PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
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University of Yahia Fares Medea



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Department of Mathematics and Computer Science

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Existence and Compactness Results for a System of Fractional Differential Equations

Presented by: Ahmed Cherif Elhalassi
Defended on: 19/06/2025
Before the jury composed of:

President:	Mr. RAFA Said	University of Medea
Examiner:	Mr. MEKDACHE Noureddine	University of Medea
Supervisor:	Mr. MALOUM khaled	University of Medea

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Abstract

In this work, we address the analysis of fractional differentiel equations involving the Caputo fractional derivative. By employing Perov's fixed point theorem within the framwork of vector Banach spaces, we establish key results concerning the existence, uniqueness, boundedness, and continous dependence of solutions. Additionally, we examine the structure of the solution set, focusing on its existence, compactness, and the upper-semi-continuity of the associated solution operator.

Key Words: fractional Differential, Existence, Uniquences, Compactness, Caputo fractional derivative, Perov's theorem.

Résumé

Dans ce travail, nous abordons l'analyse des équations différentielles fractionnaires impliquant la dérivée fractionnaire de Caputo. En utilisant le théorème du point fixe de Perov dans le cadre des espaces de Banach vectoriels, nous établissons des résultats clés concernant l'existence, l'unicité, la bornitude et la dépendance continue des solutions. De plus, nous examinons la structure de l'ensemble des solutions, en mettant l'accent sur son existence, sa compacité et la semi-continuité supérieure de l'opérateur de solution associé.

Mots-clés : équations différentielles fractionnaires, existence, unicité, compacité, dérivée fractionnaire de Caputo, théorème de Perov.

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Notations

X	A generic non-empty set .
(X,d)	Generalized metric space, where $d : X \times X \longrightarrow \mathbb{R}_+$ satisfies all the usual axioms of the metric .
${}^cD^\alpha$	The Caputo fractional derivatives.
$\Gamma(\alpha)$	The Gamma function.
$\|.\|$	A norm on a vector space X .
$L^1(\Omega)$	Lebesgue integrable functions on $\Omega \subseteq \mathbb{R}^\varkappa$.
$L^1_{\text{Loc}}(\Omega)$	Locally integrable functions.
$AC^n(J,\mathbb{R})$	Absolutely continuous functions with derivatives up to order n , absolutely continuous on $J \subseteq \mathbb{R}$.
$C(J,\mathbb{R})$	The space of all continuous functions from J into $\mathbb{R}$.
$BC_b(J,\mathbb{R})$	Space of bounded continuous functions.
$\rho(A)$	Spectral radius of matrix A.
$B(x_0,r)$	The open ball cenered in x_0 with radius r.
$\overline{B(x_0,r)}$	The closed ball cenered in x_0 with radius r.
u.s.c	Upper Semi Continous.
E_y	The Mittag- Leffler function.

Chapter 1

Foundations of Fractional Differential Systems

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1.1 Introduction

In the past twenty years, the fractional differential equation has aroused great consideration not only in its application in mathematics but also in other applications in physics, engineering, finance, fluid mechanics, viscoelastic mechan ics, electroanalytical chemistry, and biological and other sciences [1][7].

In recent decades, the Riemann-Liouville, Caputo, and Hadamard fractional calculus are paid more attention; see the monographs [5][8][13]

Applied problems requiring definitions of fractional derivatives are those that are physically interpretable for ini tial conditions containing y(0), $y'(0)$, etc. The same require ments are true for boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional deriva tives of both the Riemann-Liouville type and the Caputo type, see Podlubny [12] and Diethelm [14].
The theory of fractional differential equations and inclu sions has been extensively studied and developed by many authors; see [15][21] and the references therein.
Perov in 1964 [22] and Perov and Kibenko [23] extended the classical Banach contraction principle for contractive maps on space endowed with a vector-valued metric. Later, they attempted to generalize the Perov fixed point theorem in several directions which has a number of applications in various fields of nonlinear analysis, semilinear differential equations, and system of ordinary differential equations. In [24], Dezideriu and Precup studied the following sys tem of semilinear equations :

$$\begin{cases} A_1 x = F^1(x,y) \\ A_2 y = F^2(x,y) \end{cases} \tag{1.1}$$

where A_1, A_2 : D(A)$\subset$ X$\longrightarrow X$ are linear operators and $F^1, F^2 : J \times X \times X \longrightarrow X$ are nonlineair operators.
Precup in [25] explained the advantage of vector-valued norms and the role of matrices

that are convergent to zero in the study of semilinear operator systems.
Many authors studied the existence of solutions for a system of differential equations and impulsive differential equations by using the vector version fixed point theorem; their results aregiven in [26][30] .

Our work is primarily based on the work of the researchers of Cheikh Guendouz, Jamal Eddine Lazreg, juan J.Nieto, and Abdelghani Ouahab in [13] Where we aim to simplify the mathematical concepts and results.

Our goal of this paper is to treat the systems of fractional differential equations.Moreprecisely, we will consider the following problem:

$$\begin{cases} {}^{c}D^{\alpha}x(t) = f(t, x(t), y(t)), \\ {}^{c}D^{\beta}y(t) = g(t, x(t), y(t)), \quad t \in J, \\ x(0) = x_0, \\ y(0) = y_0, \end{cases} \tag{1.2}$$

where ${}^{c}D^{\alpha}$ and ${}^{c}D^{\beta}$ are the Caputo fractional derivatives, $\alpha, \beta \in (0,1]$, $J = [0, \infty)$, $f, g : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are given functions, and $x_0, y_0 \in \mathbb{R}$.
In the case where $\alpha = \beta = 1$, the above system was used to analyze initial value problems and boundary value problems for non linear competitive or cooperative differential systems from a the mathematical biology [31] and mathematical economics [32] which can be set in the operator from (1.2)

1.2 Preliminaries

In this section, we introduce notations,definitions,and preliminary facts which are used throughout this paper.

Definition 1. Matrix convergent to Zero
In mathematics, a matrix A is said to be convergent to 0 if its powers A^n tend to the zero matrix as $n \to \infty$, meaning:

$$\lim_{n\to\infty} A^n = 0.$$

Definition 2. matrix A converges to 0 if and only if its spectral radius is less than 1: $\rho(A) = \max|\lambda_i| < 1$ where λ_i are the eigenvalues of A.

Definition 3. Contractive Operator
An operator N is said to be contractive if it does not expand distances in a given normed space. Formally, an operator$N$ on a normed space X is contractive if:

$$\|Nx - Ny| \leq K\|x - y\| \quad \forall x, y \in X,$$

with K constant $0 \leq k < 1$.

Definition 4. Strictly Contractive Operator
If there exists a constant c with $0 \leq c < 1$ such that:

$$\|Nx\| \leq c\|x\| \quad \forall x \in X,$$

then N is called a strict contraction.

Definition 5. Generalized Banach space[41]
A generalized Banach space often refers to spaces that extend Banach spaces in various ways. Some examples include:

1.2.1 p-Banach Spaces

p-Banach Spaces (Quasi-Banach Spaces):

These are quasi-normed spaces where the norm satisfies a weaker version of the triangle inequality:

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p, \quad \text{for some } 0 < p \leq 1.$$

- When $p = 1$, it reduces to a standard Banach space.
- Examples: L^p spaces for $0 < p < 1$.

