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1 Introduction

In 1915, Albert Einstein developed the theory of general relativity, in which he generalized the Newtonian mechanics and special relativity. In this geometric theory of gravity, gravitation is no longer a force: it is just the geometric deformation that the content in matter or energy imposes on the continuum (the fabric) of space-time to four dimensions, causing it to curve and bend. Thus, a high concentration of mass such as a star, a cluster of galaxies or a galaxy distorts the spacetime around it and affects the path of light rays coming from a distant source passing nearby.The field equations of general relativity induced the theoretical existence of one of the most fascinating classes of objects in the universe (the gravitational waves), which can be derived from the linearized gravity.

In the theory of general relativity, linear gravity is the application of the theory of perturbations to the metric tensor that describes the geometry of space-time. As a result, linearised gravity is an efficient method of modeling the effects of gravity when the gravitational field is weak. The usage of linearized gravity is integral to the study of gravitational waves.

The existence of gravitational waves has been confirmed indirectly, and this at several times, by measuring their effects on the dynamics of binary systems. As with all massive objects, binary black holes rotate around each other. around others due to their mutual gravitational attraction. Unlike a conservative Newtonian two-body system, the binary hole system black has a dynamic quadrupole and a higher order mass moment, which translated by gravitational radiation. When the system produces an gravitational wave (GW), it loses energy so that the two black holes come closer together until they end by merging into a single black hole.

On February 2016, the LIGO detector announced the first observation of gravitational waves, from a signal detected at 09:50:45 GMT on 14 September 2015 of two black holes with masses of 29 and 36 solar masses merging about 1.3 billion light-years away.and The Royal Swedish Academy of Sciences has awarded the Nobel Prize in Physics 2017 to Rainer Weiss, Barry C. Barish and Kip S. Thorne for decisive contributions to the LIGO detector and the observation of gravitational waves. First predicted a century ago by Albert Einstein who did not believe that the gravitational waves could be measured.

In this thesis, we start by reviewing the main geometric concepts used in general relativity to describe the gravitational phenomena and the Einstein field equations from which we derive the Scwarzchild solution as a first application of the theory. In the second chapter, we derive the linearized version of the gravitational field equation, from which we can derive the wave-like solutions. The final chapter takes a special look at the special choice of gravitational plane waves in the transverse traceless gauge and their interactions with matter, then describes the detection process used in the LIGO detector.

2 Mathematical Formulation of General Relativity

2.1 From metric to Einstein Equation In general relativity,

The metric tensor is the fundamental object of study. It may loosely be thought of as a generalization of the gravitational potential of Newtonian gravitation. The metric captures all the geometric and causal structure of spacetime, being used to define notions such as time, distance, volume, curvature, angle, and separation of the future and the past.

Mathematically, spacetime is represented by a four-dimensional differentiable manifold M , and the metric tensor is given as a covariant, second-degree, symmetric tensor on M, conventionally denoted by g.

Moreover, the metric is required to be nondegenerate with signature $(-+++)$. A manifold M equipped with such a metric is a type of Lorentzian manifold.

Usually we work in local coordinates (i.e. coordinates defined on some local patch of M). In local coordinates x^μ (where μ is an index that runs from 0 to 3) the metric can be written in the form.

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \tag{1}$$

The metric tensor plays a key role in index manipulation. In index notation, the coefficients $g_{\mu\nu}$ of the metric tensor g provide a link between covariant and contravariant components of other tensors. Contracting the contravariant index of a tensor with one of a covariant metric tensor coefficient has the effect of lowering the index

$$g_{\mu\nu} A^\nu = A_\mu \tag{2}$$

And similarly a contravariant metric coefficient raises the index

$$g^{\mu\nu} A_\nu = A^\mu \tag{3}$$

Applying this property of raising and lowering indices to the metric tensor components themselves leads to the property

$$g_{\mu\nu} g^{\nu\lambda} = \delta^\lambda_\mu \tag{4}$$

2.2 Flat spacetime

The simplest example of a Lorentzian manifold is flat spacetime, which can be given as $R^4$with coordinates (t, x, y, z) and the metric

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{5}$$

the matrix representation of η is

$$\eta = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

Christoffel symbols are defined by

$$\Gamma^\lambda{}_{uv} = g^{\lambda\rho} \Gamma_{\rho uv} = \frac{1}{2} g^{\lambda\rho} \left[g_{\rho u,v} + g_{\rho v,u} - g_{uv,\rho} \right] \tag{7}$$

$$R^k{}_{\lambda uv} = \Gamma^k{}_{\lambda v,u} - \Gamma^k{}_{\lambda u,v} + \Gamma^k{}_{\rho u} \Gamma^\rho{}_{\lambda v} - \Gamma^k{}_{\rho v} \Gamma^\rho{}_{\lambda u} \tag{8}$$

We can derive the Ricci tensor from the Riemann tensor by contracting the first and third indices as fellow

$$\begin{aligned} R_{uv} &= R^\alpha{}_{u\alpha v} && (9) \\ &= \partial_\alpha \Gamma^\alpha{}_{uv} - \partial_v \Gamma^\alpha{}_{u\alpha} + \Gamma^\alpha{}_{\beta\alpha} \Gamma^\beta{}_{uv} - \Gamma^\alpha{}_{\beta v} \Gamma^\beta{}_{u\alpha} && (10) \\ &= \Gamma^\alpha{}_{uv,\alpha} - \Gamma^\alpha{}_{u\alpha,v} + \Gamma^\alpha{}_{\beta\alpha} \Gamma^\beta{}_{uv} - \Gamma^\alpha{}_{\beta v} \Gamma^\beta{}_{u\alpha} && (11) \end{aligned}$$

The Ricci tensor is symmetric:

$$R_{jl} = R_{lj} \tag{12}$$

Proof. $R_{ijkl} = R_{klij} \rightarrow g^{ib}R_{ijkl} = g^{ib}R_{klij}$
$\rightarrow R^{b}{}_{jkl} = R_{kl}{}^{b}{}_{j} \rightarrow R^{b}{}_{jbl} = R_{bl}{}^{b}{}_{j} \rightarrow R_{jl} = R_{lj}$ ■

Then, we define the *Ricci scalar* R as the Riemann curvature tensor contracted twice,

$$\begin{aligned} R &= R^{a}{}_{a} && (13)\\ &= g^{ab}R_{ab} && (14)\\ &= g^{ab}\left(\Gamma^{c}{}_{ab,c} - \Gamma^{c}{}_{ac,b} + \Gamma^{d}{}_{ab}\Gamma^{c}{}_{cd} - \Gamma^{d}{}_{ac}\Gamma^{c}{}_{bd}\right) && (15)\\ &= 2g^{ab}\left(\Gamma^{c}{}_{a[b,c]} + \Gamma^{d}{}_{a[b}\Gamma^{c}{}_{c]d}\right) && (16) \end{aligned}$$

Now, we cab define the Einstein tensor as

$$G^{uv} = R^{uv} - \frac{1}{2}Rg^{uv} \tag{17}$$

Which is covariantly constant. [need ref]

The Einstein tensor describes the matter and energy existing in the space-time and can be expressed in terms of Christoffel symbols as

$$\begin{aligned} G^{uv} &= R^{uv} - \frac{1}{2}Rg^{uv} && (18)\\ &= \Gamma^{\alpha}{}_{uv,\alpha} - \partial_{v}\Gamma^{\alpha}{}_{u\alpha,v} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{uv} - \Gamma^{\alpha}{}_{\beta v}\Gamma^{\beta}{}_{u\alpha}\\ &\quad -\frac{1}{2}g^{uv}g^{ab}\left(\Gamma^{c}{}_{ab,c} - \Gamma^{c}{}_{ac,b} + \Gamma^{d}{}_{ab}\Gamma^{c}{}_{cd} - \Gamma^{d}{}_{ac}\Gamma^{c}{}_{bd}\right) && (19) \end{aligned}$$

Now, we can express the Einstein field equations (EFE), without cosmological constant as

$$G^{uv} = \frac{8\pi G}{c^{4}}T^{uv} \tag{20}$$

Which represent a unification of geometry and physics, as the Einstein tensor G^{uv} describes the geometry of spacetime and T^{uv} describes the matter-energy in the spacetime. G is the Newton gravitation constant.

2.3 Schwarzschild Solution of Einstein Equation:

Solving the EFE meaning we put discription of $T_{\mu\nu}$ (mass momuntum energie) to get discription of the space time which we study

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- First we assume that we outside in the vaccum ($T_{\mu\nu} = 0$) and there is no cosmological scale($\Lambda = 0$), so

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$$

$$R_{\mu\nu}g^{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}g^{\mu\nu} = 0$$

$$R^{\mu}_{\mu} - \frac{1}{2}R\delta^{\mu}_{\mu} = 0$$

$$R - \frac{4}{2}R = 0 \Leftrightarrow R = 0$$

R=0 which we call Ricci Flat space time (no immidiate volume change) but the Riemannien coefficient $R^{\rho}_{\tau\mu\nu} \neq 0$(curved spacetime)

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(\sin\theta)^2 \end{pmatrix} \tag{21}$$

Connection coefficients : there are 30 CC and only 9 of theme are independent, we calculate by the rule:

$$\Gamma_{\alpha\beta\mu} = -\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\mu}} + \frac{1}{2}\frac{\partial g_{\beta\mu}}{\partial x^{\alpha}} + \frac{1}{2}\frac{\partial g_{\mu\alpha}}{\partial x^{\beta}} \tag{22}$$

$$\begin{aligned} \Gamma^0_{01} &= \Gamma^0_{10} = \frac{1}{2A}(\partial_r A) \\ \Gamma^1_{00} &= \frac{1}{2b}(\partial_r A) \end{aligned}$$

$$\begin{aligned} \Gamma^1_{11} &= \frac{1}{2b}(\partial_r b) \\ \Gamma^1_{22} &= \frac{-r}{b} \end{aligned}$$

$$\begin{aligned} \Gamma^1_{33} &= \frac{-r(\sin\theta)^2}{b} \\ \Gamma^2_{33} &= -\sin\theta\cos\theta \end{aligned}$$

$$\begin{aligned} \Gamma^2_{12} &= \Gamma^2_{21} = 1/r \\ \Gamma^3_{13} &= \Gamma^3_{31} = 1/r \\ \Gamma^3_{23} &= \Gamma^3_{32} = \cot\theta \end{aligned}$$

