, a sequence of polynomials $(A_n(x))_{n\geq 0}$ is referred to as being of Appell type whenever it fulfills the relation

$$\frac{d}{dx}A_n(x) = nA_{n-1}(x), \quad n \geq 1.$$

3.3.1 The p-Cauchy Polynomials Of The First Kind

Generating Function And Explicit Formula

Theorem 3.3.1. *For $p \geq 0$, let us consider the p-Cauchy polynomials of the first kind $C_{n,p}(x)$ defined by means of the following generating function*

$$\sum_{n\geq 0} C_{n,p}(x)\frac{z^n}{n!} = {}_1F_1\left(\begin{array}{c}1\\ p+2\end{array};\ln(1+z)\right)e^{xz}. \tag{3.11}$$

Theorem 3.3.2. *Let $n, k, j \in \mathbb{N}$. The explicit formula is defined by*

$$C_{n,p}(x) = \sum_{k=0}^{n}\binom{n}{k}\left[\sum_{j=0}^{k}\frac{(p+1)!k!}{(p+k+1)!}s(k,j)\right]x^{n-k}. \tag{3.12}$$

Proof. According to (3.11) we have

$$\begin{aligned}\sum_{n\geq 0} C_{n,p}\frac{z^n}{n!} &= {}_1F_1\left(\begin{array}{c}1\\ p+2\end{array};\ln(1+z)\right)\\ &= \sum_{n\geq 0}\frac{1^{\overline{n}}}{(p+2)^{\overline{n}}}\frac{(\ln(1+z))^n}{n!}\\ &= \sum_{n\geq 0}\frac{(p+1)!n!}{(p+n+1)!}\frac{(\ln(1+z))^n}{n!}\\ &= \sum_{n\geq 0}\frac{(p+1)!n!}{(p+n+1)!}\sum_{k=0}^{n}s(n,k)\frac{z^k}{k!}.\end{aligned}$$

Suppose that

$$\alpha_{n,p} = \sum_{k=0}^{n}\frac{(p+1)!n!}{(p+n+1)!}s(n,k).$$

So,

$${}_1F_1\left(\begin{array}{c}1\\ p+2\end{array};\ln(1+z)\right) = \sum_{n\geq 0}\alpha_{n,p}\frac{z^n}{n!}.$$

Thus,

$$\begin{aligned}\sum_{n\geq 0} C_{n,p}(x)\frac{z^n}{n!} &= {}_1F_1\left(\begin{array}{c}1\\ p+2\end{array};\ln(1+z)\right)e^{xz}\\ &= \left(\sum_{n\geq 0}\alpha_{n,p}\frac{z^n}{n!}\right)\left(\sum_{n\geq 0}x^n\frac{z^n}{n!}\right)\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}\alpha_{k,p}x^{n-k}\right)\frac{z^n}{n!}.\end{aligned}$$

By identification we find

$$C_{n,p}(x) = \sum_{k=0}^{n} \binom{n}{k} \alpha_{k,p} x^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left[\sum_{j=0}^{k} \frac{(p+1)!k!}{(p+k+1)!} s(k,j) \right] x^{n-k}.$$

□

Theorem 3.3.3. *For $n \geq 0$ and $p \geq 0$, we have*

1.

$$C_{n,p}(x) = \sum_{k=0}^{n} \binom{n}{k} C_{k,p} x^{n-k}, \tag{3.13}$$

2.

$$\frac{d}{dx} C_{n,p}(x) = n C_{n-1,p}(x), \tag{3.14}$$

3.

$$\int_0^x C_{n,p}(t)dt = \frac{C_{n+1,p}(x) - C_{n+1,p}}{n+1}, \tag{3.15}$$

4.

$$\int_0^1 C_{n,p}(t)dt = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} C_{k,p}, \tag{3.16}$$

5.

$$C_{n,p}(x+y) = \sum_{k=0}^{n} \binom{n}{k} C_{k,p}(x) y^{n-k}, \tag{3.17}$$

6.

$$C_{n,p}(x+1) - C_{n,p}(x) = \sum_{k=0}^{n-1} \binom{n}{k} C_{k,p}(x), \tag{3.18}$$

7.

$$C_{n,p}(mx) = \sum_{k=0}^{n} \binom{n}{k} C_{k,p}(x)(m-1)^{n-k} x^{n-k}. \tag{3.19}$$

Proof. For $n \geq 0$ and $p \geq 0$.