Definition 6. Almost Equicontinuous Functions
A family of functions $\mathscr{F} = \{f_\alpha : X \to Y\}$ is said to be almost equicontinuous at a point $x_0 \in X$ if:

For every $\epsilon > 0$, there exists a subset $U \subset X$ containing x_0 (often a neighborhood) such that:

$$\forall x \in U, \quad \sup_{f \in \mathscr{F}} d_Y(f(x), f(x_0)) < \epsilon.$$

Definition 7. Uniformly Continous Functions
A function $f : X \to Y$ is said to be uniformly continuous if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x_1, x_2 \in X, \text{ if } d_X(x_1, x_2) < \delta, \text{ then } d_Y(f(x_1), f(x_2)) < \epsilon,$$

Definition 8. Equiconvergent Set
A set (or family) of functions $\mathscr{F} = \{f_\alpha : X \to Y\}$ is said to be equiconvergent at a point $x_0 \in X$ if:

$$\lim_{x \to x_0} f_\alpha(x) = L, \text{ uniformly for all } f_\alpha \in \mathscr{F}.$$

Lemma 1.1. Let $u(t)$ be a continuous, non-negative function that satisfies the inequality:

$$u(t) \leq A + B \int_{t_0}^{t} u(s)\, ds \quad \text{for} \quad t \geq t_0,$$

where $A \geq 0$ and $B \geq 0$ are constants. Then:

$$u(t) \leq A e^{B(t-t_0)} \quad \text{for all} \quad t \geq t_0.$$

Integral Form:

If $u(t)$ satisfies:

$$u(t) \leq A + \int_{t_0}^{t} g(s)u(s)\,ds,$$

where $g(t)$ is a continuous function, then:

$$u(t) \leq A \exp\left(\int_{t_0}^{t} g(s)\,ds\right).$$

Definition 9. The Mittag-Leffler function [23] [36]
The Mittag-Leffler function is a special function that generalizes the exponential function and plays a crucial role in fractional calculus, differential equations, and complex analysis. Let's break it down clearly

Applications:

• Solutions of fractional differential equations. • Modeling of anomalous diffusion.
• Used in control theory and signal processing .

Definition 10. The Mittag-Leffler function $E_t alpha(z)$ is defined as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}.$$

Where:

- $z \in \mathbb{C}$ (a complex variable),
- $\alpha > 0$ (a parameter),
- $\Gamma(\cdot)$ is the Gamma function:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\,dt.$$

Definition 11. Locally Integrable Function
is a function that is integrable (in the lebsugue sense) over every compact subset of its domain .
Let $\Omega \subseteq \mathbb{R}^{\kappa}$ be an open set , A mesurable function $f : \Omega \longrightarrow \mathbb{R}$ is called locally integrable if for every compact subset $K \subseteq \Omega$:

$$\int_k |f(x)|\,dx < \infty .$$

the space of all such function is denoted $L^1_{\text{Loc}}(\Omega)$ we have :

1. if f $\in L^1(\Omega)$, then $f \in L^1_{\text{Loc}}(\Omega)$.
2. if $f \in L^P(\Omega)$ for $1 \leq p < \infty$ then $f \in L^1_{\text{Loc}}(\Omega)$ (since compact sets have finite measure).

Definition 12. [22]
Let X be a non empty set. The mapping $d : X \times X \longrightarrow \mathbb{R}_+^m$ which satisfies all the usual axioms of the metric is called a generalized metric in Perov's sense and (X, d) is called a generalized metric space.

In a generalized metric space in Perov's sense, the concepts of Cauchy sequence, convergent sequence completness, and open and closed subsets are similarly defined as those for usual metric space.

If $v, r \in \mathbb{R}^m, v := (v_1, v_2, ..., v_m)$ and $r := (r_1, r_2, ..., r_m)$, then by $v \leq r$, we mean $v_i \leq r_i$ for each $i \in 1, ..., m$ and by $v < r$ we mean $v_i < r_i$ for each $i \in 1, ..., m$. Also $|v| := (|v_1|..., |v_m|)$ and $\max(u, v) := (\max(u_1, v_1), ...\max(u_m, v_m))$ If $c \in \mathbb{R}$, then $v \leq c$ means $v_i \leq c$ for each $i \in 1, ..., m$ Denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\} .$$

The open ball centered in x_0 with radius r, and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\} .$$

.
the closed ball centered in x_0 with radius r.

Definition 13. . A square matrix M of real numbers is said to be convergent to zero if and only if $M^n \longrightarrow 0$ as $n \longrightarrow \infty$

Lemma 1.2. [33] .Let $M \in \mathscr{M}_{m,m}(\mathbb{R}_+)$ the following statments are equivalent :

1. M is a matrix convergent to zero .
2. the eigenvalues of M are in open disc, i.e., $|\mu| < 1$, for every $\mu \in \mathbb{C}$ with $\det(M - \mu I) = 0$.
3. the matrix $I - M$ is nonsingular and $(I - M)^{-1} = I + M + ... + M^n + ...$
4. the matrix $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements .
5. $M^n q \longrightarrow 0$ and ${}^t q M^n \longrightarrow 0$ as $n \longrightarrow \infty$, for any $q \in \mathbb{R}^{>}$.

Example : Some examples of matrix convergent to zero are :

1. $M = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$
   where $a, b \in \mathbb{R}_+$ and $a + b < 1$,

2. $$M = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$$
   where $a, b \in \mathbb{R}_+$ and $a + b < 1$,

3. $M = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$
   where $a, b, c \in \mathbb{R}_+$ and $\max(a, c) < 1$.

4. $M = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.4 \end{pmatrix}$
   we have: $\mu(M) < 1$, then M convergent to 0.

Definition 14. [34] Let (X, d) be a generalized metric space . An operator N which $N : X \longrightarrow X$ is said to be contractive if there exists a convergent to zero matrix M such that

$$d(N(x), N(y)) \leq M d(x, y) \text{ for all } x, y \in X. \tag{1.3}$$

Notice now that the Banach fixed point theorem can be extended to generalized metric spaces in the sense of Perov.

Definition 15. [5] The Riemann-Liouville fractional integral of the function $h \in L^1([0, T], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$, is defined by :

$$I^a = \frac{1}{\Gamma(a)} \int_0^t (t - s)^{a-1} h(s) ds ,$$

where Γ is the Euler gamma function defined by :

$$\Gamma(a) = \int_0^\infty t^{\alpha-1} e^{-t}, \alpha > 0,$$

Definition 16. [5]. For a function $h \in AC^n(J, \mathbb{R})$, the Caputo fractional-order derivative of order $\alpha \; of \; h$ is defined by :

$$({}^c D_0^\alpha h(t)) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} h^{(n)}(s) ds ,$$

where $n[\alpha] + 1$.

We recall Gronwall's lemma for singular kernels, whose proof can be found in Lemma 7.1.1 of [37].

Lemma 1.3. . Let $v : [0, b) \longrightarrow [0, \infty)$ be a real function; $w(.)$ is a non negative, locally integrable function on $[0, b)$ $(some \;\; b \leq +\infty)$; $a(t)$ is a non negative,non decreasing continuous function defined on $0 \leq t < b$; $a(t) \leq M$ (constant) ; and suppose $v(t)$ is non negative and locally integrable on $0 \leq t < b$. Assume $y > 0$ such that :

$$v(t) \leq w(t) + a(t) \int_0^t \frac{v(s)}{(t-s)^{1-\gamma}} ds,$$

then :

$$v(t) \leq w(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(a(t)\Gamma(\gamma))^n}{\Gamma(n\gamma)} (t-s)^{n\gamma-1} w(s) ds.$$

for every $t \in [0, b)$.

Theorem 1.1. A common version of the theorem states:

Let X be a Banach space, and let $T : X \to X$ be a compact and continuous operator. Suppose that there exists a constant $C > 0$ such that any solution x of the equation

$$x = \lambda T(x), \quad \text{for } 0 \leq \lambda \leq 1,$$

satisfies $\|x\| \leq C$. Then, T has at least one fixed point, i.e., there exists $x \in X$ such that

$$T(x) = x.$$

Theorem 1.2. [39] Let (X, d) be a *complete generalized metric space*, where $d : X \times X \to \mathbb{R}_+^n$ is a *vector-valued metric*, and let $T : X \to X$ be an operator such that $d(Tx, Ty) \leq M d(x, y), \quad \forall x, y \in X$, where $M \in \mathbb{R}^{n \times n}$ is a *matrix with nonnegative entries* that is *convergent to zero* , i.e.,

$$\lim_{k \to \infty} M^k = 0.$$

Then:

- T has a **unique fixed point** $x^* \in X$, such that $T(x^*) = x^*$,
- For any $x_0 \in X$, the iterative sequence $x_{k+1} = T(x_k)$ converges to x^*.