Ricci coefficients : Now we calculate R_{00} R_{11} R_{22}, we already know from the EFE that $R_{\mu\nu} = 0$

$$
\begin{aligned}
R^{\rho}_{\sigma\mu\nu} &= \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\rho}_{\mu\alpha} - \Gamma^{\beta}_{\mu\sigma}\Gamma^{\rho}_{\nu\beta} \\
R_{00} &= R^{\mu}_{0\mu0} = \partial_{\mu}\Gamma^{\mu}_{00} - \partial_{0}\Gamma^{\mu}_{\mu0} + \Gamma^{\alpha}_{00}\Gamma^{\rho}_{\mu\alpha} - \Gamma^{\beta}_{\mu0}\Gamma^{\mu}_{0\beta} \\
R_{00} &= \partial_{1}\Gamma^{1}_{00} - 0 + \Gamma^{1}_{00}\Gamma^{\mu=0123}_{\mu1} - \Gamma^{0}_{\mu0}\Gamma^{\mu}_{00} - \Gamma^{1}_{\mu0}\Gamma^{\mu=0}_{01} \\
&= \partial_{1}\Gamma^{1}_{00} + \Gamma^{1}_{00}\Gamma^{0}_{01} + \Gamma^{1}_{00}\Gamma^{1}_{11} + \Gamma^{1}_{00}\Gamma^{2}_{21} + \Gamma^{1}_{00}\Gamma^{3}_{31} - \Gamma^{0}_{10}\Gamma^{1}_{00} - \Gamma^{1}_{00}\Gamma^{0}_{01} \\
&= \partial_{1}\Gamma^{1}_{00} + +\Gamma^{1}_{00}\Gamma^{1}_{11} + 2\Gamma^{1}_{00}\Gamma^{2}_{21} - \Gamma^{0}_{10}\Gamma^{1}_{00} \\
&= \partial_{r}(\frac{\partial_{r}A}{2B}) + (\frac{\partial_{r}A}{2B})(\frac{\partial_{r}B}{2B}) + 2(\frac{\partial_{r}A}{2B})(\frac{1}{r}) - (\frac{\partial_{r}A}{2A})(\frac{\partial_{r}A}{2B}) \\
&= \frac{\partial_{r}\partial_{r}A}{2B} + \frac{\partial_{r}A}{2}B^{-2}(-1)(\partial_{r}B) + \frac{\partial_{r}A\partial_{r}B}{4B^{2}} + \frac{\partial_{r}A}{Br} - \frac{(\partial_{r}A)^{2}}{4AB} \\
&= \frac{\partial_{r}\partial_{r}A}{2B} - \frac{\partial_{r}A\partial_{r}B}{2B^{2}} + \frac{\partial_{r}A\partial_{r}B}{4B^{2}} + \frac{\partial_{r}A}{Br} - \frac{(\partial_{r}A)^{2}}{4AB} = 0 \\
&\quad \times(4AB^{2}r) \\
R_{00}\times(4AB^{2}r) &= 2rABA^{''} - rAA^{'}B^{'} + 4ABA^{'} - rB(A^{'})^{2} = 0
\end{aligned}
$$

Now we calculate(R_{11}) :

$$
\begin{aligned}
R_{11} &= R^{\mu}_{1\mu1} = \partial_{\mu}\Gamma^{\mu}_{11} - \partial_{1}\Gamma^{\mu}_{\mu1} + \Gamma^{\alpha}_{11}\Gamma^{\rho}_{\mu\alpha} - \Gamma^{\beta}_{\mu1}\Gamma^{\mu}_{1\beta} \\
&= \partial_{1}\Gamma^{1}_{11} - \partial_{1}\Gamma^{0}_{01} - \partial_{1}\Gamma^{1}_{11} - \partial_{1}\Gamma^{2}_{21} - \partial_{1}\Gamma^{3}_{31} + \Gamma^{1}_{11}\Gamma^{0}_{01} + \Gamma^{1}_{11}\Gamma^{1}_{11} \\
&\quad +\Gamma^{1}_{11}\Gamma^{2}_{21} + \Gamma^{1}_{11}\Gamma^{3}_{31} - \Gamma^{0}_{01}\Gamma^{0}_{10} - \Gamma^{1}_{11}\Gamma^{1}_{11} - \Gamma^{2}_{21}\Gamma^{2}_{12} - \Gamma^{3}_{13}\Gamma^{3}_{31} \\
&= -\partial_{1}\Gamma^{0}_{01} - 2\partial_{1}\Gamma^{2}_{21} + \Gamma^{1}_{11}\Gamma^{0}_{01} + 2\Gamma^{1}_{11}\Gamma^{2}_{21} - \Gamma^{0}_{01}\Gamma^{0}_{10} - 2\Gamma^{2}_{21}\Gamma^{2}_{12} \\
&= -(\frac{\partial_{r}\partial_{r}A}{2A} + \frac{\partial_{r}A}{2A^{2}}(-1)(\partial_{r}A)) + \frac{\partial_{r}A\partial_{r}B}{4AB} - 2(\frac{1}{r^{2}})(-1) + \frac{2\partial_{r}B}{2Br} - \frac{(\partial_{r}A)^{2}}{4A^{2}} - \frac{2}{r^{2}} \\
&= -\frac{\partial_{r}\partial_{r}A}{2A} + \frac{(\partial_{r}A)^{2}}{4A^{2}} + \frac{\partial_{r}A\partial_{r}B}{4AB} + \frac{\partial_{r}B}{Br} \\
&\quad \times 4A^{2}Br \\
R_{11}\times 4A^{2}Br &= -2rABA^{''} + rB(A^{'})^{2} + rAA^{'}B^{'} + 4A^{2}B^{'} = 0
\end{aligned}
$$

Calculate N $\mathbf{R}_{22}$:

$$\begin{aligned}
R_{22=}R^{\mu}_{2\mu 2} &= \partial_\mu\Gamma^{\mu}_{22} - \partial_2\Gamma^{\mu}_{\mu 2} + \Gamma^{\alpha}_{22}\Gamma^{\rho}_{\mu\alpha} - \Gamma^{\beta}_{\mu 2}\Gamma^{\mu}_{2\beta} \\
&= \partial_1\Gamma^{1}_{22} - \partial_2\Gamma^{3}_{32} + \Gamma^{1}_{22}\Gamma^{\mu}_{\mu 1} - \Gamma^{1}_{\mu 2}\Gamma^{\mu}_{21} - \Gamma^{2}_{2\mu}\Gamma^{\mu}_{22} - \Gamma^{3}_{\mu 2}\Gamma^{\mu}_{23} \\
&= \partial_1\Gamma^{1}_{22} - \partial_2\Gamma^{3}_{32} + \Gamma^{1}_{22}\Gamma^{0}_{01} + \Gamma^{1}_{22}\Gamma^{1}_{11} + \Gamma^{1}_{22}\Gamma^{2}_{21} + \Gamma^{1}_{22}\Gamma^{3}_{31} - \Gamma^{1}_{22}\Gamma^{2}_{21} - \Gamma^{2}_{21}\Gamma^{1}_{22} - \Gamma^{3}_{32}\Gamma^{3}_{23} \\
&= \partial_1\Gamma^{1}_{22} - \partial_2\Gamma^{3}_{32} + \Gamma^{1}_{22}\Gamma^{0}_{01} + \Gamma^{1}_{22}\Gamma^{1}_{11} - \Gamma^{3}_{32}\Gamma^{3}_{23} \\
R_{22} &= \partial_1\Gamma^{1}_{22} - \partial_2\Gamma^{3}_{32} + \Gamma^{1}_{22}(\Gamma^{0}_{01} + \Gamma^{1}_{11}) - \Gamma^{3}_{32}\Gamma^{3}_{23} \\
&= \partial_r(-rB^{-1}) - \partial_\theta(\cos\theta(\sin\theta)^{-1}) + \left(\frac{-r}{B}\right)\left(\frac{\partial_r A}{2A} + \frac{\partial_r B}{2B}\right) - (\cot\theta)^2 \\
&= -B^{-1} - \left(rB^{-2}\right)(-1)\left(\partial_r B\right) - (-\sin\theta)\left(\sin\theta\right)^{-1} - (\cos\theta)\left(\sin\theta\right)^{-2}(-1)(\cos\theta) \\
&\quad -\frac{r}{B}\left(\frac{\partial_r A}{2A} + \frac{\partial_r B}{2B}\right) - (\cot\theta)^2 \\
&= \frac{-1}{B} + \frac{r\partial_r B}{B^2} + 1 + (\cot\theta)^2 - \frac{r}{B}\left(\frac{\partial_r A}{2A} + \frac{\partial_r B}{2B}\right) \\
R_{22} &= \frac{-1}{B} + 1 - r\frac{\partial_r A}{2AB} + \frac{r\partial_r B}{2B^2} = 0 \\
R_{22}\times 2AB^2 &= -2AB + 2AB^2 - rA^{'}B + rAB^{'} = 0
\end{aligned}$$

$$\begin{aligned}
R_{00} &= 2rABA^{''} - rAA^{'}B^{'} + 4ABA^{'} - rB(A^{'})^2 \\
R_{11} &= -2rABA^{''} + rB(A^{'})^2 + rAA^{'}B^{'} + 4A^2B^{'} \\
R_{22} &= -2AB + 2AB^2 - rA^{'}B + rAB^{'}
\end{aligned}$$

$$R_{00} + R_{11} = 4ABA^{'} + 4A^2B^{'}$$

we get : $\mathrm{BA}^{'}+\mathrm{AB}^{'}=0 \Rightarrow \partial_r(AB) = 0.$
so $(\mathrm{AB})=\omega$

we already know that the shwarzchild metric approche to the minkowski metric when (r$\rightarrow\infty$)

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(\sin\theta)^2 \end{pmatrix} \tag{23}$$

$$r\rightarrow\infty \Longrightarrow g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(\sin\theta)^2 \end{pmatrix} \tag{24}$$

so :$A(r) \Rightarrow 1$ and $B(r) \Rightarrow 1$

$$A(r) \times B(r) = \omega$$

$$\omega = 1$$

$$B(r) = \frac{1}{A(r)} \text{ for all r}$$

we have

$$R_{22} = -2AB + 2AB^2 - rA^{'}B + rAB^{'} \tag{25}$$

and

$$B = A^{-1}; B^{'} = \partial_r(A^{-1}) = \frac{-A^{'}}{A^2} \tag{26}$$

by offset (26) in (25) we get:

$$rA^{'} = 1 - A$$

by solving the previous diffrential equation we get

$$A(r) = 1 - \frac{k}{r}$$

$$\begin{aligned} \frac{\partial}{\partial_r}A &= \frac{\partial}{\partial_r}\left(1 - \frac{k}{r}\right) \\ A^{'} &= -k\frac{\partial}{\partial_r}r^{-1} \\ A^{'} &= 0 - k(-1)r^{-2} = \frac{k}{r^2} \end{aligned}$$

plug in and check:

$$\begin{aligned} rA^{'} &= 1 - A \\ r(\frac{k}{r^2}) &= 1 - (1 - \frac{k}{r}) \\ k/r &= k/r \end{aligned}$$

so this soulution satisfy the differntail equation:
so

$$A(r) = 1 - \frac{k}{r} \text{ and } \mathrm{B(r)}=\frac{1}{A(r)}$$

the shwarzchild metric become :

