People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

University of Dr. Yahia Fares Medea

Department of Mathematics and Computer Sciences



THESIS

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Specialty: Mathematical Analysis and Applications
For the preparation of:
Master degree

Entitled

EXPLORING TEACHING MATHEMATICAL PROOF AT THE UNIVERSITY LEVEL

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2019-2020

ACKNOWLEDGEMENT

Firstly, I would like to express my sincere appreciation to my advisor Dr. AZROU Nadia for her full support and her great patience for guiding my research for the past months. She gave me the opportunity to understand a brand-new field of research.

I owe my deepest gratitude to my parents, and my family. They could provide such suitable conditions, which allowed me to pursuit my education, they supported me and encouraged me being always patient during all my whole life.

I am also thankful to all students who contributed in accomplishing this dissertation. Moreover, I would like to thank my teachers of Dr. Yahia Fares University of Medea for giving me knowledge and skills through my education, which helped me in writing this dissertation.

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EXPLORING TEACHING MATHEMATICAL PROOF AT THE UNIVERSITY LEVEL

ABSTRACT

Much of research on proofs has concentrated on content, rather than on enhancing students' ability to construct their own proofs (Selden, 1995). The fact that students are coming to the Algerian university without the culture of proof (what proof means, what is the proof structure), and still have difficulties with proof and proving even after several years of studying mathematics and practicing proof, is an important issue, which needs to be examined. In order to know the origin of this problem, we interviewed twenty-one undergraduate students of mathematics from different universities and levels. Results show that what students learn before, in the school, seems not serving them for learning proving and making proofs in the university. Moreover, students who are aware about many of their difficulties with proof and proving feel the necessity to be taught how to make proofs, how proofs are constructed and how to write them.

Key words: proof and proving, writing proof, undergraduate students, teaching of proof.

INTRODUTION

A major reason that proof, proving and understanding the role of reasoning in mathematics have been given increased attention in recent years is because they are fundamental to doing and knowing mathematics. 'What is a mathematical proof?' This question, and variations on it have been debated for some time, and many answers have been proposed' (CadwalladerOlsker, 2011).

A traditional view of many mathematicians and mathematics teachers is that a mathematical proof is a set of logical lines of reasoning that begin with basic or original assumptions known as axioms that would guarantee the conclusion by moving through logical steps (Weber, 2003). Rota describes it as: "a proof of a mathematical theorem is a sequence of steps which leads to the desired conclusion" (Rota, 1997, p. 134), while according to Hersh (1997), "practical mathematical proof is what we do to make each other believe our theorems" (Hersh, 1997, p. 49). Proof is fundamental to mathematics; we do not know whether a mathematical proposition is true or false until we have proved (or disproved) it (CadwalladerOlsker, 2011, p. 34). According to another position, a

proof confirms truth for a mathematician the way experiment or observation does for the natural scientist (Weber, 2003). Such views are usually held by mathematics teachers and are passed along to our students. However, many mathematics educators and some mathematicians believe that proofs are much more than this (Weber, 2003).

Hence, a proof has different definitions in mathematics literature. But it has also different roles and functions and students have many difficulties with proof and proving.

Our motivation for this research is the failure of students in the university with proof and proving and the increasing incapacity of students with proof and proving. University teachers witness every day, students producing incorrect proving processes and unclear written proof texts, mostly in students' exams sheets. Many forms of this phenomenon indicate that something is deficient or rather absent in teaching mathematics, regarding how to make proof. Although proof at Algerian school curriculum is officially present since middle school level, students' difficulties at the university level, indicate as if students have never encountered such activity before. Moreover, many students graduate from the university, even with a master degree without overcoming their difficulties with proof and proving.

It is very common that university teachers teach advanced mathematics, where proof and proving are regular activities, supposing that students know what is proof and how to make proofs, and if it is not the case, teachers believe that students would learn these important activities with time and practice with proving activities. But the number of students in mathematics discipline and particularly their marks and their true mathematics level show a different reality. Our contribution is to examine the teaching of proof and proving at the university level. Our aim is to study how teaching of proof is done at the university level, in order to identify the origin of students' difficulties with proof and proving, but also to examine in which ways students are introduced to proof and proving, before the university, at different school levels and during the university level. This study might help in overcoming some of students' difficulties with proof and proving.

The present dissertation is organized into seven parts, after an introductory part, we have introduced in the second part 'different methods of teaching of proof'

that presented in research, and we presented students' difficulties with proof and proving in the point of view of researchers.