1.3 Lipschitz Matrix Fixed Point Theorem

Theorem 1.3. [22][28], Let (X, d) be a complete generalized metric space and $N : X \longrightarrow X$ be a contractive operator with Lipschitz matrix M.Then, N has a unique fixed point x^* , and for each $x_0 \in X$, we have

$$d(N^k(x_0), x^*)) \leq M^k (I - M)^{-1} d(x_0, N(x_0)) \text{ for all } k \in \mathbb{N}. \tag{1.4}$$

We recall now the following Leary-Schauder type theorem :

1.4 Leary-Schauder type theorem

Theorem 1.4. [28][35].Let X be a generalized Banach space and let $N : X \longrightarrow X$ be a completely continuous operator.Then, either

1. the equation $N(x) = x$ has at least one solution, or
2. the set $\mathscr{M} = \{x \in X \mid \mu N(x) = x, \mu \in (0,1)\}$ is unbounded

We will use the following notations .Let (X,d) and (Y,ρ) be two metrics spaces and : $G : X \longrightarrow \mathscr{P}(Y)$.

$$\begin{aligned}\mathscr{P}(X) &= \{Y \subset X : Y \neq \emptyset\}, \mathscr{P}_{cl}(X)\\ &= \{Y \in \mathscr{P}(X) : Y \text{ is closed}\}, \mathscr{P}_b(X)\\ &= \{Y \in \mathscr{P}(X) : Y \text{ bounded}\}.\end{aligned}$$

Definition 17. [28][36] .A multivalued map $\varphi : X \longrightarrow \mathscr{P}(Y)$ is called upper semicontinuous (u.s.c.) at a point $x_0 \in X$ provided that for every open subset $V \subset Y \; with \; V \supseteq \varphi(x_0)$, there exists $U \in \mathscr{V}$ such that

$$\forall x \in U, \varphi(x) \subset V;$$

φ is called uppersemicontinuous if it is u.s.c.(Upper Semi Continuous)at every point $x \in X$.

The mapping G is said to be completely continuous if it is u.s.c. (Upper Semi Continuous),and for every bounded subset $C \subseteq X, G(C)$ is relatively compact, i.e, there exists a relatively compact set $K = K \subset X$ such that :

$$G(C) = \cup\{G(x) : x \in C\} \subset K.$$

Also, G is compact if $G(X)$ is relatively compact,and it is called locally compact if for each $x \in X$, there exists an open set W containing x,such that $G(W)$ is relatively compact.

Theorem 1.5. [36].
Let $G : X \longrightarrow \mathscr{P}_{cp}(Y)$ be a closed locally compact multifunction.Then , G is Upper Semi Continous .

Chapter 2

Existence. Uniqueness. and Bounded Solutions

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In order to define a solution for problem (1.2), consider the following functional spaces: Let $J=[0,\infty)\ and\ C(J,\mathbb{R})$ be the space of all continuous functions from J into $\mathbb{R}$

$$C_b=\{y\in C(J,\mathbb{R}):\ \text{y is bounded }\}.$$

C_b is a Banach with norm .

$$||y||C_b=\sup\{|y(t)|:t\in J\},$$

We need the following auxiliary result :

Lemma 2.1. [14]. concerning the problem :

$$\begin{cases} {}^cD^{\alpha}x(t)=f(t,x),0<\alpha<1\\ x(0)=x_0\in\mathbb{R}\end{cases} \tag{2.1}$$

where the function $f:\mathbb{R}_+\times\mathbb{R}\longrightarrow\mathbb{R}$ is contiouns. the function $x:\mathbb{R}_+\times\mathbb{R}\longrightarrow\mathbb{R}$ is the unique solution of the problem (2.1) if and only if:

$$x(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds,t\in\mathbb{R}_+.$$

Definition 18. . A function $x,y\in C_b$ is said to be a solution of (1.2) if and only if :

$$\begin{cases} x(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s),y(s))ds,\\ y(t)=y_0+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}g(s,x(s),y(s))ds,\end{cases}$$

In this section, we assume the following conditions :

(H1) . There exists functions $h_{i,\alpha}, h_{j,\beta} \in L^1(J,\mathbb{R}^+), i=1,2\ j=3,4$ such that ;

$$|f(s,x,y)-f(s,\bar{x},\bar{y})| \leq h_{i,\alpha}(s)|x-\bar{x}|+h_{2,\alpha}(s)|y-\bar{y}| \text{ for all} x,\bar{x},y,\bar{y} \in \mathbb{R},$$

$$|g(s,x,y)-g(s,\bar{x},\bar{y})| \leq h_{3,\beta}(s)|x-\bar{x}|+h_{4,\beta}(s)|y-\bar{y}| \text{ for all} x,\bar{x},y,\bar{y} \in \mathbb{R},$$

where :

$$||h_{i,\alpha}||_\infty = \sup_{t\in J}\int_0^t \theta^\alpha(t-s)h_{i,\alpha}(s)ds < \infty \quad i=1,2,$$
$$||h_{j,\beta}||_\infty = \sup_{t\in J}\int_0^t \theta^\beta(t-s)h_{i,\beta}(s)ds < \infty \quad j=3,4,$$

where for $y=\alpha,\beta$,

$$\theta_y(t)=\begin{cases} t^{y-1}, & t>0,\\ 0, & t\leq 0\end{cases}$$

(H2). The functions $f_\alpha, g_\beta : J \longrightarrow \mathbb{R}$ are defined by :

$$f_\alpha(t)=\theta_\alpha(t-s)f(s,0,0).$$
$$g_\beta(t)=\theta_\beta(t-s)g(s,0,0);\ t\in J.$$

Satifies :

$$||f_\alpha||_\infty = \sup_{t\in J}\int_0^t \theta_\alpha(t-s)|f(s,0,0)|ds<\infty;$$
$$||g_\beta||_\infty = \sup_{t\in J}\int_0^t \theta_\beta(t-s)|g(s,0,0)|ds<\infty;$$

Now, we are in a position to prove our existence and uniqueness solution for the problem (1.2) using the Perov fixed point theorem and show that for each initial condition (x_0, y_0), the solution is bounded.

2.1 Matrix-Based Existence and Uniqueness Theorem

Theorem 2.1. Assume that $(H1)-(H2)$ are satisfied. If the matrix

$$M=\begin{pmatrix} \frac{||h_{1,\alpha}||_\infty}{\Gamma(\alpha)} & \frac{||h_{2,\alpha}||_\infty}{\Gamma(\alpha)} \\ \frac{||h_{3,\beta}||_\infty}{\Gamma(\beta)} & \frac{||h_{4,\beta}||_\infty}{\Gamma(\beta)}\end{pmatrix} \in \mathscr{M}_{2\times 2}(\mathbb{R}^+) \tag{2.2}$$

converges to zero. Then, the problem (1.2) has a unique bounded solution.