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{k}{r} & 0 & 0 & 0 \\ 0 & -(1 - \frac{k}{r})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(\sin\theta)^2 \end{pmatrix} \tag{27}$$

or k called the shwarzchild radius A.K.A the event horizon of the black hole.

solve for the shwarzschild radius (k) :

we can solve the radius by forcing the shwarzschild metric to reproduice newtonien gravity (in limit of low velocity and weak gravity)

in the limite of low velocity the speed tensor writen :

$$\overrightarrow{u} = (c, 0, 0, 0) \tag{28}$$

we going to use the minkowski metric in cartisien coordinate

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{29}$$

by forcing the geodisique equation to match the newtonien equation for acceleration due to gravity

$$\begin{aligned} m\overrightarrow{a} &= m\overrightarrow{g} \\ \overrightarrow{a} &= \overrightarrow{g} \\ \frac{d^2x}{dt^2} &= -(\nabla\Phi) \\ \frac{d^2x^i}{dt^2} &= \frac{-\partial\Phi}{\partial x^i} \\ \frac{d^2x^i}{dt^2} + \frac{\partial\Phi}{\partial x^i} &= 0 \leftrightarrow \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \end{aligned}$$

we have

$$\overrightarrow{u} = (c, 0, 0, 0) \tag{30}$$

$$\begin{aligned} \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta &= 0 \\ \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta &= 0 \\ \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}u^0 u^0 &= 0 \\ \frac{d^2x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta}c^2 &= 0 \\ \frac{d^2x^\mu}{dt^2} + \Gamma^i_{00}c^2 &= 0 \leftrightarrow \frac{d^2x^i}{dt^2} + \frac{\partial\Phi}{\partial x^i} = 0 \\ \Gamma^i_{00} &= \frac{\partial\Phi}{\partial x^i}\frac{1}{c^2} \end{aligned}$$

now when we saying take the weak gravity this mean that the metric can be written as a summation:

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\ \text{or } \eta_{\mu\nu} &: \text{minkowski metric} \\ & \|h_{\mu\nu}\| \ll 1 \end{aligned}$$

so when doing the summation we get

$$\begin{aligned} T^{\mu\alpha}g_{\alpha\nu} &= T^{\mu\alpha}(\eta_{\alpha\nu}+h_{\alpha\nu}) \\ &= T^{\mu\alpha}\eta_{\alpha\nu}+T^{\mu\alpha}h_{\alpha\nu} \\ &= T^{\mu\alpha}\eta_{\alpha\nu}+0 \end{aligned}$$

and when doing derivation we get

$$\begin{aligned} \partial_\sigma g_{\mu\nu} &= \partial_\sigma(\eta_{\mu\nu}+h_{\mu\nu}) \\ &= \partial_\sigma\eta_{\mu\nu}+\partial_\sigma h_{\mu\nu} \\ &= 0+\partial_\sigma h_{\mu\nu} \qquad \text{(we ignored the minkowski constant in cartisien part)} \end{aligned}$$

The connection coefficiants can be written as :

$$\begin{aligned} \Gamma^\sigma_{\mu\nu} &= \frac{1}{2}g^{\sigma\alpha}(\partial_\nu g_{\alpha\mu}+\partial_\mu g_{\alpha\nu}-\partial_\alpha g_{\mu\nu}) \\ \Gamma^i_{00} &= \frac{1}{2}g^{i\alpha}(\partial_0 g_{\alpha 0}+\partial_0 g_{\alpha 0}-\partial_\alpha g_{00}) \\ \Gamma^i_{00} &= \frac{1}{2}\eta^{i\alpha}(\partial_0 h_{\alpha 0}+\partial_0 h_{\alpha 0}-\partial_\alpha h_{00}) \\ \Gamma^i_{00} &= \frac{1}{2}\eta^{ii}(\partial_0 h_{i0}+\partial_0 h_{i0}-\partial_i h_{00}) \\ \Gamma^i_{00} &= \frac{1}{2}(-1)(-\partial_i h_{00}) = \frac{1}{2}\partial_i h_{00} \\ \frac{1}{2}\partial_i h_{00} &= \frac{1}{c^2}\partial_i\phi \end{aligned}$$

or

$$\begin{aligned} \frac{1}{2}h_{00} &= \frac{1}{c^2}\phi \\ h_{00} &= \frac{2\phi}{c^2} \text{ or } \phi = \frac{-GM}{r} \\ h_{00} &= h_{tt} = \frac{-2GM}{c^2 r} \qquad \text{(no change in the spherical metric)} \end{aligned}$$

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{k}{r} & 0 & 0 & 0 \\ 0 & -(1-\frac{k}{r})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(\sin\theta)^2 \end{pmatrix} \tag{31}$$

$$\begin{aligned} g_{00} &= \eta_{00}+h_{00} \\ &= 1-\frac{2GM}{c^2 r} \end{aligned}$$

so we get :

$$k = \frac{2GM}{c^2}$$

or

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{2GM}{c^2} & 0 & 0 & 0 \\ 0 & -(1 - \frac{2GM}{c^2 r})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(\sin\theta)^2 \end{pmatrix} \tag{32}$$

3 Linearized theory of gravity

As a first step towards the gravitational wave solution, we wish to study how the Einstein's field equations expand around the flat spacetime metric. We consider therefore a flat background with the Minkowski metric $\eta_{\mu\nu}$ with a small perturbation $h_{\mu\nu}$ with $\mid h_{\mu\nu} \mid \ll 1$ We can write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{33}$$

From this arises the linearized theory. Its basic interpretation is the following: the sources of perturbations are assumed to move on "classical" trajectories, i.e. on flat spacetime metric $\eta_{\mu\nu}$ along paths determined by their mutual influence e.g. the description of a binary system would be done using Newtonian dynamics, rather than full General Relativity. On the otherhand, the response of a test mass to these perturbations is assumed to happen on a general relativistic metric, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and where we only consider terms up to first order in $h_{\mu\nu}$ when evaluating the Christoffel symbols or the Riemann tensor. To compute the linearized field equations we evaluate one by one these tensors at linear order, i.e. neglecting terms of the order (h^2) .

3.1 Linearizd Einstein tensor

Recall that in linearized theory we raise and lower indices with the flat metric $\eta_{\mu\nu}$. The linearized Christoffel symbols are

$$\Gamma_{\rho\mu\nu} = \frac{1}{2}\eta^{\sigma}_{\rho}\left(\partial_\nu h_{\mu\sigma} + \partial_\mu h_{\nu\sigma} - \partial_\sigma h_{\mu\nu}\right) \tag{34}$$

while the linearized Riemann tensor becomes

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_{\mu\rho} h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}\right) \tag{35}$$

For the field equations one still needs to evaluate the linearized Ricci tensor and scalar curvature. We define therefore the flat space d'Alembert operator
$\Box = \partial_\mu\partial^\mu = \eta_{\mu\nu}\partial^\mu\partial^\nu$ and the trace of $h_{\mu\nu}$ as

$$h = \eta^{\mu\nu} h_{\mu\nu} \tag{36}$$

Contracting the linearized Riemann tensor one gets then the Ricci tensor

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu=} = \frac{1}{2}\left(\partial^\rho\partial_\mu h_{\rho\nu} + \partial^\rho\partial_\nu h_{\rho\mu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) \tag{37}$$

and the scalar curvature

$$R = R^{\mu}_{\mu} = \partial^{\rho}\partial_{\mu}h_{\rho\mu} - \Box h \tag{38}$$

In order to write the field equations in a compact way, we define the trace-reversed metric

In weak-field limit of general relativity, the metric is almost Minkowskian

$$g_{uv} = \eta_{uv} + h_{uv} \tag{39}$$

where $|h_{uv}| \ll 1$ everywhere in spacetime.

The linearized Christoffel symbols can be found from the equation (7):

$$\begin{aligned}
\Gamma^{\lambda}_{\ uv} &= \frac{1}{2}g^{\lambda\rho}\left[g_{\rho u,v} + g_{\rho v,u} - g_{uv,\rho}\right] && (40)\\
&= \frac{1}{2}\left(\eta^{\lambda\rho} + h^{\lambda\rho}\right)\left[\left(\eta_{\rho u,v} + h_{\rho u,v}\right) + \left(\eta_{\rho v,u} + h_{\rho v,u}\right) - \left(\eta_{uv,\rho} + h_{uv,\rho}\right)\right] && (41)\\
&= \frac{1}{2}\left(\eta^{\lambda\rho} + h^{\lambda\rho}\right)\left[\eta_{\rho u,v} + h_{\rho u,v} + \eta_{\rho v,u} + h_{\rho v,u} - \eta_{uv,\rho} - h_{uv,\rho}\right] && (42)\\
&= \frac{1}{2}\left(\eta^{\lambda\rho} + h^{\lambda\rho}\right)\left[h_{\rho u,v} + h_{\rho v,u} - h_{uv,\rho}\right] && (43)\\
&= \frac{1}{2}\eta^{\lambda\rho}\left[h_{\rho u,v} + h_{\rho v,u} - h_{uv,\rho}\right] && (44)
\end{aligned}$$

$$\Gamma^{\lambda}_{\ uv} = \frac{1}{2}\eta^{\lambda\rho}\left[h_{\rho u,v} + h_{\rho v,u} - h_{uv,\rho}\right]$$

Riemann curvature tensor can derived from equation (8)

$$R^{k}_{\ \lambda uv} = \Gamma^{k}_{\ \lambda v,u} - \Gamma^{k}_{\ \lambda u,v} + \Gamma^{k}_{\ \rho u}\Gamma^{\rho}_{\ \lambda v} - \Gamma^{k}_{\ \rho v}\Gamma^{\rho}_{\ \lambda u} \tag{45}$$

we use

$$\begin{aligned}
\Gamma^{k}_{\ \lambda v} &= \frac{1}{2}\eta^{k\rho}\left[h_{\rho\lambda,v} + h_{\rho v,\lambda} - h_{\lambda v,\rho}\right] && (46)\\
\Gamma^{k}_{\ \lambda u} &= \frac{1}{2}\eta^{k\rho}\left[h_{\rho\lambda,u} + h_{\rho u,\lambda} - h_{\lambda u,\rho}\right] && (47)\\
\Gamma^{k}_{\ \rho u} &= \frac{1}{2}\eta^{km}\left[h_{m\rho,u} + h_{mu,\rho} - h_{\rho u,m}\right] && (48)\\
\Gamma^{\rho}_{\ \lambda v} &= \frac{1}{2}\eta^{\rho m}\left[h_{m\lambda,v} + h_{mv,\lambda} - h_{\lambda v,m}\right] && (49)\\
\Gamma^{k}_{\ \rho v} &= \frac{1}{2}\eta^{km}\left[h_{m\rho,v} + h_{mv,\rho} - h_{\rho v,m}\right] && (50)\\
\Gamma^{\rho}_{\ \lambda u} &= \frac{1}{2}\eta^{\rho m}\left[h_{m\lambda,u} + h_{mu,\lambda} - h_{\lambda u,m}\right] && (51)
\end{aligned}$$