In the third part, we have briefly presented the teaching of proof in the Algerian school as presented in textbooks of the three levels (primary, middle and high school). The fourth part is the description of the experiment, which report in details, the interviews presented to students. The fifth part reports the responses of students, of each level, to the interviews, with some excerpts of students. The following part is the conclusion we made by extracting the main ideas emerged from students' responses. Finally, in the discussion, we have discussed the conclusion regarding the purpose of this study.

LITERATURE REVIEW

Different methods for teaching proof

Teaching how to construct proofs is not very present in research. Much of the research on proof has focused on increasing students' understandings of teacher, textbook proofs and the content of proof, rather than on enhancing students' ability to construct their own proofs and validate them (Selden, 1995). We will present some of the few interesting studies focusing on exploring new methods for teaching proof.

The Moore method

(Also called inquiring based learning), it is one of the teaching methods which reinforced the students' ability of constructing proof and helping them developing their proof-written skills and proof techniques. Moore and proponents of this method believe that the traditional method, where the professor proves theorems and mathematical concepts, and students write down passively what the professor presents on the blackboard without any interaction (asking questions, discussing or sharing ideas), unable to learn a little or nothing about mathematical concepts. In order to solve this problem, the teachers need to give the opportunity to their students to construct proofs by themselves. The following paragraph is a brief explanation of Moore method:

The teacher gives the definition of a mathematical concept, and the student is required to prove a set of propositions about this concept. When the student proves it, he/she presents the proof on the board. Then, if the proof has been completely done, the teacher and fellow students may critique the student's proof, or they ask him/her to explain to them to elucidate his/her arguments and

clarify his/her reasoning. If the proof is not completed yet, someone of his/her fellow students finishes what was done, and so forth until the final text of the proof is clear enough, well organized and convinced for the outside reader. What is critical in this method is that students will never be provided with the actual proof of the concept that has been proved. 'All proofs are generated by the students themselves' (weber, 2003). Beside of constructing a proof, formulating conjectures is another fundamental aspect of the teaching of mathematics. And for that, the researchers have created a space for discussion of ideas and sharing arguments, that space is called "scientific debate". 'The role of the latter is to facilitate the expression of ideas and to allow opposing views to be clearly stated' (Legrand, M. 2001, p. 127). The first step of this debate is similar to Moore's method in presenting mathematical statements to students. What is different in this method that when the teacher proposes a proposition, the students are invited to formulate conjectures, and then they are required to prove if this conjecture is true or not by constructing a proof and giving arguments in order to convince their classmates of the validity of their opinion. Thus, a student in scientific debate should assume the responsibility of his/her conjecture. Eventually, if everyone in the classroom is convinced (including the teacher), then, the teacher passes to another mathematical statement, and so forth.

Consequently, during this type of courses, student will learn much more about math concepts and proof-writing skills. Moreover, the course, which is led by students instead of being led, by the teacher is easier to remember for students than the course which is led by teachers. What matters and is absent from the traditional mathematics course is creativity. Such method makes student more creative and an active participant rather than being a passive spectator.

Marty's introductory proof courses

Marty's introductory proof courses, is another method of teaching mathematics. Marty (1986) believes that a shortcoming of introductory proof is that mathematical educators focus on the meaning of the statements that are proven and on the content rather than on techniques. In this case, Marty created an introductory proof course where she emphasized the proof techniques and she minimized the importance of mathematical content (Weber, 2003). The following summarizes the principles that Marty used in her course:

1. ‘The purpose of the course was to cover a set of proof techniques (e.g., direct proof, proof by contradiction). Each proof technique was focused on one at a time and in great detail’ (Weber, 2003).
2. Students faced each proof technique in variety of mathematical concepts.
3. Students were asked to apply proof techniques in various contexts and to present their proof on the board. The problems the students were asked to solve were basic and did not involve cleverness or tricks.
4. Students were encouraged to use alternative proofs of an already proved statement (Weber, 2003).

In addition, it has been suggested that teachers might first present global overviews of proofs, followed by structured top-down approaches introducing concepts as needed, to avoid the appearance of ‘pulling a rabbit out of a hat’ (Leron, 1983, p.183).