Proof. . Transform the problem (1.2) into a fixed point theorem of the operator N : $C_b \times C_b \longrightarrow C_b \times C_b$ defined by $N(x,y)=(N_1(x,y),N_2(x,y))$, where

$$N_1(x,y)(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s),y(s))ds,$$
$$N_2(x,y)(t)=y_0+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g(s,x(s),y(s))ds$$
$$t\in[0,\infty)$$

First, we show that the operator N is well-defined. Let $(x,y)\in C_b\times C_b \; and \; t\in[0,\infty)$ then we have :

$$|N_1(x,y)(t)|\leq |x_0|+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),y(s))ds|.$$

and

$$\begin{aligned}|f(s,x(s),y(s))| &= |f(s,x(s),y(s))-f(s,0,0)+f(s,0,0)|\\ &\leq |f(s,x(s),y(s))-f(s,0,0)|+|f(s,0,0)|\end{aligned}$$

Thus

$$\int_0^t (t-s)^{\alpha-1}|f(s,x(s),y(s))|ds\leq \int_0^t (t-s)^{\alpha-1}|f(s,x(s),y(s))-f(s,0,0)|ds+\int_0^t (t-s)^{\alpha-1}|f(s,0,0)|ds$$

multiplying both terms by $\frac{1}{\Gamma(\alpha)}$

$$|N_1(x,y)(t)|\leq |x_0|+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),y(s))-f(s,0,0)|ds+\frac{1}{\Gamma(\alpha)}\int_0^t \frac{|f(s,0,0)|}{(t-s)^{1-\alpha}}ds$$

use the notation $||h_{i,\alpha}||_\infty$ and define $||x||_{C_b}=\sup\{|y(t)|\}$

Then

$$||N_1(x,y)||_{c_b}\leq |x_0|+\frac{||h_{1,\alpha}||_\infty}{\Gamma(\alpha)}||x||c_b+\frac{||h_{2,\alpha}||_\infty}{\Gamma(\alpha)}||y||c_b+\frac{||f_\alpha||_\infty}{\Gamma(\alpha)};$$
$$||N_2(x,y)||_{c_b}\leq |y_0|+\frac{||h_{3,\beta}||_\infty}{\Gamma(\beta)}||x||c_b+\frac{||h_{4,\beta}||_\infty}{\Gamma(\beta)}||y||c_b+\frac{||g_\beta||_\infty}{\Gamma(\beta)};$$

Hence, the operator N is well-defined.

Clearly, the fixed points of operator N are solutions of problem (1.2) Now, we show that N is a contraction.For all $(x,y),(\bar{x},\bar{y})\in C_b\times C_b$, we have :

$$\begin{aligned}\left|N_1(x,y)(t)-N_1(\bar{x},\bar{y})(t)\right| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,x(t),y(t))-f(s,\bar{x}(t),\bar{y}(t))\right| ds\\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^t \left[h_{1,\alpha}(s)|x(t)-\bar{x}(t)|+h_{2,\alpha}(s)\left|y(t)-\bar{y}(t)\right|\right] ds.\end{aligned}$$

Then :

$$||N_1(x,y)-N_1(\bar{x},\bar{y})||_{c_b}\leq \frac{||h_{1,\alpha}||_\infty}{\Gamma(\alpha)}||x-\bar{x}||c_b+\frac{||h_{2,\alpha}||_\infty}{\Gamma(\alpha)}||y-\bar{y}||c_b;$$

Similiary, we have :

$$||N_2(x,y)-N_2(\bar{x},\bar{y})||_{c_b}\leq \frac{||h_{3,\beta}||_\infty}{\Gamma(\beta)}||x-\bar{x}||c_b+\frac{||h_{4,\beta}||_\infty}{\Gamma(\beta)}||y-\bar{y}||c_b;$$

Therebefore

$$||N(x,y)-N(\bar{x},\bar{y})||_{c_b\times c_b}\leq M\begin{pmatrix}||x-\bar{x}||c_b\\ \\ ||y-\bar{y}||c_b\end{pmatrix}$$

According to Theorem 1.3, we deduce that the operator N has unique fixed point which is a solution of problem (1.2) Now, we will prove that the solution (x,y) of problem 1.2 is bounded. For all $t\in(0.\infty]$, we have :

$$\begin{aligned}|x(t)| &\leq |x_0|+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,x(s),y(s))ds|\\ &\leq |x_0|+\frac{1}{\Gamma(\alpha)}\int_0^t \left[h_{1,\alpha}(s)|x(s)|+h_{2,\alpha}(s)|y(s)\right] ds+\frac{1}{\Gamma(\alpha)}\int_0^t |f_\alpha(s,0,0)|ds\\ &\leq |x_0|+\frac{1}{\Gamma(\alpha)}\int_0^t \left[h_{1,\alpha}(s)|x(s)|+h_{2,\alpha}(s)|y(s)\right] ds+\frac{||f_\alpha||_\infty}{\Gamma(\alpha)},\end{aligned}$$

$$\begin{aligned}|y(t)| &\leq |y_0|+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}|g(s,x(s),y(s))ds|\\ &\leq |y_0|+\frac{1}{\Gamma(\beta)}\int_0^t \left[h_{3,\beta}(s)|x(s)|+h_{4,\beta}(s)|y(s)\right] ds+\frac{1}{\Gamma(\beta)}\int_0^t |g_\beta(s,0,0)|ds\\ &\leq |y_0|+\frac{1}{\Gamma(\beta)}\int_0^t \left[h_{3,\beta}(s)|x(s)|+h_{4,\beta}(s)|y(s)\right] ds+\frac{||g_\beta||_\infty}{\Gamma(\beta)}\end{aligned}$$

Therefore

$$\|x\|_{c_b} \leq |x_0| + \frac{\|h_{1,\alpha}\|_\infty}{\Gamma(\alpha)}\|x\|c_b + \frac{\|h_{2,\alpha}\|_\infty}{\Gamma(\alpha)}\|y\|c_b + \frac{\|f_\alpha\|_\infty}{\Gamma(\alpha)};$$
$$\|y\|_{c_b} \leq |y_0| + \frac{\|h_{3,\beta}\|_\infty}{\Gamma(\beta)}\|x\|c_b + \frac{\|h_{4,\beta}\|_\infty}{\Gamma(\beta)}\|y\|c_b + \frac{\|g_\beta\|_\infty}{\Gamma(\beta)};$$

Hence

$$\begin{pmatrix} \|x\|_{c_b} \\ \\ \|y\|_{c_b} \end{pmatrix} \leq M \begin{pmatrix} \|x\|_{c_b} \\ \\ \|y\|_{c_b} \end{pmatrix} + \begin{pmatrix} \sigma_1 \\ \\ \sigma_2 \end{pmatrix}$$

where :

$$\sigma_1 = |x_0| + \frac{\|f_\alpha\|_\infty}{\Gamma(\alpha)},\ \sigma_2 = \frac{\|g_\beta\|_\infty}{\Gamma(\beta)}$$

Then

$$\begin{pmatrix} \|x\|_{c_b} \\ \\ \|y\|_{c_b} \end{pmatrix} \leq (I-M)^{-1} \begin{pmatrix} \sigma_1 \\ \\ \sigma_2 \end{pmatrix}$$

From (H1) and (H2), we deduce that the solution (x, y) is bounded. □

For the next result, we prove the continuous dependence of solutions on initial conditions.

2.2 Continuous Dependence on Initial Conditions

Theorem 2.2. Assume that (H1) and (H2) hold. If $f(t,0,0) = g(t,0,0) = 0, t \in J$ and the matrix M defined in (2.2) converges to zero.

For every $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$, we denoted by $(x(t, x_0), y(t, y_0))$. the solution of problem (1.2) . Then, the map $(x_0, y_0) \longrightarrow (x(., x_0), y(., y_0))$ is countinous.