we get

$$R^{k}{}_{\lambda uv} = \left\{ \begin{array}{c} \Gamma^{k}{}_{\lambda v,u} \\ -\Gamma^{k}{}_{\lambda u,v} \\ +\Gamma^{k}{}_{\rho u}\Gamma^{\rho}{}_{\lambda v} \\ -\Gamma^{k}{}_{\rho v}\Gamma^{\rho}{}_{\lambda u} \end{array} \right\} \tag{52}$$

$$= \frac{1}{2}\left\{ \begin{array}{c} \eta^{k\rho}\left[h_{\rho\lambda,vu}+h_{\rho v,\lambda u}-h_{\lambda v,\rho u}\right] \\ -\eta^{k\rho}\left[h_{\rho\lambda,uv}+h_{\rho u,\lambda v}-h_{\lambda u,\rho v}\right] \\ +\frac{1}{2}\eta^{km}\left[h_{m\rho,u}+h_{mu,\rho}-h_{\rho u,m}\right]\eta^{\rho m}\left[h_{m\lambda,v}+h_{mv,\lambda}-h_{\lambda v,m}\right] \\ -\frac{1}{2}\eta^{km}\left[h_{m\rho,v}+h_{mv,\rho}-h_{\rho v,m}\right]\eta^{\rho m}\left[h_{m\lambda,u}+h_{mu,\lambda}-h_{\lambda u,m}\right] \end{array} \right\} \tag{53}$$

so,

$$R_{\ k\lambda uv} = \frac{1}{2}\left\{ \begin{array}{c} \eta_{k}{}^{\rho}\left[h_{\rho\lambda,vu}+h_{\rho v,\lambda u}-h_{\lambda v,\rho u}\right] \\ -\eta_{k}{}^{\rho}\left[h_{\rho\lambda,uv}+h_{\rho u,\lambda v}-h_{\lambda u,\rho v}\right] \\ +\frac{1}{2}\eta_{k}{}^{m}\left[h_{m\rho,u}+h_{mu,\rho}-h_{\rho u,m}\right]\eta^{\rho m}\left[h_{m\lambda,v}+h_{mv,\lambda}-h_{\lambda v,m}\right] \\ -\frac{1}{2}\eta_{k}{}^{m}\left[h_{m\rho,v}+h_{mv,\rho}-h_{\rho v,m}\right]\eta^{\rho m}\left[h_{m\lambda,u}+h_{mu,\lambda}-h_{\lambda u,m}\right] \end{array} \right\}$$

Dropping higher order terms of h_{uv}, we have the Riemann curvature tensor

$$R_{\ k\lambda uv} = \frac{1}{2}\left\{\eta_{k}{}^{\rho}\left[h_{\rho\lambda,vu}+h_{\rho v,\lambda u}-h_{\lambda v,\rho u}\right]-\eta_{k}{}^{\rho}\left[h_{\rho\lambda,uv}+h_{\rho u,\lambda v}-h_{\lambda u,\rho v}\right]\right\} \tag{54}$$

$$= \frac{1}{2}\left\{\left[h_{k\lambda,vu}+h_{kv,\lambda u}-h_{\lambda v,ku}\right]-\left[h_{k\lambda,uv}+h_{ku,\lambda v}-h_{\lambda u,kv}\right]\right\} \tag{55}$$

$$= \frac{1}{2}\left\{h_{k\lambda,vu}+h_{kv,\lambda u}-h_{\lambda v,ku}-h_{k\lambda,uv}-h_{ku,\lambda v}+h_{\lambda u,kv}\right\} \tag{56}$$

where we used the relation $\eta^{uv}A_v = A^u \rightarrow \eta_{mu}\eta^{uv} = \eta_m{}^v \rightarrow \eta_m{}^v A_v = \eta_{mu}\eta^{uv}A_v = A_m$

and the Riemann tensor becomes

$$R_{\ k\lambda uv} = \frac{1}{2}\left(h_{k\lambda,vu}+h_{kv,\lambda u}-h_{\lambda v,ku}-h_{k\lambda,uv}-h_{ku,\lambda v}+h_{\lambda u,kv}\right) \tag{57}$$

$$\tag{58}$$

and the Ricci tensor :

$$\rightarrow \quad R_{\lambda v} = R^{u}{}_{\lambda uv} \tag{59}$$

$$= \eta^{ku}R_{\ k\lambda uv} \tag{60}$$

$$= \frac{1}{2}\eta^{ku}\left(h_{k\lambda,vu}+h_{kv,\lambda u}-h_{\lambda v,ku}-h_{k\lambda,uv}-h_{ku,\lambda v}+h_{\lambda u,kv}\right) \tag{61}$$

$$R_{\lambda v} = \frac{1}{2}\left(\eta^{ku}h_{k\lambda,vu}+\eta^{ku}h_{kv,\lambda u}-\eta^{ku}h_{\lambda v,ku}-\eta^{ku}h_{k\lambda,uv}-\eta^{ku}h_{ku,\lambda v}+\eta^{ku}h_{\lambda u,kv}\right) \tag{62}$$

$$= \frac{1}{2}\left(\partial_u\partial_v h^{u}{}_{\lambda}+\partial_u\partial_\lambda h^{u}{}_{v}-\Box h_{\lambda v}-\partial_u\partial_v h^{u}{}_{\lambda}-\partial_\lambda\partial_v h+\partial_k\partial_v h_{\lambda}{}^{k}\right) \tag{63}$$

where $\Box = \partial_u\partial^u$ and $h = h^u_u$ is the trace.

So, the Ricci scalar will takes the form

$$\begin{aligned} R &= R^{v}{}_{v} && (64)\\ &= \partial_u\partial_v h^{uv} - \Box h && (65)\end{aligned}$$

If we define the trace reversed field tensor

$$\bar{h}_{uv} = h_{uv} - \frac{1}{2}h\eta_{uv} \tag{66}$$

the resultant Einstein tensor will be

$$G^{(1)}_{uv} = \frac{1}{2}\left[\partial^\alpha\partial_v\bar{h}_{\alpha u} + \partial^\rho\partial_\mu\bar{h}_{\rho\nu} - \Box\bar{h}_{\mu\nu} - \eta_{uv}\partial^a\partial^b\bar{h}_{ab}\right] \tag{67}$$

or

$$G^{(1)}_{uv} = \frac{1}{2}\left[\partial^\alpha\partial_v h_{\alpha u} + \partial_\mu\partial^\rho h_{\rho\nu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{uv}\partial^a\partial^b h_{ab} + \eta_{uv}\Box h\right] \tag{68}$$

$$\begin{aligned} G^{(1)}_{uv} &= R^{(1)}_{uv} - \frac{1}{2}R^{(1)}\eta_{uv} = \frac{1}{2}\left(\partial_\alpha\partial_\nu h_\mu{}^\alpha + \partial_\mu\partial_\rho h^\rho{}_\nu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right) - \frac{1}{2}\left(\partial_a\partial_b h^{ab} - \Box h\right)\eta_{uv}\\
&= \frac{1}{2}\left[\partial^\alpha\partial_v h_{\alpha u} + \partial_\mu\partial^\rho h_{\rho\nu} - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h - \eta_{uv}\partial^a\partial^b h_{ab} + \eta_{uv}\Box h\right]\\
&= \frac{1}{2}\left[\partial^\alpha\partial_v\left(\bar{h}_{\alpha u} + \frac{1}{2}\eta_{\alpha u}h\right) + \partial_\mu\partial^\rho\left(\bar{h}_{\rho\nu} + \frac{1}{2}\eta_{\rho\nu}h\right) - \Box\left(\bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}h\right) - \partial_\mu\partial_\nu h - \eta_{uv}\partial^a\partial^b\left(\bar{h}_{ab} + \frac{1}{2}\eta_{ab}h\right)\right.\\
&= \frac{1}{2}\left[\begin{array}{c}\partial^\alpha\partial_v\bar{h}_{\alpha u} + \frac{1}{2}\partial^\alpha\partial_v\eta_{\alpha u}h + \partial_\mu\partial^\rho\bar{h}_{\rho\nu} + \frac{1}{2}\partial_\mu\partial^\rho\eta_{\rho\nu}h - \Box\bar{h}_{\mu\nu}\\ -\frac{1}{2}\eta_{\mu\nu}\Box h - \partial_\mu\partial_\nu h - \eta_{uv}\partial^a\partial^b\bar{h}_{ab} - \frac{1}{2}\eta_{uv}\partial^a\partial^b\eta_{ab}h + \eta_{uv}\Box h\end{array}\right]\\
&= \frac{1}{2}\left[\begin{array}{c}\partial^\alpha\partial_v\bar{h}_{\alpha u} + \underbrace{\frac{1}{2}\partial^\alpha\partial_v\eta_{\alpha u}h + \frac{1}{2}\partial_\mu\partial^\rho\eta_{\rho\nu}h - \partial_\mu\partial_\nu h}_{=0} + \partial_\mu\partial^\rho\bar{h}_{\rho\nu}\\ -\Box\bar{h}_{\mu\nu} - \eta_{uv}\partial^a\partial^b\bar{h}_{ab}\underbrace{-\frac{1}{2}\eta_{\mu\nu}\Box h - \frac{1}{2}\eta_{uv}\partial^a\partial^b\eta_{ab}h + \eta_{uv}\Box h}_{=0}\end{array}\right]\\
&= \frac{1}{2}\left[\partial^\alpha\partial_v\bar{h}_{\alpha u} + \partial^\rho\partial_\mu\bar{h}_{\rho\nu} - \Box\bar{h}_{\mu\nu} - \eta_{uv}\partial^a\partial^b\bar{h}_{ab}\right]\end{aligned}$$

The linearized Einstein equations are then

$$\begin{aligned} G_{uv} &= -\frac{8\pi G_N}{c^4}T_{uv} && (75)\\
\partial^2 h_{uv} - \left(\partial_u\partial^\alpha h_{\alpha v} + \partial_v\partial^\alpha h_{\alpha u}\right) + \partial_u\partial_v h + \eta_{uv}\partial_\rho\partial_\sigma h^{\rho\sigma} - \eta_{uv}\Box h &= -16\pi G T_{uv} && (76)\end{aligned}$$

and in terms of the trace reversed field tensor:

$$\partial^\alpha\partial_{(u}\bar{h}_{v)\alpha} - \frac{1}{2}\left(\partial_\alpha\partial^\alpha\bar{h}_{uv} + \eta_{uv}\partial^\alpha\partial^\beta\bar{h}_{\alpha\beta}\right) = 8\pi T_{uv} \tag{77}$$

$$\partial^2\bar{h}_{uv} - \left(\partial_u\partial^\rho\bar{h}_{\rho v} + \partial_v\partial^\rho\bar{h}_{\rho u}\right) + \eta_{uv}\partial_\rho\partial_\sigma\bar{h}^{\rho\sigma} = -16\pi G T_{uv}c^4 \tag{78}$$