1. The proof of formula (3.13), according to (3.11), we have

$$\sum_{n\geq 0} C_{n,p}(x) \frac{z^n}{n!} = {}_1F_1\left(\begin{matrix} 1 \\ p+2 \end{matrix} ; \ln(1+z) \right) e^{xz}$$

$$= \left(\sum_{n\geq 0} C_{n,p} \frac{z^n}{n!} \right) \left(\sum_{n\geq 0} x^n \frac{z^n}{n!} \right)$$

$$= \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{C_{k,p}}{k!} \frac{x^{n-k}}{(n-k)!} \right) z^n$$

$$= \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} C_{k,p} x^{n-k} \right) \frac{z^n}{n!}$$

By identification of coefficients $\frac{z^n}{n!}$, we find (3.13).

2. The proof of formula (3.14), according to (3.11), we have

$$\begin{aligned}\frac{d}{dx}\sum_{n\geq 0}C_{n,p}(x)\frac{z^n}{n!} &= \sum_{n\geq 0}\frac{d}{dx}C_{n,p}(x)\frac{z^n}{n!}\\ &= \frac{d}{dx}\left({}_1F_1\left(\begin{array}{c}1\\ p+2\end{array};\ln(1+z)\right)e^{xz}\right)\\ &= z\left({}_1F_1\left(\begin{array}{c}1\\ p+2\end{array};\ln(1+z)\right)e^{xz}\right)\\ &= z\left(\sum_{n\geq 0}C_{n,p}(x)\frac{z^n}{n!}\right)\\ &= \sum_{n\geq 0}C_{n,p}(x)\frac{z^{n+1}}{n!}.\end{aligned}$$

By change of variables $n+1=N$, $n=N-1$ we find

$$\sum_{n\geq 0}\frac{d}{dx}C_{n,p}(x)\frac{z^n}{n!} = \sum_{n\geq 0}C_{n-1,p}(x)\frac{z^n}{(n-1)!}$$

$$\sum_{n\geq 0}\frac{d}{dx}C_{n,p}(x)\frac{z^n}{n!} = \sum_{n\geq 0}nC_{n-1,p}(x)\frac{z^n}{n!}.$$

By identification of coefficients $\frac{z^n}{n!}$, we find (3.14).

3. The proof of formula (3.15), according to (3.14), we have

$$\frac{d}{dx}C_{n+1,p}(x) = (n+1)C_{n,p}(x).$$

By integration we find

$$\begin{aligned}\int_0^x\frac{d}{dx}C_{n+1,p}(x)dx &= (n+1)\int_0^x C_{n,p}(x)dx\\ C_{n+1,p}(x)-C_{n+1,p}(0) &= (n+1)\int_0^x C_{n,p}(x)dx\\ \int_0^x C_{n,p}(x)dx &= \frac{C_{n+1,p}(x)-C_{n+1,p}}{(n+1)},\end{aligned}$$

where $C_{n+1,p}(0)=C_{n+1,p}$.

4. The proof of formula (3.16), according to (3.13), we have

$$C_{n,p}(x) = \sum_{k=0}^{n}\binom{n}{k}C_{k,p}x^{n-k}.$$

By integration, we find

$$\begin{aligned}\int_0^1 C_{n,p}(x)dx &= \int_0^1\sum_{k=0}^{n}\binom{n}{k}C_{k,p}x^{n-k}dx\\ \int_0^1 C_{n,p}(x)dx &= \sum_{k=0}^{n}\binom{n}{k}C_{k,p}\int_0^1 x^{n-k}dx\\ \int_0^1 C_{n,p}(x)dx &= \sum_{k=0}^{n}\binom{n}{k}C_{k,p}\frac{1}{n-k+1}.\end{aligned}$$