Proof. From Theorem 2.1. for each initial condition $(x_0, y_0), (\bar{x}, \bar{y}) \in \mathbb{R} \times \mathbb{R}$, there exists unique solution $(x(., x_0), y(., y_0)), (\bar{x}(., \bar{x}_0), \bar{y}(., \bar{y}_0))$, then we get :

$$\begin{cases} x(t,x_0) = x_0 + \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),y(s))\,ds \\ \\ y(t,y_0) = y_0 + \dfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,x(s),y(s))\,ds \end{cases}$$

$$\begin{cases} \bar{x}(t,\bar{x}_0) = \bar{x}_0 + \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\bar{x}(s),\bar{y}(s))\,ds \\ \\ \bar{y}(t,\bar{y}_0) = \bar{y}_0 + \dfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,x(s),\bar{y}(s))\,ds \end{cases}$$

Therefore

$$|x(t,x_0)-\bar{x}(t,\bar{x}_0)| \le |x_0-\bar{x}_0| + \frac{1}{\Gamma(\alpha)}\int_0^t \left[h_{1,\alpha}(s)|x(s)-\bar{x}(s)| + h_{2,\alpha}(s)|y(s)-\bar{y}(s)\right] ds$$

$$|y(t,y_0)-\bar{y}(t,\bar{y}_0)| \le |y_0-\bar{y}_0| + \frac{1}{\Gamma(\beta)}\int_0^t \left[h_{3,\beta}(s)|x(s)-\bar{x}(s)| + h_{4,\beta}(s)|y(s)-\bar{y}(s)\right] ds$$

Hence

$$\begin{pmatrix} \|x(\cdot,x_0)-\bar{x}(\cdot,\bar{x}_0)\|_{C_b} \\ \\ \|y(\cdot,y_0)-\bar{y}(\cdot,\bar{y}_0)\|_{C_b} \end{pmatrix} \le (I-M)^{-1} \begin{pmatrix} |x_0-\bar{x}_0| \\ \\ |y_0-\bar{y}_0| \end{pmatrix} \longrightarrow 0, \quad \text{as } (x_0,y_0) \to (\bar{x}_0,\bar{y}_0).$$

□

Chapter 3

Existence and Compactness of Solutions

For the existence and compactness result of problem (1.2), consider the following Banach space :

$$\mathrm{BC}_b(J,\mathbb{R}) = \left\{ u \in C(J,\mathbb{R}) : \sup_{t\in J} \tfrac{|u(t)|}{1+t} < \infty \right\}.$$

with norm :

$$\|u\|_B = \sup_{t\in J} \frac{|u(t)|}{1+t}.$$

It is evident that $BC_b(J,\mathbb{R})$ is a Banach space.The following compactness criterion on unbounded domains is called Corduneanu compactness criterion in which the proof is easy and similar to the classical one in $C_b(\mathbb{R}^+,\mathbb{R})$ (see [38]).

Lemma 3.1. Let $H \subset BC_b(\mathbb{R}^+,\mathbb{R})$. Then, H is relatively compact if the following conditions hold :

(a) H is *uniformly bounded* in $\mathrm{BC}_b(\mathbb{R}^+,\mathbb{R}^n)$.

(b) *The functions belonging to* H *are almost equicontinuous on* $\mathbb{R}^+$, *i.e., for all* $I \subset \mathbb{R}$, *compact interval, for any* $\varepsilon > 0$, *there exists* $\delta(\varepsilon) > 0$ *such that for every* $t_1, t_2 \in J$ *with* $|t_1 - t_2| < \delta(\varepsilon)$, *we have for all* $u \in H$:

$$\left| \frac{u(t_2)}{1+t_2} - \frac{u(t_1)}{1+t_1} \right| < \varepsilon; \tag{3.1}$$

(c) *The functions from* H *are equiconvergent, that is, given* $\varepsilon > 0$, *there corresponds* $T(\varepsilon) > 0$ *such that*

$|\frac{u(t_1)}{1+t_1} - \frac{u(t_2)}{1+t_2}| < \varepsilon$ for any $t_1, t_2 \geq T(\varepsilon)$ and $u \in H$.

.

In the sequel of this section,we will consider the following assumption.
(H3). For every $l>0$, the functions $(x,y)\longrightarrow f(t,x,y), g(t,x,y)$ are uniformly continuous on the sets $[-l,l]\times[-l,l]$ uniformly with respect to $t\in\mathbb{R}_+$,i.e., f and g satisfied the following condition: for all $l>0\epsilon>0$, there exists $\delta(\epsilon)>0$ such that for all $(x,y),(\bar{x},\bar{y})\in[-l,l]\times[-l,l]$ and for all $t\in\mathbb{R}_+\ with\ |x(1+t)-\bar{x}(1+t)|<\delta(\epsilon),|y(1+t)-\bar{y}(1+t)|<\delta(\epsilon)$, we have :

$$|f(t,x,y)-f(t,\bar{x},\bar{y})<\epsilon\ .$$
$$|g(t,x,y)-g(t,\bar{x},\bar{y})<\epsilon\ .$$

(H4).There exist $a_i,b_i,c_i>0, i=1,2$, such that

$$|f(t,x,y)|\leq a_1\frac{|x|}{1+t}+b_1\frac{|y|}{1+t}+c_1$$

$$|g(t,x,y)|\leq a_2\frac{|x|}{1+t}+b_2\frac{|y|}{1+t}+c_2$$

$\forall x,y\in\mathbb{R},t\in\mathbb{R}$

Theorem 3.1. Assume that (H3) and (H4) hold. Then, the problem (1.2) has at least one bounded solution. Moreover, the solution set

$$S(x_0,y_0)=\{(x,y)\in BC_b(J,\mathbb{R})\times BC_b(J,\mathbb{R}):(x,y)\text{ is solution of }1.2\},$$

is compact and the multivalued map

$$S:(x_0,y_0)\longrightarrow S(x_0,y_0)\text{ is u.s.c}$$

Proof. Let $N=(N_1,N_2)$ is defined in the proof of Theorem 2.1.

Step 1. N is well defined. Let $(x,y)\in BC_b(J,\mathbb{R})\times BC_b(J,\mathbb{R})$ then :

$$\begin{aligned}\frac{|N_1(x(t),y(t))|}{1+t}&\leq\frac{|x_0|}{1+t}+\frac{1}{(1+t)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f(s,x(s),y(s))\right|ds\\&\leq\frac{|x_0|}{1+t}+\frac{1}{(1+t)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(a_1\frac{|x(s)|}{1+s}+b_1\frac{|y(s)|}{1+s}+c_1\right)ds\\&\leq\frac{|x_0|}{1+t}+\frac{c_1t^\alpha}{\Gamma(\alpha+1)(1+t)}+\frac{1}{(1+t)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(a_1\frac{|x(s)|}{1+s}+b_1\frac{|y(s)|}{1+s}\right)ds\\&\leq|x_0|+\frac{c_1}{\Gamma(\alpha+1)}+\frac{1}{(1+t)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\left(a_1\|x\|_B+b_1\|y\|_B\right).\end{aligned}$$

Bound the integrand using (H4):

$|f(s,x(s),y(s))| \leq a_1 \frac{|x(s)|}{1+s} + b_1 \frac{|y(s)|}{1+s} + c_1$

and we have :

the integral:

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha}, \quad \text{and} \quad \frac{t^\alpha}{1+t} \leq 1 \quad \text{for } t \geq 0.$$

Use the norm $\|x\|_B = \sup_{t\in J} \frac{|x(t)|}{1+t}$:

$$\frac{N_1(x(t),y(t))}{1+t} \leq |x_0| + \frac{c_1}{\Gamma(\alpha+1)} + \frac{a_1\|x\|_B + b_1\|y\|_B}{\Gamma(\alpha+1)}.$$

Hence

$$\begin{aligned} \sup_{t\in J} \frac{|N_1(x(t),y(t))|}{1+t} &\leq |x_0| + \frac{c_1}{\Gamma(\alpha+1)} + \frac{a_1\|x\|_B + b_1\|y\|_B}{\Gamma(\alpha+1)} \\ \sup_{t\in J} \frac{|N_2(x(t),y(t))|}{1+t} &\leq |y_0| + \frac{c_1}{\Gamma(\beta+1)} + \frac{a_2\|x\|_B + b_2\|y\|_B}{\Gamma(\beta+1)}. \end{aligned} \tag{3.2}$$

Step 2. N is countiouns.