These equations have an infinite dimensional kernel consisting of fields of the form

$$h_{uv} = \partial_u \epsilon_v + \partial_v \epsilon_u \tag{79}$$

or equivalently

$$\bar{h}_{uv} = \partial_u \epsilon_v + \partial_v \epsilon_u - \eta_{uv} \partial_\rho \epsilon^\rho \tag{80}$$

3.2 Wave equation in the Lorentz gauge

Lorentz gauge (also known as the harmonic gauge) condition holds:

$$\partial^u \bar{h}_{uv} = 0 \tag{81}$$

with the trace reversed field tensor

$$\bar{h}_{uv} = h_{uv} - \frac{1}{2} h \eta_{uv} \tag{82}$$

then

$$\partial^u h_{uv} = \frac{1}{2} \partial_v h \tag{83}$$

whitch implyies

$$R^{(1)}_{uv} = -\frac{1}{2} \Box h_{uv} \tag{84}$$

$$R^{(1)} = -\frac{1}{2} \Box h \tag{85}$$

thus turning the linearized Einstein equation into the form of a standard wave equation:

$$\Box \bar{h}_{uv} = \left(\frac{16 \pi G_N}{c^4} \right) T^{(0)}_{uv} \tag{86}$$

From the field equations in the weak-field limit, eqns.(86), we obtain the linearised field equations in vacuum,

$$\Box \bar{h}_{uv} = 0$$

whitch are similar to the wave equations in electromagnetism. These equations admit the plane-wave solutions,

$$\bar{h}_{uv} = \varepsilon_{uv} e^{i k_\alpha x^\alpha}$$

where, ε_{uv} is a constant, symmetric, rank-2 tensor (polarization tensor) and k_α is a constant four-vector known as the wave vector.

4 Plane waves solutions in the tracless transverse gauge(TT)

In order to fully exploit gauge freedom and study the spread of WG and their interaction with the test masses, we consider that these $\overline{h_{\mu\nu}}$ waves are already generated and far from the source, i.e: $T_{\mu\nu} = 0$. and the linearized Einstein equations take the forme a homogeneous wave equation

$$\Box \overline{h}_{\mu\nu} = 0 \tag{87}$$

which admits a class of homogeneous solutions which are superpositions of monochromatic plane waves

$$\overline{h}_{\mu\nu} = \int d^3 \overrightarrow{k}\, A_{\mu\nu}\left(\overrightarrow{k}\right) e^{ik_\alpha x^\alpha} \tag{88}$$

where the integral is taken over all values of $\vec{k}$. Physical solutions are obtained by taking the real part of ϕ^{uv}.

$$\begin{aligned} \phi^{uv} &= \varepsilon^{uv} \cos k_\alpha x^\alpha & (89)\\ &= \mathrm{Re}\left[\varepsilon^{uv} \exp\left(ik_\alpha x^\alpha\right)\right] = \varepsilon^{uv} \exp\left(ik_\alpha x^\alpha\right) & (90) \end{aligned}$$

where the ε^{uv} are constant (and, in general, complex) components of a symmetric tensor and k_u are the constant (real) components of a vector.

Physical solutions corresponding to propagating plane gravitational waves in empty space may be obtained by taking the real part

$$\phi^{uv} = \mathrm{Re}\left[\varepsilon^{uv} \exp\left(ik_\alpha x^\alpha\right)\right] = \frac{1}{2}\varepsilon^{uv} e^{ik_\alpha x^\alpha} + \frac{1}{2}\left(\varepsilon^{uv}\right)^* e^{-ik_\alpha x^\alpha} \tag{91}$$

which is clearly just a superposition of two plane waves.

The Lorenz gauge condition is satisfied provided that the additional constraint

$$\begin{aligned} \partial_u \phi^{uv} &= 0 & (92)\\ &\rightarrow \quad \varepsilon^{uv} k_u = 0 & (93) \end{aligned}$$

the wave equ also lead to

$$\begin{aligned} \partial_\lambda \partial^\lambda \phi^{uv} &= 0 & (94)\\ &\rightarrow \quad k_\alpha k^\alpha = 0 & (95) \end{aligned}$$

The nullity of k implies that

$$\begin{aligned} &\rightarrow \quad k_0 = \frac{\omega}{c} = \left(k_x^2 + k_y^2 + k_z^2\right)^{\frac{1}{2}} = |\mathbf{k}| & (96)\\ &\rightarrow \quad \frac{\omega}{|\mathbf{k}|} = 1 & (97) \end{aligned}$$

so both the group and phase velocity of a gravitational wave are equal to the speed of light.

Since ε^{uv} is symmetric, the amplitude tensor has 10 different (complex) components, but the four Lorenz gauge conditions ($\varepsilon^{uv}k_u = 0$) reduce the number of independent components to 6. Moreover, we still have the freedom to make a further gauge transformation of the form

$$h'_{uv} = h_{uv} - \xi_{u;v} - \xi_{v;u} \tag{98}$$

which will preserve the Lorenz gauge provided that we choose the four functions $\xi^u(x)$ so that they satisfy $\Box^2\xi^u = 0$. This may be used to reduce the number of independent components in the amplitude matrix from 6 to just 2. This results in two possible polarisations for plane gravitational waves.

It is convenient to consider the concrete example of a plane gravitational wave propagating in the z-direction, in which case the components of the 4-wavevector are

$$\begin{aligned} k^\alpha &= \left(\frac{\omega}{c}, 0, 0, \frac{\omega}{c}\right) && (99)\\ &= (k, 0, 0, k) && (100)\end{aligned}$$

Now, the Lorenz (Hilbert) gauge condition ($\varepsilon^{uv}k_u = 0$) then immediately gives

$$\begin{aligned} \varepsilon^{uv}k_u &= 0 && (101)\\ &= \varepsilon^{0v}k_0 + \varepsilon^{1v}k_1 + \varepsilon^{2v}k_2 + \varepsilon^{3v}k_3 && (102)\\ &= \varepsilon^{0v}k_0 + \varepsilon^{3v}k_3 = \left(\varepsilon^{0v} + \varepsilon^{3v}\right)k = 0 && (103)\\ \rightarrow\ \varepsilon^{0v} = -\varepsilon^{3v} \rightarrow & \begin{array}{r} \varepsilon^{00} = -\varepsilon^{30} \\ \varepsilon^{01} = -\varepsilon^{31} \\ \varepsilon^{02} = -\varepsilon^{32} \\ \varepsilon^{03} = -\varepsilon^{33} = -\varepsilon^{00} \end{array} && (104)\end{aligned}$$

and the traceless condition gives

$$\begin{aligned} \varepsilon^u{}_u &= 0 \rightarrow -\varepsilon_{00} + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0 && (105)\\ &\rightarrow -\varepsilon_{00} + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{00} = 0 \rightarrow \varepsilon_{11} + \varepsilon_{22} = 0 && (106)\\ &\rightarrow \varepsilon_{11} = -\varepsilon_{22} && (107)\end{aligned}$$

$$\varepsilon^{uv} = \begin{pmatrix} \varepsilon^{00} & \varepsilon^{01} & \varepsilon^{02} & \varepsilon^{03} \\ \varepsilon^{10} & \varepsilon^{11} & \varepsilon^{12} & \varepsilon^{13} \\ \varepsilon^{20} & \varepsilon^{21} & \varepsilon^{22} & \varepsilon^{23} \\ \varepsilon^{30} & \varepsilon^{31} & \varepsilon^{32} & \varepsilon^{33} \end{pmatrix} \underset{\varepsilon^{uv}=\varepsilon^{vu}}{\rightarrow} \begin{pmatrix} \varepsilon^{00} & \varepsilon^{01} & \varepsilon^{02} & \varepsilon^{03} \\ \varepsilon^{01} & \varepsilon^{11} & \varepsilon^{12} & \varepsilon^{13} \\ \varepsilon^{02} & \varepsilon^{12} & \varepsilon^{22} & \varepsilon^{23} \\ \varepsilon^{03} & \varepsilon^{13} & \varepsilon^{23} & \varepsilon^{33} \end{pmatrix} \tag{108}$$

$$\underset{\varepsilon^{uv}k_u=0}{\rightarrow} \begin{pmatrix} \varepsilon^{00} & \varepsilon^{01} & \varepsilon^{02} & \varepsilon^{00} \\ \varepsilon^{01} & \varepsilon^{11} & \varepsilon^{12} & -\varepsilon^{01} \\ \varepsilon^{02} & \varepsilon^{12} & \varepsilon^{22} & -\varepsilon^{02} \\ \varepsilon^{00} & -\varepsilon^{01} & -\varepsilon^{02} & \varepsilon^{00} \end{pmatrix} \underset{\varepsilon^u{}_u=0}{\rightarrow} \begin{pmatrix} \varepsilon^{00} & \varepsilon^{01} & \varepsilon^{02} & \varepsilon^{00} \\ \varepsilon^{01} & \varepsilon^{11} & \varepsilon^{12} & -\varepsilon^{01} \\ \varepsilon^{02} & \varepsilon^{12} & -\varepsilon^{11} & -\varepsilon^{02} \\ \varepsilon^{00} & -\varepsilon^{01} & -\varepsilon^{02} & \varepsilon^{00} \end{pmatrix} \tag{109}$$

We may now perform a gauge transformation of the form

$$h'_{uv} = h_{uv} - \xi_{u;v} - \xi_{v;u} \tag{110}$$

to simplify the amplitude tensor still further. To preserve the Lorenz gauge condition we must ensure that

$$\begin{aligned} \partial^u h'_{uv} &= \partial^u h_{uv} - \partial_v \partial^u \xi_u - \partial^u \partial_u \xi_v && (111) \\ &\rightarrow \partial^u \partial_u \xi_v = 0 && (112) \end{aligned}$$

A suitable transformation, which satisfies this condition, is given by

$$\xi^u = \epsilon^u e^{ik_v x^v} \tag{113}$$

where the ϵ^u are constants. Substituting this expression into the transformation law

$$\begin{aligned} h'_{uv} &= h_{uv} - (\epsilon_{u;v} + \epsilon_{v;u}) && (114) \\ & && (115) \end{aligned}$$