We have

$$\binom{n}{k}\frac{1}{n-k+1} = \frac{n!}{k!(n-k)!(n-k+1)} = \frac{n!}{k!(n-k+1)!} = \frac{1}{1+n}\frac{(n+1)!}{k!(n-k+1)!} = \frac{1}{1+n}\binom{n+1}{k}.$$

So

$$\int_0^1 C_{n,p}(x)dx = \frac{1}{1+n}\sum_{k=0}^{n}\binom{n+1}{k}C_{k,p}.$$

5. The proof of formula (3.17), according to (3.11), we have

$$\sum_{n\geq 0} C_{n,p}(x+y)\frac{z^n}{n!} = {}_1F_1\left(\begin{array}{c}1\\ p+2\end{array};\ln(1+z)\right)e^{z(x+y)}.$$

By integration we find

$$\begin{aligned}\sum_{n\geq 0} C_{n,p}(x+y)\frac{z^n}{n!} &= \left(\sum_{n\geq 0} C_{n,p}(x)\frac{z^n}{n!}\right)e^{zy}\\ &= \left(\sum_{n\geq 0} C_{n,p}(x)\frac{z^n}{n!}\right)\left(\sum_{n\geq 0} y^n\frac{z^n}{n!}\right)\\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}C_{k,p}(x)y^{n-k}\right)\frac{z^n}{n!}.\end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find (3.17).

6. The proof of formula (3.18), according to (3.17) by $y=1$, we have

$$C_{n,p}(x+1) = \sum_{k=0}^{n}\binom{n}{k}C_{k,p}(x).$$

For $k=n$ we find

$$\binom{n}{n}C_{k,p}(x) = C_{k,p}(x).$$

Thus

$$C_{n,p}(x+1) = \sum_{k=0}^{n-1}\binom{n}{k}C_{k,p}(x) + C_{k,p}(x),$$

$$C_{n,p}(x+1) - C_{n,p}(x) = \sum_{k=0}^{n-1}\binom{n}{k}C_{k,p}(x).$$

7. The proof of formula (3.19), according to (3.17) by $y=(m-1)x$, we have

$$\begin{aligned}C_{n,p}(x+(m-1)x) &= C_{n,p}(mx)\\ &= \sum_{k=0}^{n}\binom{n}{k}C_{k,p}(x)((m-1)x)^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}C_{k,p}(x)(m-1)^{n-k}x^{n-k}.\end{aligned}$$

□

Example 3.3.1. *For $n=0$, according to (3.12)*

$$C_{0,p}(x) = 0!s(0,0)\frac{(p+1)!}{(p+1)!}x^0 = 1.$$

For $n=1$

$$C_{1,p}(x) = \sum_{k=0}^{1}\binom{1}{k}\left[\sum_{j=0}^{k}\frac{(p+1)!k!}{(p+k+1)!}s(k,j)\right]x^{1-k} = x + \frac{1}{p+2}.$$

The First Few p-Cauchy Polynomials Of The First Kind

$$C_{0,p}(x) = 1,$$
$$C_{1,p}(x) = x + \frac{1}{p+2},$$
$$C_{2,p}(x) = x^2 + \frac{2x}{p+2} - \frac{p+1}{(p+2)(p+3)},$$
$$C_{3,p}(x) = x^3 + \frac{3x^2}{p+2} - \frac{3(p+1)x}{(p+2)(p+3)} + \frac{2(p+1)}{(p+2)(p+4)},$$
$$C_{4,p}(x) = x^4 + \frac{4x^3}{p+2} - \frac{6(p+1)x^2}{(p+2)(p+3)} + \frac{8(p+1)x}{(p+2)(p+4)} - \frac{2(p+1)(3p^2+22p+38)}{(p+2)(p+3)(p+4)(p+5)}.$$

Lemma 3.3.1.

$$\sum_{k=0}^{n} k\binom{n}{k} C_{k,p} x^{n-k} = \sum_{k=0}^{n-1} \frac{n!}{k!}(-1)^{n-k}\left(\frac{p+1}{p+2}C_{k,p+1}(x) - C_{k,p}(x)\right).$$