Let $(x_n, y_n) \longrightarrow (x,y)\ in\ BC_b(J,\mathbb{R}) \times BC_b(J,\mathbb{R})$ .Then; there exists $l > 0$ such that for any $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, we have :

$$\frac{x_n(t)}{1+t}, \quad \frac{y_n(t)}{1+t}, \quad \frac{x(t)}{1+t}, \quad \frac{y(t)}{1+t} \in [-l,l].$$

By (H3), for every $\epsilon > 0\ there\ exists\ \delta(\epsilon) > 0$ such that for all $x, y, \bar{x}, \bar{y} \in [-l,l]$ and for all $t \in \mathbb{R}_+$ with

$$\left|\frac{x(t)}{1+t} - \frac{\bar{x}(t)}{1+t}\right| < \delta(\epsilon)\,, \left|\frac{y(t)}{1+t}| - \frac{\bar{y}(t)}{1+t}\right| < \delta(\epsilon).$$

we have

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| < \epsilon\Gamma(\alpha+1), |g(t,x,y) - g(t,\bar{x},\bar{y})| < \epsilon\Gamma(\beta+1).$$

Since, (x_n, y_n) converge to (x,y), then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\left|\frac{x_n(t)}{1+t} - \frac{x(t)}{1+t}\right| < \delta(\epsilon), \left|\frac{y_n(t)}{1+t} - \frac{y(t)}{1+t}\right| < \delta(\epsilon), \ \ \forall t \in \mathbb{R}_+$$

Hence

$$\left|\frac{N_1(x_n,y_n)(t)-N_1(x,y)(t)}{1+t}\right| \leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t}\left|f(s,x_n(s),y_n(s))-f(s,x(s),y(s))\right| ds$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t} ds\Gamma(\alpha+1)\varepsilon \leq \frac{t^\alpha}{1+t}\varepsilon \leq \varepsilon.$$

Thus

$$\left\|N_1(x_n,y_n)-N_1(x,y)\right\|_B \longrightarrow 0,$$

$$\left\|N_2(x_n,y_n)-N_2(x,y)\right\|_B \longrightarrow 0,$$

as $n \to \infty$.

Step 3. We will show that N maps bounded sets into bounded sets in $BC_b(J,\mathbb{R}) \times BC_b(J,\mathbb{R})$ Let $B_r := \{(x,y) \in BC_b(J,\mathbb{R}) \times BC_b(J,\mathbb{R}) \| (x,y)\|_\infty \leq r\}$, where $r = (r_1, r_2)$ and if $(x,y) \in B_r$ then we obtain

$$\frac{|N_1(x(t),y(t))|}{1+t} \leq \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t}\left|f(s,x(s),y(s))\right| ds$$
$$\leq \frac{1}{\Gamma(\alpha)}[a_1 r_1 + b_1 r_2 + c_1].$$

Similarly, we have

$$\frac{|N_2(x(t),y(t))|}{1+t} \leq \frac{1}{\Gamma(\beta)}[a_2 r_1 + b_2 r_2 + c_2].$$

Hence

$$\left(\|N_1(x,y)\|, \|N_2(x,y)\|\right) \leq (l_1, l_2),$$

where

$$l_1 = \frac{1}{\Gamma(\alpha)}[a_1 r_1 + b_1 r_2 + c_1], \quad l_2 = \frac{1}{\Gamma(\beta)}[a_2 r_1 + r_2 b_2 + c_2].$$

Step 4. Now, we prove that N maps bounded sets in $BC_b(J,\mathbb{R}) \times BC_b(J,\mathbb{R})$ into almost equicontinuous sets of $BC_b(J,\mathbb{R}) \times BC_b(J,\mathbb{R})$.

Let $(x,y)\in B_r$. Then for all $t_1<t_2$,

$$\begin{aligned}\left|\frac{N_1(x(t_2),y(t_2))}{1+t_2}-\frac{N_1(x(t_1),y(t_1))}{1+t_2}\right| &\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2}-\frac{(t_1-s)^{\alpha-1}}{1+t_1}\right||f(s,x(s),y(s))|ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2}|f(s,x(s),y(s))|ds\\ &\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2}-\frac{(t_1-s)^{\alpha-1}}{1+t_1}\right|(a_1r_1+b_1r_2+c_1)ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2}(a_1r_1+b_1r_2+c_1)ds\\ &\leq \frac{a_1r_1+b_1r_2+c_1}{\Gamma(\alpha+1)}\left(\left|\frac{t_2^\alpha}{1+t_2}-\frac{t_1^\alpha}{1+t_1}\right|+\frac{2(t_2-t_1)^\alpha}{1+t_2}\right).\end{aligned}$$

Thus

$$\left|\frac{N_1(x(t_2),t(t_2))}{1+t_2}-\frac{N_1(x(t_1),t(t_1))}{1+t_1}\right|\to 0 \quad \text{as} \quad t_1\to t_2.$$

Similarly, we have

$$\left|\frac{N_2(x(t_2),t(t_2))}{1+t_2}-\frac{N_2(x(t_1),t(t_1))}{1+t_1}\right|\to 0 \quad \text{as} \quad t_1\to t_2.$$

Step 5. The set $N(\bar{B}(0,r))$ is equiconvergente, i.e, for every $\epsilon>0$, there exists $T(\epsilon)>0$ such that $|N(x(t_2),y(t_2))-N(x(t_1),y(t_1))|<\epsilon$, for every $t_1,t_2>T(\epsilon)$ and each $(x,y)\in\bar{B}(0,r)$.

$$\left|\frac{N_1(x(t_2),y(t_2))}{1+t_2}-\frac{N_1(x(t_1),y(t_1))}{1+t_1}\right|\leq\frac{a_1r_1+b_1r_2+c_1}{\Gamma(\alpha+1)}\left(\frac{3t_2^\alpha}{1+t_2}+\frac{t_1^\alpha}{1+t_1}\right) \tag{3.3}$$

$$\begin{aligned}&\left|\frac{N_2(x(t_2),y(t_2))}{1+t_2}-\frac{N_2(x(t_1),y(t_1))}{1+t_1}\right|\\ &\leq\frac{a_2r_1+b_1r_2+c_1}{\Gamma(\alpha+1)}\left(\frac{3t_2^\beta}{1+t_2}+\frac{t_1^\beta}{1+t_1}\right)\end{aligned}$$

It is clear that :

$$\frac{3t_2^\beta}{1+t_2}\to 0,\quad \frac{t_1^\beta}{1+t_1}\to 0,\quad \text{as } t_2,t_1\to\infty. \tag{3.4}$$

Then, for any $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that for all $t_1, t_2 > T(\epsilon)$, we have

$$\left| \frac{N_1(x(t_2), y(t_2))}{1+t_2} - \frac{N_1(x(t_1), y(t_1))}{1+t_1} \right| \leq \epsilon \text{ for all } (x, y) \in \bar{B}(0, r) \tag{3.5}$$

$$\left| \frac{N_2(x(t_2), y(t_2))}{1+t_2} - \frac{N_2(x(t_1), y(t_1))}{1+t_1} \right| \leq \epsilon \text{ for any } (x, y) \in \bar{B}(0, r)$$

Step 6 . The set is defined as follows: $\mathscr{M} = \{(x, y) \in BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R}) : (x, y) = \mu N(x, y), \mu \in (0, 1)\}$ is bounded. Let $(x, y) \in \mathscr{M}$, then

$$\frac{|x(t)|}{1+t} \leq \frac{|x_0|}{1+t} + \frac{\mu}{(1+t)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, x(s), y(s))| ds$$

$$\leq \frac{|x_0|}{1+t} + \frac{\mu}{(1+t)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cdot \left(a_1 \frac{|x(s)|}{1+s} + b_1 \frac{|y(s)|}{1+s} + c_1 \right) ds$$

$$\leq \frac{|x_0|}{1+t} + \frac{c_1 t^\alpha}{\Gamma(\alpha+1)(1+t)} + \frac{1}{(1+t)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \cdot \left(a_1 \frac{|x(s)|}{1+s} + b_1 \frac{|y(s)|}{1+s} \right) ds.$$