Taking $\overrightarrow{u} = c\overrightarrow{e_t} \Rightarrow u^\mu = (c; 0; 0; 0)$

and take $\mathrm{K}^\mu = (\frac{\omega}{c}; 0; 0; \frac{\omega}{c})$ wave travlling the (oz) direction, the general form of the polarization tensor has 16 dégres of freedom

$$\begin{pmatrix} A_{tt} & A_{tx} & A_{ty} & A_{tz} \\ A_{xt} & A_{xx} & A_{xy} & A_{xz} \\ A_{yt} & A_{yx} & A_{yy} & A_{yz} \\ A_{zt} & A_{zx} & A_{zy} & A_{zz} \end{pmatrix}$$

in the other hand the metric is symmetric ($g_{uv} = g_{vu}$), so we have

$$\begin{pmatrix} A_{tt} & A_{xt} & A_{yt} & A_{zt} \\ A_{tx} & A_{xx} & A_{yx} & A_{zx} \\ A_{ty} & A_{xy} & A_{yy} & A_{zy} \\ A_{tz} & A_{xz} & A_{yz} & A_{zz} \end{pmatrix} \tag{116}$$

then we have 10 degres of freedom. Now, using the transversalty condition $A_{\mu\nu}k^\mu = 0$ with $\mathrm{K}^\mu = (\frac{\omega}{c}; 0; 0; \frac{\omega}{c})$ gives

$$\begin{aligned} A_{\mu\nu}k^\mu &= 0 \rightarrow A_{\mu\nu}k^\mu = A_{0\nu}k^0 + A_{1\nu}k^1 + A_{2\nu}k^2 + A_{3\nu}k^3 = A_{0\nu}k^0 + A_{3\nu}k^3 = 0 && (117) \\ &\rightarrow (A_{0\nu} + A_{3\nu})\, k = 0 \rightarrow A_{0\nu} = -A_{3\nu} \rightarrow \begin{matrix} A_{00} = -A_{30} \\ A_{01} = -A_{31} \\ A_{02} = -A_{32} \\ A_{03} = -A_{33} \end{matrix} \rightarrow \begin{matrix} A_{00} = -A_{30} = A_{33} \\ A_{01} = -A_{31} \\ A_{02} = -A_{32} \\ A_{03} = -A_{33} \end{matrix} && (118) \end{aligned}$$

so, we get

$$\begin{aligned} A^{uv} &= \begin{pmatrix} A_{tt} & A_{xt} & A_{yt} & A_{zt} \\ A_{tx} & A_{xx} & A_{yx} & A_{zx} \\ A_{ty} & A_{xy} & A_{yy} & A_{zy} \\ -A_{tt} & -A_{xt} & -A_{yt} & A_{tt} \end{pmatrix} && (119) \\ A^u{}_u &= A^{uv}\eta_{vu} = -A_{tt} + A_{xx} + A_{yy} + A_{tt} = 0 \rightarrow A_{xx} = -A_{yy} && (120) \end{aligned}$$

$$A^{uv} = \begin{pmatrix} A_{tt} & A_{xt} & A_{yt} & A_{zt} \\ A_{tx} & A_{xx} & A_{yx} & A_{zx} \\ A_{ty} & A_{xy} & -A_{xx} & A_{zy} \\ -A_{tt} & -A_{xt} & -A_{yt} & A_{tt} \end{pmatrix} \tag{121}$$

gives $\mathrm{A}_{t\nu}u^t = A_{t\nu}c = 0 \;\Rightarrow A_{\nu t} = A_{t\nu} = 0$ so all the compenent with t goes to 0

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & A_{xz} \\ 0 & A_{yx} & A_{yy} & A_{yz} \\ 0 & A_{zx} & A_{zy} & A_{zz} \end{pmatrix}$$

using $A_{\mu\nu}k^\mu = 0$ with $\mathrm{K}^\mu = (\frac{\omega}{c}; 0; 0; \frac{\omega}{c})$ gives $A_{t\nu}k^t + A_{z\nu}k^z = 0$

already know $A_{t\nu} = 0 \Rightarrow 0(\frac{\omega}{c}) + A_{z\nu}(\frac{\omega}{c}) = 0 \Rightarrow A_{z\nu} = A_{\nu z} = 0$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{yx} & A_{yy} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

since the trace of $A^\mu_\mu = 0$

$$\begin{aligned} A^\mu_\mu &= \eta^{\mu\nu}A^\mu_\nu \\ \eta^{xx}A_{xx} + \eta^{yy}A_{yy} &= 0 \\ -A_{xx} - A_{yy} &= 0 \end{aligned}$$

gives

$$A_{xx} = -A_{yy}$$

and finally by semmetry

$$A_{xy} = A_{yx}$$

so

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we gonna rename this

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & 0 & 0 \\ 0 & 0 & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A_\times & 0 \\ 0 & A_\times & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

4.1 Interaction of gravitational waves with matter

4.1.1 Interaction of gravitational waves with single particle

Once we described the propagation of gravitational waves, now, we can discuss their interactions with matter, which will help us o to establish an appropriate detecting process. Starting from the geodesic equation

$$\frac{du^\mu}{d\tau} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = 0$$

and we consider a test particle to be at rest, $u^\mu = (1, 0, 0, 0)$, at an initial time, let's say, $\tau = \tau_0$. Evaluating the geodesic equation at this time, we have

$$\frac{du^\mu}{d\tau} + \Gamma^\rho_{00} u^0 u^0 = 0$$

which give us

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu_{00} \mid_{\tau=\tau_0}$$

by using the diffnition of cristoffle symbole (n e) in the linearized gravity we get

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu_{00} = \frac{1}{2}(2\partial_0 h^{TT}_{0i} - \partial_i h^{TT}_{00}) \mid_{\tau=\tau_0} \tag{122}$$

which is identically zero since both h_{0i} and h_{00} are zero in the transverse-traceless gauge (n e). The equation (122) means that the particle does not experience any acceleration, this reflects the fact the Riemannian spacetime is locally flat at any given point.

4.1.2 Interaction of gravitational waves with ring of particles

To detect gravitational waves we must go beyond a single particule and explore its neighborhood. Consider the wave passing through a ring of test particles in x - y plane. Let's denote by v^μ the distance of a test particles to the center of the ring and use the geodesic deviation equation [6]

$$\frac{D^2 v^\mu}{d\tau^2} = \eta^{\mu\lambda} R_{\lambda\nu\rho\sigma} u^\nu u^\rho v^\sigma \tag{123}$$

$$\frac{D^2 v^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho v^\sigma \tag{124}$$

and The linearized Riemanien tensor is given by :

$$R_{\lambda\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho h_{\lambda\sigma} + \partial_\sigma\partial_\lambda h_{\nu\rho} - \partial_\nu\partial_\sigma h_{\lambda\rho} + \partial_\rho\partial_\lambda h_{\nu\sigma}) \tag{125}$$

where

$$R^\rho_{\sigma\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}\left(\partial_\alpha\partial_\nu h_{\sigma\mu} + \partial_\sigma\partial_\mu h_{\alpha\nu} - \partial_\alpha\partial_\mu h_{\sigma\nu} - \partial_\sigma\partial_\nu h_{\alpha\mu}\right) \tag{126}$$

$$R^x_{0x0} = \frac{1}{2}\eta^{xx}\left(\partial_x\partial_0 h_{0x} + \partial_0\partial_x h_{x0} - \partial_x\partial_x h_{00} - \partial_0\partial_0 h_{xx}\right)$$
$$= -\frac{1}{2}\partial_0\partial_0 h_{xx} \quad (127)$$

$$R^y_{0x0} = \frac{1}{2}\eta^{yy}\left(\partial_y\partial_0 h_{0x} + \partial_0\partial_x h_{y0} - \partial_y\partial_x h_{00} - \partial_0\partial_0 h_{yx}\right) \quad (128)$$
$$= -\frac{1}{2}\partial_0\partial_0 h_{yx} \quad (129)$$

$$R^y_{0y0} = \frac{1}{2}\eta^{y\alpha}\left(\partial_\alpha\partial_0 h_{0y} + \partial_0\partial_y h_{\alpha 0} - \partial_\alpha\partial_y h_{00} - \partial_0\partial_0 h_{\alpha y}\right) \quad (130)$$
$$R^y_{0y0} = \frac{1}{2}\eta^{yy}\left(\partial_y\partial_0 h_{0y} + \partial_0\partial_y h_{y0} - \partial_y\partial_y h_{00} - \partial_0\partial_0 h_{yy}\right) \quad (131)$$
$$= -\frac{1}{2}\partial_0\partial_0 h_{yy} \quad (132)$$

is a gauge invariant quantity we can compute it in any frame without affecting the result

Clearly the best choice is the TT gauge since the form of $h_{\mu\nu}$ in this frame is extremely simple. Assuming the particles to be moving slowly, $U^\nu \approx (1, 0, 0, 0)$ so the deviation equation become

$$\frac{D^2 v^\mu}{d\tau^2} = \eta^{i\lambda} R_{\lambda 00\sigma} u^0 u^0 v^\sigma = R^i{}_{00j} v^j$$

using the definition of linearized Riemannian tensor (35), we find

$$\frac{D^2 v^\mu}{d\tau^2} = \frac{1}{2}\frac{d^2 h^{TT}_{ij}}{dt^2} v^j \quad (133)$$

The resulting equation is extremely simple. The response of the particles can be understood in purely Newtonian terms, without any further reference to General Relativity. Since h^{TT}_{ij} is traceless, the effective Newtonian force per unit mass

$$F_i = \frac{1}{2}\frac{d^2 h^{TT}_{ij}}{dt^2} u^j$$

$$\partial^i F_i = 0$$

meaning that there are no sources for the gravitational lines.

also , as in the electromagnetic case, only the transverse directions (v_x and v_y) to the wave propagation are affected

If a particle is initially at z = 0, it will remain at z = 0.

4.2 Gravitational waves' sources:

In general terms, gravitational waves are radiated by objects whose motion involves acceleration and its change, provided that the motion is not perfectly spherically symmetric (like an expanding or contracting sphere) or rotationally symmetric (like a spinning disk or sphere). A simple example of this principle is a spinning dumbbell. If the dumbbell spins around its axis of symmetry, it will not radiate gravitational waves; if it tumbles end over end, as in the case of two planets orbiting each other, it will radiate gravitational waves. The heavier the dumbbell, and the faster it tumbles, the greater is the gravitational radiation it will give off. In an extreme case, such as when the two weights of the dumbbell are massive stars like neutron stars or black holes, orbiting each other quickly, then significant amounts of gravitational radiation would be given off.

4.2.1 Binaries

Gravitational waves carry energy away from their sources and, in the case of orbiting bodies, this is associated with an in-spiral or decrease in orbit [7]. Imagine for example a simple system of two masses – such as the Earth–Sun system – moving slowly compared to the speed of light in circular orbits. Assume that these two masses orbit each other in a circular orbit in the x–y plane. To a good approximation, the masses follow simple Keplerian orbits. However, such an orbit represents a changing quadrupole moment. That is, the system will give off gravitational waves.