Proof. Let us denote $\sum_{k=0}^{n} k\binom{n}{k} C_{k,p} x^{n-k}$ by $p_n(x)$, since $k\binom{n}{k} = n\binom{n-1}{k-1}$, and using (3.9), we have

$$\begin{aligned}
p_n(x) &= \sum_{k=0}^{n} k\binom{n}{k} C_{k,p} x^{n-k} \\
&= n\sum_{k=0}^{n-1} \binom{n-1}{k} C_{k+1,p} x^{n-k-1} \\
&= n\sum_{k=0}^{n-1} \binom{n-1}{k} \left((1-k)C_{k,p} - \frac{p+1}{p+2}C_{k,p+1}\right) x^{n-k-1} \\
&= nC_{n-1,p}(x) - \frac{p+1}{p+2} nC_{n-1,p+1}(x) - np_{n-1}(x) \\
&= n\sum_{k=0}^{n-1} \frac{(-1)^k}{k!}\left[C_{n-1-k,p}(x) - \frac{p+1}{p+2}C_{n-1-k,p+1}(x)\right] - (-1)^n.np_{n-1-n}(x) \\
&= n!\sum_{k=0}^{n-1} \frac{(-1)^k}{k!}\left(\frac{p+1}{p+2}C_{k,p+1}(x) - C_{k,p}(x)\right) \\
&= \sum_{k=0}^{n-1} \frac{n!}{k!}(-1)^k\left(\frac{p+1}{p+2}C_{k,p+1}(x) - C_{k,p}(x)\right).
\end{aligned}$$

Applying this recursively we get the desired result. □

Theorem 3.3.4. *The p-Cauchy polynomials of the first kind $C_{n,p}(x)$ satisfy the following recurrence relation*

$$C_{n+1,p}(x) = xC_{n,p}(x) - n!\sum_{j=0}^{n} \frac{(-1)^{n-j}}{j!}\left(\frac{p+1}{p+2}C_{j,p+1}(x) - C_{j,p}(x)\right). \tag{3.20}$$

With the initial sequence $C_{0,p}(x) = 1$.

Proof. We have

$$x\frac{d}{dx}C_{n,p}(x) = nC_{n,p}(x) - \sum_{k=0}^{n} k\binom{n}{k} C_{k,p} x^{n-k}.$$

And by (3.14)

$$xnC_{n-1,p}(x) = nC_{n,p}(x) - p_n(x)$$

$$xC_{n-1,p}(x) = C_{n,p}(x) - \frac{p_n(x)}{n}.$$

For $n \Rightarrow n+1$,

$$xC_{n,p}(x) = C_{n+1,p}(x) - \frac{p_{n+1}(x)}{n+1}.$$

So,

$$\begin{aligned} C_{n+1,p}(x) &= xC_{n,p}(x) + \frac{p_{n+1}(x)}{n+1} \\ &= xC_{n,p}(x) - n!\sum_{j=0}^{n} \frac{(-1)^{n-j}}{j!}\left(\frac{p+1}{p+2}C_{j,p+1}(x) - C_{j,p}(x)\right). \end{aligned}$$

□

3.3.2 The p-Cauchy Polynomials Of The Second Kind

Generating Function And Explicit Formula

Theorem 3.3.5. *For $p \geq 0$, let us consider the p-Cauchy polynomials of the second kind $C_{n,p}(x)$ defined by means of the following generating function*

$$\sum_{n\geq 0} \widehat{C}_{n,p}(x)\frac{z^n}{n!} = {}_1F_1\left(\begin{array}{c} 1 \\ p+2 \end{array}; -\ln(1+z)\right)e^{xz}. \tag{3.21}$$

Theorem 3.3.6. *Let $n, k, j \in \mathbb{N}$. The explicit formula is defined by*

$$\widehat{C}_{n,p}(x) = \sum_{k=0}^{n}\binom{n}{k}\left[\sum_{j=0}^{k}\frac{(p+1)!k!}{(p+k+1)!}(-1)^k s(k,j)\right]x^{n-k}. \tag{3.22}$$