Similarly, we get

$$\frac{|y(t)|}{1+t} \leq \frac{|y_0|}{1+t} + \frac{c_2 t^\beta}{\Gamma(\beta+1)(1+t)} + \frac{1}{(1+t)\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \cdot \left(a_2 \frac{|x(s)|}{1+s} + b_2 \frac{|y(s)|}{1+s} \right) ds.$$

Thus

$$\frac{|x(t)|}{1+t} + \frac{|y(t)|}{1+t} \leq \frac{X_0}{1+t} + \frac{C t^\gamma}{1+t} + \frac{C_*}{1+t} \int_0^t (t-s)^{\gamma-1} \cdot \left(a_1 \frac{|x(s)|}{1+s} + b_2 \frac{|y(s)|}{1+s} \right) ds.$$

Where

$$X_0 = |x_0| + |y_0|,$$

$$C = \frac{c_1}{\Gamma(\alpha+1)} + \frac{c_2}{\Gamma(\beta+1)},$$

$$C_* = \frac{a_1+b_1}{\Gamma(\alpha)} + \frac{a_2+b_2}{\Gamma(\beta)},$$

$$\gamma = \max(\alpha, \beta).$$

Hence

$$\frac{|x(t)|}{1+t} + \frac{|y(t)|}{1+t} \leq X_0 + C + \frac{C_*}{1+t}\int_0^t (t-s)^{\gamma-1}\cdot\left(\frac{|x(s)|}{1+s} + \frac{|y(s)|}{1+s}\right) ds.$$

By the Gronwall-Bellman lemma 1.3 , we have

$$\begin{aligned}\frac{|x(t)|}{1+t} + \frac{|y(t)|}{1+t} &\leq X_0 + C + \int_0^t \sum_{n=1}^{\infty} \frac{(C_*\Gamma(\gamma))^n}{\Gamma(n\gamma)}(t-s)^{n\gamma-1} ds \\ &\leq X_0 + C + \sum_{n=1}^{\infty} \frac{(t^\gamma C_*\Gamma(\gamma))^n}{(1+t)^n\Gamma(n\gamma+1)}(X_0+C) \\ &\leq X_0 + C + \sum_{n=0}^{\infty} \frac{(C_*\Gamma(\gamma))^n}{\Gamma(n\gamma+1)}(X_0+C)\end{aligned}$$

$v(t) = \frac{|x(t)|}{1+t} + \frac{|y(t)|}{1+t}$, $\quad w(t) = X_0 + C$, $\quad$ and $\quad a(t) = C_*$.

The solution is given by:

$$v(t) \leq w(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(C_*\Gamma(\gamma))^n}{\Gamma(n\gamma)}(t-s)^{n\gamma-1} w(s)\, ds.$$

Simplify using the Mittag-Leffler function:

$$v(t) \leq (X_0 + C)\big(1 + E_\gamma(C_*\Gamma(\gamma))\big),$$

Then, for every $t \in [0, \infty)$, we have :

$$\frac{|x(t)|}{1+t} + \frac{|y(t)|}{1+t} \leq (X_0 + C)(1 + E_\gamma(C_*\Gamma(\gamma)) := M$$

Where E_γ is thr Mittag-Lefller function defined by :

$$E_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\gamma+1)}.$$

Then

$$\|x\|_{C_b} \leq M, \|y\|_{C_b} \leq M. \tag{3.6}$$

According to theorem (1.4), problem 1.2 has at least one solution.

Now, we show that the set

$$S(x_0, y_0) = \{(x, y) \in BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R}) : (x, y) \text{ is solution of } 1.2\},$$

is compact. It is clear that $S(x_0, y_0) \subset NS(x_0, y_0)$. From 3.6 we deduced that $S(x_0, y_0)$ is bounded sets in $BC_b(J, \mathbb{R})$. Since N is compact, then $S(x_0, y_0)$ is compact if and only if $S(x_0, y_0)$ is closed. Let $\{(x_n, y_n)\}_{n\geq 1} \subset S(x_0, y_0)$ be a sequence converge to (x, y). Thus,

$$\begin{aligned} x_n(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_n(s), y_n(s))\, ds, \\ y_n(t) &= y_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x_n(s), y_n(s))\, ds, \\ & t \in J. \end{aligned}$$

Similarly to Step 2, we can prove that to (x, y). Thus,

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s))\, ds, \\ y(t) &= y_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s), y(s))\, ds, \\ & t \in J. \end{aligned}$$

This implies that $S(x_0, y_0)$ is compact. The solution operator S is u.s.c .

(a) S has a closed graph:

To see this, first note that the graph of S is the set :

$$G_S = \left\{((a, b), (x, y)) \in (\mathbb{R} \times \mathbb{R}) \times (BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R})) \mid (x, y) \in S(a, b)\right\}.$$

Let $((a_q, b_q), (x_q, y_q))_q$ be a sequence in G_S and let $((a_q, b_q), (x_q, y_q))_q \longrightarrow ((a, b), (x, y))\, as\, q \longrightarrow \infty$.

Since $(x_q, y_q) \in S(a_q, b_q)$, then we have :

$$x_q(t) = a_q + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x_q(s), y_q(s))\, ds,$$
$$y_q(t) = b_q + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x_q(s), y_q(s))\, ds,$$
$$t \in J.$$

Let :

$$\begin{aligned} Z(t) &= (Z_1(t), Z_2(t)) \\ &= \left(a + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), y(s))\, ds,\ b + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s, x(s), y(s))\, ds \right) \end{aligned}$$

$t \in J$. As in step 2, we can prove that

$$(x_q, y_q) \longrightarrow (x, y),$$

(b) Using the same method as Steps 3 to 4, we find that for each bounded set $C \subset BC_b(J, \mathbb{R}) \times BC_b(J, \mathbb{R})$, we can show that $S(C)$ is compact.

From (a) and (b), we conclued that, S is u.s.c. □

Chapter 4

Applications

In this section, we show the applicability of our main result. We begin by illustrating by Theorem 2.1.

Example 01. Consider the problem:

$$\begin{cases} {}^{c}D^{\frac{1}{2}}x = h_1(t)(1+k_1\sin|x|+k_2\cos|x|), \\ {}^{c}D^{\frac{1}{2}}y = h_2(t)(1+k_3|x|+k_4|x|)^{\frac{1}{2}}, \quad t\in\mathbb{R}, \\ x(0)=x_0, \\ y(0)=y_0, \end{cases} \tag{4.1}$$

where :

$$h_1(t)=\frac{\sqrt{t}\int_0^t s e^{-s^2}\,ds}{100(1+t^2)} \quad \text{and} \quad h_2(s)=\frac{t\sqrt{t}}{200(1+t^2)}, \quad t\in\mathbb{R}_+,\ k_i\geq 0,\ i=1,2,3,4,$$

$$f(t,x,y)=h_1(t)(1+k_1\sin|x|+k_2\cos|y|), \quad g(t,x,y)=h_2(t)(1+k_3|x|+k_4|y|).$$