In theory, the loss of energy through gravitational radiation could eventually drop the Earth into the Sun. However, the total energy of the Earth orbiting the Sun (kinetic energy + gravitational potential energy) is about 1.14×10^{36} joules of which only 200 watts (joules per second) is lost through gravitational radiation [7], leading to a decay in the orbit by about 1×10^{-15} meters per day or roughly the diameter of a proton. At this rate, it would take the Earth approximately 3×10^{13} times more than the current age of the Universe to spiral onto the Sun. This estimate overlooks the decrease in r over time, but the radius varies only slowly for most of the time and plunges at later stages,[7]

$$r\left(t\right)=r_{0}\left(1-\frac{t}{t_{c}}\right) \tag{134}$$

with t_c the time to fully coelesce,and r_0 is the initial radius. The rate of orbital decay can be approximated by [8]

$$\frac{dr}{dt}=-\frac{64}{5}\frac{G^{3}}{c^{5}}\frac{\left(m_{1}m_{2}\right)\left(m_{1}+m_{2}\right)}{r^{3}} \tag{135}$$

with r is the distance between corps, G the gravitational contant.

This leads to an expected time to merger of

$$t=\frac{5}{256}\frac{c^{5}}{G^{3}}\frac{r^{4}}{\left(m_{1}m_{2}\right)\left(m_{1}+m_{2}\right)} \tag{136}$$

4.2.2 Binary black hole:

A binary black hole is a system consisting of two black holes in close orbit around each other.formed either as remnants of high-mass binary star systems or by dynamic processes and mutual capture; and binary

supermassive black holes, believed to be a result of galactic mergers, during the late 20th and early 21st century, binary black holes became of great interest scientifically as a potential source of such waves and a means by which gravitational waves could be proven to exist. Binary black hole mergers would be one of the strongest known sources of gravitational waves in the universe, The black holes will merge once they are close enough. Once merged, the single hole settles down to a stable form, via a stage called ringdown, where any distortion in the shape is dissipated as more gravitational waves. In the final fraction of a second the black holes can reach extremely high velocity, and the gravitational wave amplitude reaches its peak.

Black hole binaries emit gravitational waves during their in-spiral, merger, and ring-down phases. The largest amplitude of emission occurs during the merger phase, which can be modeled with the techniques of numerical relativity.The first direct detection of gravitational waves, GW150914, came from the merger of two black holes.

LIGO detected GW150914 on September 2015, a distinctive gravitational wave signature of two merging stellar-mass black holes of around 30 solar masses each, GW150914 released around 3 solar masses as gravitational energy, peaking at a rate of 3.6×10^{49}watts, more than the combined power of all light radiated by all the stars in the observable universe put together [10].

4.2.3 Spinning neutron stars:

as we mentiond befor, a mass distribution will emit gravitational radiation only when there is spherically asymmetric motion among the masses. A spinning neutron star will generally emit no gravitational radiation,because neutron stars have huge dense objects with a big gravitational field that keeps them spherical,

there might be teny deformities on the surface called "mountains", which are bumps extending no more than 10 centimeters above the surface,[11] that make the spinning spherically asymmetric. This gives the star a quadrupole moment that changes with time, and it will emit gravitational waves until the deformities are smoothed out.

4.3 Gravitational waves generation:

After solving the equations for plane gravitational waves in vacuum and seeing how they affect test bodies we now turn to understanding some, possible sources of such waves. For this we use the linearized equations,

$$\overline{h}_{\mu\nu}{}^{,\lambda}{}_{,\lambda} = \Box^2 \overline{h}_{\mu\nu} = -\left(\frac{16\pi G}{c^4}\right) T_{\mu\nu} \tag{137}$$

The solution of the previous equation

$$\overline{h}\left(\overrightarrow{x}, t\right) = -\left(\frac{4G}{c^4}\right) \int \frac{1}{r} T\left(\overrightarrow{x}'; t_{ret}\right)_{\mu\nu} d^3x' \tag{138}$$

or $t_{ret} = t - \frac{r}{c}; r = \left|\overrightarrow{x} - \overrightarrow{x}'\right|$

For the small source approximation, in which the source size and characteristic frequency obey $\mu = \nu = 0$,the radiation from all parts of the source is in phase, and the solution far from the source reduces to an integral over the source at a single retarded time,

$$\overline{h}\left(\overrightarrow{x}, t\right) = -\left(\frac{4G}{c^4}\right) \frac{1}{r} \int T\left(\overrightarrow{x}'; t_{ret}\right)_{\mu\nu} d^3x' \tag{139}$$

$r = |\overrightarrow{x}|\,, t_{ret} = t - \frac{r}{c}$

since we treat the enregy metric tensor below

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{140}$$

and the integral of the density is the mass M:

$$\overline{h}_{00} = -\left(\frac{4G}{c^4 r}\right)\int T_{00} d^3x' = \frac{-4GM}{c^2 r} \tag{141}$$

last equation is approximation solution, the metric field from small distance source, and known energy momentum distribution ,it's not convinient for typical sources and secondly it's clearly not in the traceless transverse gauge, for that it's convinient for studying in a detector systeme ,

Diffrential the equation 139and set $\mu = 0$ we get

$$\frac{\partial}{\partial t}\int T^{0\nu} d^3x' = \int T^{0\nu}{}_{;0} d^3x' \tag{142}$$

Consider first $P^\sigma P_\sigma$so the zero divergence of the energy-momentum

$$T^{0\beta}{}_{,\beta} = T^{00}{}_{,0} + T^{0i}{}_{,i} = 0 \rightarrow T^{00}{}_{,0} = -T^{0i}{}_{,i} \tag{143}$$

and with using Gauss's theorem we evaluate142 as a surface integral

$$\frac{\partial}{\partial t}\int T^{00} d^3x' = -\int T^{0k}{}_{;k} d^3x' = -\int T^{0k}\ dS_k \tag{144}$$

but the source is of limited extent by assumption, so we can choose the surface S to be outside the source region and the surface integral is thus zero and does not correspond to gravitational waves

Only the space components of the energy-momentum tensor produce waves

after ignoring the $\mu = 0$ parts of the solution138 we are left with, in obvious abbreviated notation

$$\overline{h}_{ij} = -\left(\frac{4G}{c^4}\right)\frac{1}{r}\int T_{ij} d^3x' \tag{145}$$

There is a wonderful theorem that will let us calculate the integral145 in a simple and physically meaningful way in terms of the quadrupole nature of the source. The theorem is

$$I_{ij}^T = \int \rho(x)\left[r_i r_j - \frac{1}{3}r^2\delta_{ij}\right] d^3r \tag{146}$$

or

$$\int T^{kl} d^3x = \frac{1}{2c^2}\frac{\partial}{\partial t^2}\int T^{00} x^k x^l d^3x \tag{147}$$

As in the previous manipulations the theorem is based on the symmetry and zero divergence of the energy-momentum tensor

$$T^{\mu\nu}{}_{,\nu} = 0 \rightarrow T^{00}{}_{,0} = -T^{0l}{}_{,l}\ , T^{k0}{}_{,0} = -T^{kl}{}_{,l} \tag{148}$$

we sibstue 147 into 145 to get a simple formula for the field

$$\overline{h}_{ij} = -\left(\frac{4G}{c^4 r}\right)\frac{\partial^2}{\partial t^2}\underbrace{\int T^{00} x'^i x'^j d^3x'}_{I_{ij}} \tag{149}$$

where the I_{ij} represent the quadripole moment. The $(0,0)$ component of the energy-momentum tensor is simply the energy density u , so the integral in149 has a clear physical meaning and is easy to calculate. It's called the quadrupole integral.

we thus arrive at the famous quadrupole formula,

$$h_{kl}^{TT}(x,t) = -\frac{2G}{c^4 r}\ddot{I}_{kl}^{TT}(t - r/c) \tag{150}$$

Our final manipulation is to consider the metric field 149 at large distances from the source, where it is asymptotically a plane wave, and to transform it to the traceless transverse gauge we used previously to calculate the motion of particles. There we found that the components of the field with an index equal to 0 or 3 could be transformed away; we have also just shown that these components are constant in time and are not relevant to a wave analysis, so no more need be said about them. It only remains to transform the 1,2 block into traceless form which we already did before

we may write the wave solution components in the new system,

$$h^{11} = -h^{22} = -\left(\frac{G}{c^2 r}\right)\frac{\partial^2}{\partial t^2}\int T^{00}\left(x'^2 - y'^2\right) d^3x' \tag{151}$$

$$h^{12} = -\left(\frac{2G}{c^2 r}\right)\frac{\partial^2}{\partial t^2}\int T^{00}\left(x'.y'\right) d^3x' \tag{152}$$

from this expression, we can see that the wave amplitude is proportional to $\left(\frac{G}{C^2} = 7.414\,0 \times 10^{-28}\right)$, which reflect the weak coupling between GW and masses.

4.4 Energy carried by a gravitational wave:

The amplitude of the gravitational waves that pass the Earth regularly (once a month or more often) is likely to be well below 10^{-21}. The smallness of this number is a measure of the weakness of its interaction with the matter it passes through. This is at once a benefit and a curse because it makes the waves hard to detect, a benefit because one can be sure that the radiation has not been significantly affected by anything it may have passed through since leaving its source. Unlike photons, which are easily scattered and reprocessed, gravitational waves carry their original information intact.

The weakness of the effect of gravitational waves on our detectors should not make us overlook the fact that they carry enormous amounts of energy. Using the Isaacson stress-energy tensor , one can easily calculate that the energy flux in a gravitational wave of amplitude h and frequency f is [12]

$$F_{gw} = 3,2 \times 10^{-3}\left[\frac{f}{1KHZ}\right]\left[\frac{h}{10^{-22}}\right]\ w/m^2 \tag{153}$$

In astronomers' language, a 1 kHz wave with amplitude 10^{-22} is as bright as a star of apparent magnitude -13, which is twice as bright as the full moon! This may last only a millisecond, but since the wave

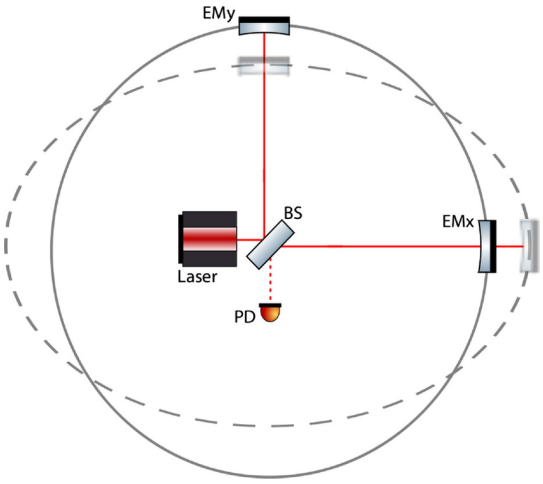
originates in a distant galaxy, it is clear that enormous energies are involved. By integrating previous equation over a sphere of radius r for a time t, we find the relation between the amplitude h and the energy E carried by the wave during t is [12]

$$h = 1.4 \times 10^{-21} \left[\frac{E}{0.01 M_{\odot} C^2}\right]^{\frac{1}{2}} \left[\frac{f}{1KHZ}\right]^{-1} \left[\frac{\tau}{1ms}\right]^{-\frac{1}{2}} \left[\frac{r}{15Mpc}\right] \tag{154}$$

4.5 Detection of gravitational waves

4.6 Detection principle for interferometric detectors

After showing the effect of a gravitational wave on a ring of test masses, now we look for an appropriate apparatus able to detect this waves on the ground, the most known is the Michelson interferometer, within which the two test masses are replaced by the arm-end mirrors, as shown in the next Fig



Michelson interferometer acting G W detector and the effect of polarised wave[8]

The passage of a gravitational wave will change the separating-mirror distances and the optical path difference between the two arms will be measured. For this to work, the elements of the interferometer (mirrors, injection and detection systems) are suspended, we will see how, and behave like masses in free fall in the plane of the interferometer, at least for the frequencie

$$f \gg f_{pend} \tag{155}$$

the resonant frequency of pendulums.