Proof. According to (3.21), we have

$$\begin{aligned} \sum_{n\geq 0}\widehat{C}_{n,p}\frac{z^n}{n!} &= {}_1F_1\left(\begin{array}{c} 1 \\ p+2 \end{array}; -\ln(1+z)\right) \\ &= \sum_{n\geq 0}\frac{1^{\overline{n}}}{(p+2)^{\overline{n}}}\frac{(-\ln(1+z))^n}{n!} \\ &= \sum_{n\geq 0}\frac{(p+1)!n!}{(p+n+1)!}(-1)^n\frac{(\ln(1+z))^n}{n!} \\ &= \sum_{n\geq 0}\frac{(p+1)!n!}{(p+n+1)!}(-1)^n\sum_{k=0}^{n}s(n,k)\frac{z^k}{k!}. \end{aligned}$$

Suppose that

$$\widehat{\alpha}_{n,p} = \sum_{k=0}^{n}\frac{(p+1)!n!}{(p+n+1)!}(-1)^n s(n,k).$$

So

$${}_1F_1\left(\begin{array}{c} 1 \\ p+2 \end{array}; -\ln(1+z)\right) = \sum_{n\geq 0}\widehat{\alpha}_{n,p}\frac{z^n}{n!}.$$

Thus

$$\begin{aligned} \sum_{n\geq 0}\widehat{C}_{n,p}(x)\frac{z^n}{n!} &= {}_1F_1\left(\begin{array}{c} 1 \\ p+2 \end{array}; -\ln(1+z)\right)e^{xz} \\ &= \left(\sum_{n\geq 0}\widehat{\alpha}_{n,p}\frac{z^n}{n!}\right)\left(\sum_{n\geq 0}x^n\frac{z^n}{n!}\right) \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}\widehat{\alpha}_{k,p}x^{n-k}\right)\frac{z^n}{n!}. \end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find

$$C_{n,p}(x) = \sum_{k=0}^{n} \binom{n}{k} \widehat{\alpha}_{k,p} x^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left[\sum_{j=0}^{k} \frac{(p+1)!k!}{(p+k+1)!} (-1)^k s(k,j) \right] x^{n-k}.$$

□

Theorem 3.3.7. *For $n \geq 0$ and $p \geq 0$, we have*

1.

$$\widehat{C}_{n,p}(x) = \sum_{k=0}^{n} \binom{n}{k} \widehat{C}_{k,p} x^{n-k}, \tag{3.23}$$

2.

$$\frac{d}{dx} \widehat{C}_{n,p}(x) = n \widehat{C}_{n-1,p}(x), \tag{3.24}$$

3.

$$\int_0^x \widehat{C}_{n,p}(t)dt = \frac{\widehat{C}_{n+1,p}(x) - \widehat{C}_{n+1,p}}{n+1}, \tag{3.25}$$

4.

$$\int_0^1 \widehat{C}_{n,p}(t)dt = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} \widehat{C}_{k,p}, \tag{3.26}$$

5.

$$\widehat{C}_{n,p}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \widehat{C}_{k,p}(x) y^{n-k}, \tag{3.27}$$

6.

$$\widehat{C}_{n,p}(x+1) - \widehat{C}_{n,p}(x) = \sum_{k=0}^{n-1} \binom{n}{k} \widehat{C}_{k,p}(x), \tag{3.28}$$

7.

$$\widehat{C}_{n,p}(mx) = \sum_{k=0}^{n} \binom{n}{k} \widehat{C}_{k,p}(x)(m-1)^{n-k} x^{n-k}. \tag{3.29}$$

Proof. For $n \geq 0$ and $p \geq 0$.

1. The proof of formula (3.23), according to (3.21), we have

$$\sum_{n\geq 0} \widehat{C}_{n,p}(x) \frac{z^n}{n!} = {}_1F_1 \left(\begin{array}{c} 1 \\ p+2 \end{array} ; -\ln(1+z) \right) e^{xz}$$

$$= \left(\sum_{n\geq 0} \widehat{C}_{n,p} \frac{z^n}{n!} \right) \left(\sum_{n\geq 0} x^n \frac{z^n}{n!} \right)$$

$$= \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{\widehat{C}_{k,p}}{k!} \frac{x^{n-k}}{(n-k)!} \right) z^n$$

$$= \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \widehat{C}_{k,p} x^{n-k} \right) \frac{z^n}{n!}$$

By identification of coefficients $\frac{z^n}{n!}$, we find (3.23).