It is clear that :

$$\begin{aligned} \int_0^t \frac{h_1(s)}{(t-s)^{1/2}}\,ds &= \int_0^t \frac{\sqrt{s}\int_0^s r e^{-r^2}\,dr}{100(1+s^2)(t-s)^{1/2}}\,ds \\ &\leq \frac{\int_0^\infty e^{-r^2}\,dr}{100(1+s^2)(t-s)^{\frac{1}{2}}}\int_0^t \frac{s\sqrt{s}}{(1+s^2)(t-s)^{1/2}}\,ds \\ &= \frac{\int_0^\infty e^{-r^2}\,dr}{100}\int_0^t \frac{t^2\sqrt{l}}{(1+t^2l^2)(1-l)^{1/2}}\,dl \\ &\leq \frac{\int_0^\infty e^{-r^2}\,dr}{100}\int_0^t (1-l)^{-1/2}\,dl \\ &\leq \frac{\int_0^\infty e^{-r^2}\,dr}{50} = \frac{\sqrt{\pi}}{100}, \end{aligned}$$

$$\int_0^t \frac{h_2(s)}{(t-s)^{1/2}}\,ds = \int_0^t \frac{s\sqrt{s}}{200(1+s^2)(t-s)^{1/2}}\,ds \leq \frac{1}{100}.$$

Then,

$$\sup_{t\in\mathbb{R}_+}\int_0^t \frac{h_i(s)}{(t-s)^{1/2}}\,ds<\infty,\quad i=1,2.$$

Observe that every $x,y,\bar{x},\bar{y}\in\mathbb{R}\ and\ t\in\mathbb{R}$, we have

$$|f(t,x,y)-f(t,\bar{x},\bar{y})|\le k_1h_1(t)|x-\bar{x}|+k_2h_1(t)|y-\bar{y}|,$$
$$|g(t,x,y)-g(t,\bar{x},\bar{y})|\le k_3h_2(t)|x-\bar{x}|+k_4h_2(t)|y-\bar{y}|.$$

Hence the condition (H1) holds.
We see that :

$$|f(t,0,0)|=h_1(t),\quad |g(t,0,0)|=h_2(t),\quad t\in\mathbb{R}_+.$$

Hence the condition (H2) holds. Assume that $k_1k_4-k_2k_3\neq 0$. For this , we havr

$$\tilde{M}=\begin{pmatrix} \frac{k_1\sqrt{\pi}}{\Gamma(1/2)100} & \frac{k_2\sqrt{\pi}}{\Gamma(1/2)100}\\ \frac{k_3\sqrt{\pi}}{\Gamma(1/2)100} & \frac{k_4\sqrt{\pi}}{\Gamma(1/2)100}\end{pmatrix}.$$

$\det\tilde{M}\neq 0$. If we add that $k_2=0\ or\ k_3\ and\ 0<k_1<\dfrac{(\Gamma(\frac{1}{2})100)}{\sqrt{\pi}}, 0<k_4<\Gamma(\frac{1}{2})100\le 1$. the $\rho(\tilde{M})<1$. By Theorem 2.1 . it follows that problem 4.1 has a unique solution.

In this last exemple, we illustrate the applicability of Theoreme 3.1 .

Lemma 4.1. Let us consider the function $f:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ definded by

$$f(t,x,y)=a(t)\phi(x,y),$$

where $a:\mathbb{R}_+\longrightarrow\mathbb{R}, \phi:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfy the following assumptions :

(S1) $a\in C_b(\mathbb{R})$
(S2) the function ϕ is continous and bounded. Then, the function

$$f_*(t,x,y)=a(t)\phi\left(\frac{x}{1+t},\frac{y}{1+t}\right),\quad t\in\mathbb{R}_+,\ (x,y)\in\mathbb{R}\times\mathbb{R},$$

satisfied the condition (H3)

Proof. Since a is bounded, then there exist $A>0$ such that

$$|a(t)|<A, \forall\, t\in\mathbb{R}_+$$

Let $l>0$, then for any $(x,y),(\bar{x},\bar{y})\in[-l,l]\times[-l,l]$, we obtain

$$\left|f(t,x,y)-f(t,\bar{x},\bar{y})\right|=|a(t)|\left|\phi(x,y)-\phi(\bar{x},\bar{y})\right|.$$

Hence,

$$\left|f(t,x,y)-f(t,\bar{x},\bar{y})\right|\leq A\left|\phi(x,y)-\phi(\bar{x},\bar{y})\right|.$$

It is clear that form (S2) ϕ is uniformly continuous on the $[-l,l]\times[-l,l]$. Thus, for any $\epsilon>0$ there exsits $\delta(\epsilon)>0$ such that for all $(x,y),(\bar{x},\bar{y})\in[-l,l]\times[-l,l]$ with $|x-\bar{x}|<\delta(\epsilon),|y-\bar{y}|<\delta(\epsilon)$ then,

$$\left|f(t,x,y)-f(t,\bar{x},\bar{y})\right|\leq A\epsilon.$$

This implies that for every $t\in\mathbb{R}_+$ such that for all $(x,y),(\bar{x},\bar{y})\in[-l,l]\times[-l,l]$ with $\left|\frac{x}{(1+t)}-\frac{\bar{x}}{(1+t)}\right|<\delta(\epsilon),\left|\frac{y}{(1+t)}-\frac{\bar{y}}{(1+t)}\right|<\delta(\epsilon)$, we get

$$\left|f(t,x,y)-f(t,\bar{x},\bar{y})\right|\leq A\epsilon.$$

□

Theorem 4.1. Let $a_1,a_2:\mathbb{R}_+\longrightarrow\mathbb{R},\phi_1,\phi_2:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ be functions that satisfy the conditions (S1) and (S2).

Assume
(S3) there exists $d_i>0, i=1,6$ such that

$$\begin{aligned}\left|\phi_1(x,y)\right|&\leq d_1|x|+d_2|y|+d_3,\quad \forall x,y\in\mathbb{R},\ t\in\mathbb{R}_+,\\ \left|\phi_2(x,y)\right|&\leq d_4|x|+d_5|y|+d_6,\quad \forall x,y\in\mathbb{R},\ t\in\mathbb{R}_+.\end{aligned}$$

Then, the following probleme

$$\begin{cases} {}^{c}D^{\alpha}x(t) = f(t,x(t)), \\ {}^{c}D^{\beta}x(t) = g(t,x(t)), \quad t \in I, \\ x(0) = x_0, \\ y(0) = y_0, \end{cases} \tag{4.2}$$

Where

$$f(t,x,y) = a_1(t)\phi_1\left(\frac{x}{1+t}, \frac{y}{1+t}\right), \quad t \in \mathbb{R}_+, (x,y) \in \mathbb{R}\times\mathbb{R},$$

$$g(t,x,y) = a_2(t)\phi_2\left(\frac{x}{1+t}, \frac{y}{1+t}\right), \quad t \in \mathbb{R}_+, (x,y) \in \mathbb{R}\times\mathbb{R},$$

Proof. From Lemma 4.1, we deduce that f and g satisfies(H3). By (S1) and (S3), there exist $A_1, A_2 > 0$ such that

$$\begin{aligned} f(t,x,y) &\leq A_1\left(\frac{|x|}{1+t}, \frac{|y|}{1+t}\right), \\ g(t,x,y) &\leq A_2\left(\frac{x}{1+t}, \frac{y}{1+t}\right), \quad t \in \mathbb{R}_+, \ x,y \in \mathbb{R}. \end{aligned}$$

Therefore, all the conditions of Theorem 3.1 hold. Then, problem (4.2) has at least one solution in BC_b □

Remark 4.1. For another application, we can replace the con dition (S2) with (S4). The functions φ_1 and φ_2 are uniformly continuous and bounded.

Conclusion

In this work we investigated a system of fractional differ ential equations under various assumptions on the right hand-side nonlinearity and we obtain a number of results regarding the existence and uniqueness of solutions in an appropriate space of continuous functions. In this paper, we have focused on the dependence continuity of a solution, compactness of solution sets, and upper semicontinuity of operator solutions. We hope this paper can provide some contribution to the questions of existence and topological structure for the system of fractional differential equations [1] on unbounded domains.

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