To better understand the principle of detection, let us evaluate the round trip time of the photons in an arm. In the general case, we can write the element of invariant length for a photon in a field:

$$ds^2 = 0 = g_{\alpha\beta} dx^{\alpha} dx^{\beta} = \eta_{\alpha\beta} dx^{\alpha} dx^{\beta} + h_{\alpha\beta} dx^{\alpha} dx^{\beta} \tag{156}$$

To simplify the calculation, consider the special case of a wave propagating along z (perpendicular to the plane of the arms), of "+" polarization. We then have

$$ds^2 = 0 = -c^2dt^2 + (1 + h_+(t))\,dx^2 + (1 - h_-(t))\,dy^2 + dz^2 \tag{157}$$

We evaluate the travel time of the photons by integrating on the path, for example to the arm according to "x":

$$\frac{1}{c}\int_0^l dx = \int_0^{\tau_{travel}} \frac{1}{\sqrt{1 + h_+(t)}} dt \approx \int_0^{\tau_{travel}} \left(1 - \frac{1}{2}h_+(t)\right) dt \tag{158}$$

We must now take into account

– round trip in one arm

– the wavelength of the gravitational wave, which we will assume to be much greater than the arm length. Thus, we can consider that on the path of the photon, the amplitude $h_+(t)$ is independent of the position along the arm

– the period of the gravitational wave, which must be much less than the round trip time light in an arm. Thus, we can consider that $h_+(t) = cte = h_+$.

We can then integrate on the arm according to "x" in two ways:

$$\int_0^{\tau_{round.x}} \left(1 - \frac{1}{2}h_+(t)\right) dt \approx \frac{1}{c}\left(\int_0^l dx - \int_l^0 dx\right) = \frac{2l_x}{c} \tag{159}$$

$$= \tau_{round.x} - \frac{1}{2}\int_0^{round.x} h_+(t)\,dt = \tau_{round.x} - \frac{1}{2}\int_0^{\frac{2l_x}{c}} h_+(t)\,dt \tag{160}$$

which calculates the round trip time

$$\tau_{round.x} = \frac{2l_x}{c} + \frac{1}{2}\int_0^{\frac{2l_x}{c}} h_+(t)\,dt \tag{161}$$

We can do the same calculation for the arm according to "y":

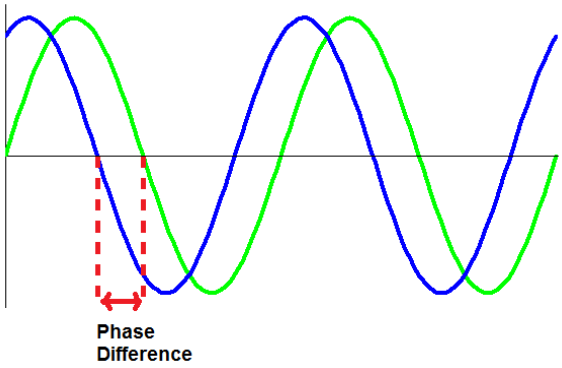
$$\tau_{round.y} = \frac{2l_y}{c} - \frac{1}{2}\int_0^{\frac{2l_y}{c}} h_+(t)\,dt \tag{162}$$

And what interests us is the time difference, where we take into account the constancy of h and where it is assumed that $L_x = L_y = L$

$$\delta\tau_{round} = \frac{1}{2}h_+\left(\frac{2l_x}{c} + \frac{2l_y}{c}\right) = h_+\frac{2l}{c} \tag{163}$$

This time difference can be smaller than the period of the laser light wave.The interferometric device is actually sensitive to the phase difference between the two waves. We consider therefore rather the phase difference accumulated on the path of the photons [5]:

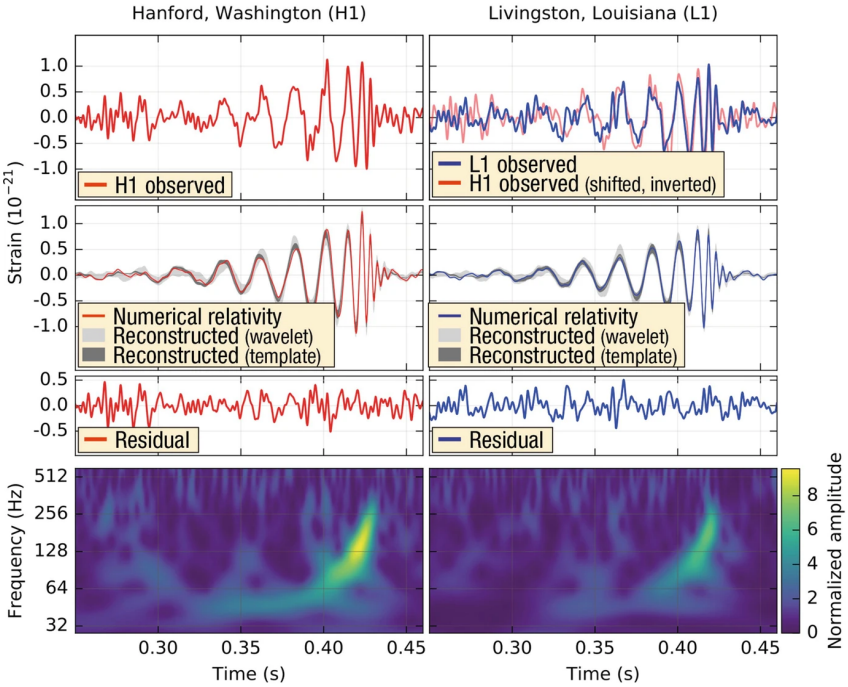
$$\delta\phi = \omega_{laser}\delta\tau_{round} = \frac{4\pi}{\lambda_{laser}}Lh_+ \tag{164}$$



(165)

It is interesting to notice that this phase difference is proportional to h and L. We will therefore seek to produce devices with the greatest possible arm length, giving light time to "feel" the gravitational wave

The chirp of the first gravitational-wave event detected GW150914 is shown in Fig.



the first observation of GW150914[4]

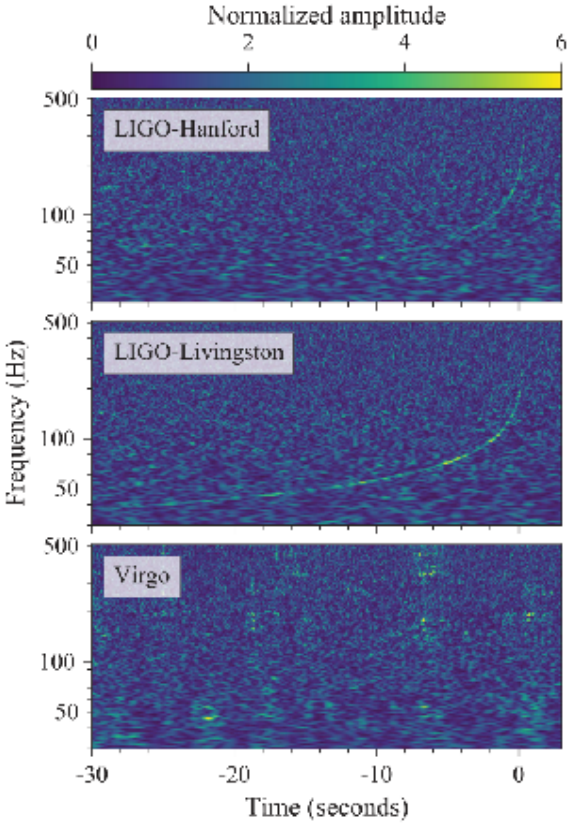
The data of the first gravitational wave, detected by aLIGO on September 14th 2015. The upper panel left is the Hanford signal, while the upper panel right is the Livinston signal shifted by 7 milliseconds, which is the time difference between the detections at the two different sites. Upper middle panel is comparisons of the two signals with a theoretical model predicted by general relativity, lower middle panel residual differences between the signals and the model. The lower panel represent the frequency spectra of the two signals [5].

4.7 GW170817

GW 170817 was a gravitational wave (GW) signal observed by the LIGO and Virgo detectors on 17 August 2017, originating from the shell elliptical galaxy NGC 4993. The signal was produced by the last minutes

of a binary pair of neutron stars' inspiral process, ending with a merger. It is the first GW observation that has been confirmed by non-gravitational means Unlike the five previous GW detections, which were of merging black holes not expected to produce a detectable electromagnetic signal, the aftermath of this merger was also seen by 70 observatories on 7 continents and in space, across the electromagnetic spectrum, marking a significant breakthrough for multi-messenger astronomy

The gravitational wave signal, designated GW 170817, had a duration of approximately 100 seconds, and shows the characteristics in intensity and frequency expected of the inspiral of two neutron stars,



GW170817 signal measured by the LIGO and Virgo[16]

5 Conclusion

The central lesson of general relativity is the fact that mass produces spacetime curvature. As an example, the static spherical mass of the Sun produces outside it a Schwarzschild geometry. Moreover, mass in nonspherical, nonuniform motion can generate gravitational waves, whaich are ripples of curved spacetime, they which propagate at the speed of light, they are transverse waves • have two independent polarizations, and can be detected by their effect on the relative motion of test masses, but their weak coupling to matter makes theme so difficult to be detected.; they also carry energy and have an angular momentume.[14]

detect is also what makes them so interesting astrophysically. Once produced,

There are many important sources of gravitational waves in the univers binary star systems, supernova

explosions, collapse to black holes, and the big bang are all examples. Gravitational waves provide a window for exploring these astronomical phenomena

In this work we derived two important solutions of the gravitational field equations, the shwarzchils metric which describe the field produced by spherically symetric sorces, then we moved to investigate the linearized Einstein field equations, from which we derived the transverse tracless wave solution in order to build a bridge between theory and experiment. We studied the first gravitational wave detection experiment, which is important as a test for general relativity as well as a confirmation of the existence of the black hole binary systems and the coalescence phenomena arising between them.

The Royal Swedish Academy of Sciences has decided to award the Nobel Prize in Physics 2017 with one half to Rainer Weiss, Barry C. Barish and Kip S. Thorne[15], Together they have realised a vision that is almost fifty years old. The 2017 Nobel Laureates have, with their enthusiasm and determination, each been invaluable to the success of LIGO

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