2. The proof of formula (3.24), according to (3.21), we have

$$\frac{d}{dx}\sum_{n\geq 0}\widehat{C}_{n,p}(x)\frac{z^n}{n!}=\frac{d}{dx}\left({}_1F_1\left(\begin{array}{c}1\\p+2\end{array};-\ln(1+z)\right)e^{xz}\right)$$

$$\sum_{n\geq 0}\frac{d}{dx}\widehat{C}_{n,p}(x)\frac{z^n}{n!}=z\left({}_1F_1\left(\begin{array}{c}1\\p+2\end{array};-\ln(1+z)\right)e^{xz}\right)$$

$$\sum_{n\geq 0}\frac{d}{dx}\widehat{C}_{n,p}(x)\frac{z^n}{n!}=z\left(\sum_{n\geq 0}\widehat{C}_{n,p}(x)\frac{z^n}{n!}\right)$$

$$\sum_{n\geq 0}\frac{d}{dx}\widehat{C}_{n,p}(x)\frac{z^n}{n!}=\sum_{n\geq 0}\widehat{C}_{n,p}(x)\frac{z^{n+1}}{n!}.$$

By change of variables $n+1=N$, $n=N-1$, we find

$$\sum_{n\geq 0}\frac{d}{dx}\widehat{C}_{n,p}(x)\frac{z^n}{n!}=\sum_{n\geq 0}\widehat{C}_{n-1,p}(x)\frac{z^n}{(n-1)!}$$

$$\sum_{n\geq 0}\frac{d}{dx}\widehat{C}_{n,p}(x)\frac{z^n}{n!}=\sum_{n\geq 0}n\widehat{C}_{n-1,p}(x)\frac{z^n}{n!}.$$

By identification of coefficients $\frac{z^n}{n!}$, we find (3.24).

3. The proof of formula (3.25), according to (3.24), we have

$$\frac{d}{dx}C_{n+1,p}(x)=(n+1)C_{n,p}(x).$$

By integration we find

$$\int_0^x\frac{d}{dx}\widehat{C}_{n+1,p}(x)dx=(n+1)\int_0^x\widehat{C}_{n,p}(x)dx$$

$$\widehat{C}_{n+1,p}(x)-\widehat{C}_{n+1,p}(0)=(n+1)\int_0^x\widehat{C}_{n,p}(x)dx$$

$$\int_0^x\widehat{C}_{n,p}(x)dx=\frac{\widehat{C}_{n+1,p}(x)-\widehat{C}_{n+1,p}}{(n+1)},$$

where $\widehat{C}_{n+1,p}(0)=\widehat{C}_{n+1,p}$.

4. The proof of formula (3.26), according to (3.23), we have

$$\widehat{C}_{n,p}(x)=\sum_{k=0}^{n}\binom{n}{k}\widehat{C}_{k,p}x^{n-k}.$$

By integration we find

$$\int_0^1\widehat{C}_{n,p}(x)dx=\int_0^1\sum_{k=0}^{n}\binom{n}{k}\widehat{C}_{k,p}x^{n-k}dx$$

$$\int_0^1\widehat{C}_{n,p}(x)dx=\sum_{k=0}^{n}\binom{n}{k}\widehat{C}_{k,p}\int_0^1x^{n-k}dx$$

$$\int_0^1\widehat{C}_{n,p}(x)dx=\sum_{k=0}^{n}\binom{n}{k}\widehat{C}_{k,p}\frac{1}{n-k+1}.$$

We have

$$\binom{n}{k}\frac{1}{n-k+1}=\frac{n!}{k!(n-k)!(n-k+1)}=\frac{n!}{k!(n-k+1)!}=\frac{1}{1+n}\frac{(n+1)!}{k!(n-k+1)!}=\frac{1}{1+n}\binom{n+1}{k}.$$

So,

$$\int_0^1\widehat{C}_{n,p}(x)dx=\frac{1}{1+n}\sum_{k=0}^{n}\binom{n+1}{k}\widehat{C}_{k,p}.$$

5. The proof of formula (3.27), according to (3.21), we have

$$\sum_{n\geq 0}\widehat{C}_{n,p}(x+y)\frac{z^n}{n!}={}_1F_1\left(\begin{array}{c}1\\ p+2\end{array};-\ln(1+z)\right)e^{z(x+y)}.$$

By integration, we find

$$\begin{aligned}\sum_{n\geq 0}\widehat{C}_{n,p}(x+y)\frac{z^n}{n!}&=\left(\sum_{n\geq 0}\widehat{C}_{n,p}(x)\frac{z^n}{n!}\right)e^{zy}\\&=\left(\sum_{n\geq 0}\widehat{C}_{n,p}(x)\frac{z^n}{n!}\right)\left(\sum_{n\geq 0}y^n\frac{z^n}{n!}\right)\\&=\sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}\widehat{C}_{k,p}(x)y^{n-k}\right)\frac{z^n}{n!}.\end{aligned}$$

By identification of coefficients $\frac{z^n}{n!}$, we find (3.27).

6. The proof of formula (3.28), according to (3.27) by $y=1$, we have

$$\widehat{C}_{n,p}(x+1)=\sum_{k=0}^{n}\binom{n}{k}\widehat{C}_{k,p}(x).$$

For $k=n$ we find

$$\binom{n}{n}\widehat{C}_{k,p}(x)=\widehat{C}_{k,p}(x).$$

Thus

$$\widehat{C}_{n,p}(x+1)=\sum_{k=0}^{n-1}\binom{n}{k}\widehat{C}_{k,p}(x)+\widehat{C}_{k,p}(x),$$

$$\widehat{C}_{n,p}(x+1)-\widehat{C}_{n,p}(x)=\sum_{k=0}^{n-1}\binom{n}{k}\widehat{C}_{k,p}(x).$$

7. The proof of formula (3.29), according to (3.27) by $y=(m-1)x$, we have

$$\begin{aligned}\widehat{C}_{n,p}(x+(m-1)x)&=\widehat{C}_{n,p}(mx)\\&=\sum_{k=0}^{n}\binom{n}{k}\widehat{C}_{k,p}(x)((m-1)x)^{n-k}\\&=\sum_{k=0}^{n}\binom{n}{k}\widehat{C}_{k,p}(x)(m-1)^{n-k}x^{n-k}.\end{aligned}$$

□

Example 3.3.2. *For $n=0$, according to (3.21)*

$$\widehat{C}_{0,p}(x)=0!(-1)^0s(0,0)\frac{(p+1)!}{(p+1)!}x^0=1.$$

For $n=1$

$$\widehat{C}_{1,p}(x)=\sum_{k=0}^{1}\binom{1}{k}\left[\sum_{j=0}^{k}\frac{(p+1)!k!}{(p+k+1)!}(-1)^k s(k,j)\right]x^{1-k}=x-\frac{1}{p+2}.$$

The First Few p-Cauchy Polynomials Of The Second Kind

$$\widehat{C}_{0,p}(x) = 1,$$

$$\widehat{C}_{1,p}(x) = x - \frac{1}{p+2},$$

$$\widehat{C}_{2,p}(x) = x^2 - \frac{2x}{p+2} + \frac{p+1}{(p+2)(p+3)},$$

$$\widehat{C}_{3,p}(x) = x^3 - \frac{3x^2}{p+2} + \frac{3(p+1)x}{(p+2)(p+3)} - \frac{2(p+1)}{(p+2)(p+4)},$$

$$\widehat{C}_{4,p}(x) = x^4 - \frac{4x^3}{p+2} + \frac{6(p+1)x^2}{(p+2)(p+3)} - \frac{8(p+1)x}{(p+2)(p+4)} + \frac{2(p+1)(3p^2+22p+38)}{(p+2)(p+3)(p+4)(p+5)}.$$

Conclusion

In this work, we were able to enter the field of enumerative combinatorics by recognizing many sequences of numbers and polynomials and their combinatorial and analytical properties, particularly the generalization of Cauchy numbers and polynomials through a parameter p, as well as their connections with Stirling numbers, Bernoulli numbers, and r-Stirling numbers.

We hope in the future to be able to continue deepening our knowledge in this very enjoyable field.

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