The method of presenting proof that a math teacher follows is not the only factor influencing teaching proof and clarifying ideas to students, but the type of proof is also important. According to Hanna (1990), there are two types of proof, proofs that explain and proofs that prove. The difference between those two types is that; a proof that proves shows only that mathematical claim is true: it provides evidential reasons alone. While a proof that explains, shows why a mathematical claim is true. And it is better to present to students proofs, which explain. Some researchers argue that it is the explanatory power of proofs that makes proofs meaningful to students; so, it is better to teachers to select “proofs that explain” over “proofs that prove only” (Movshovitz-Hada, 1988; Hanna, 1990; Hersh, 1993). “Generic proofs” is another type of proofs, which works at example level but as they are generalizable, they are easier for students to understand (Selden, 1995).

Students’ difficulties with proofs and proving

Although proving is a very important part of advanced mathematics, many studies (Moore, 1994; Harel & Sowder, 1998; Almeida, 2000; Weber, 2001; Selden & Selden, 2003; Baker & Campbell, 2004; Edwards & Ward, 2004; Weber, 2004; Stylianides and Philippou, 2007), indicate that undergraduate students have many difficulties regarding proof and proving, particularly, their conception of proof is deficient and they have difficulties in constructing proofs. Thus, understanding what constitutes a proof plays a pivotal role in the ability to construct proofs in advanced mathematics (cited in Sari, 2008).

Moore (1994), found seven main sources of students' difficulties in constructing proofs:

1. The students did not know the definitions, that is, they were unable to state the definitions: Because most of the proofs throughout the course depended largely on the definitions, therefore, not knowing the definitions was often a reason for the students' failure to produce a proof.

2. The students had little intuitive understanding of the concepts. They are unable of doing a proof because they do not understanding the concepts involved, but it is important to have intuitive understanding before they could get started.

3. The students' concept images were inadequate for doing the proofs. They had difficulty finding or creating mental pictures of the concepts, and without an informal understanding of a concept they could not learn the written form of the definition.

4. The students were unable, or unwilling, to generate and use their own examples. As a consequence, the inability to generate and use examples hindered their progress in understanding concepts and doing proofs.

5. The students did not know how to use definitions to obtain the overall structure of proofs. Moore meant by overall structure of proofs the organization or skeleton of a proof, particularly how to begin a proof how to end it, and how the beginning is linked to the ending by rules of logic and a definition, axiom or theorem.

6. The students were unable to understand and use mathematical language and notation, and this inability inhibited their ability to understand, remember and use the definitions.

7. The students did not know how to begin proofs.

'Knapp (2005) concluded that student's difficulties in constructing proof fit into two categories. First, students struggle with the logic, language and culture of the proof as determined by the community. Second, students lack the domain specific knowledge, such as definitions, theorems, heuristics and the ability to generate examples' (cited in Sari, 2008).

'Edwards and Ward (2004) investigated the students' use of definitions in constructing proofs and found that many students do not use and categorize

mathematical definitions the way mathematicians do. Also they cannot understand the distinction between everyday language and mathematical language' (cited in Sari, 2008).

Baker and Campbell (2004) noted that students struggle with understanding the process of proof construction and the precision in writing mathematics. They also found that students struggle with the correct use of logical arguments.

'According to Weber (2006), research related to reasons of students' difficulties with mathematical proof can be categorized into three classes. The first cause of students' difficulties is that they often possess an inaccurate conception of what constitutes a mathematical proof. A second cause is that they do not have an understanding of a theorem or a concept and systematically misapply it. Third reason is that they do not have the decision making strategies to do so' (cited in Sari, 2008).

'Gibson (1998) concluded that students' difficulties with mathematical proof are related to the following factors:

- Understanding of the rules and nature of proof.
- Conceptual understanding.
- Proof techniques and strategies.
- Cognitive load' (cited in Sari, 2008).

'Selden and Selden (2003) described a number of types of reasoning errors and underlying misconceptions. They also classified reasoning errors according to their logical characteristics, that is, according to whether they arise from difficulties in generalization, use of theorems, notation and symbols, nature of proof, or quantification. Some of these errors are beginning with the conclusion, using inverse of a theorem, overextended symbols, weakening the theorem, notational inflexibility, misuse of theorems, and ignoring and extending quantifiers' (cited in Sari, 2008).

Other researchers pointed out that it is not clear for students what to be proved and what can be presupposed while building a proof (Ball; Hoyles; Jahnke; & Movshovitz-Hadar, 2002).

TEACHING PROOF IN THE ALGERIAN SCHOOL TEXTBOOKS

In this section, we have examined forms of proof and proving in school mathematics textbooks and also in teachers' mathematics lessons when presented to students. We have also talked with some teachers of every level to find out how do students deal with learning proof and proving. We aimed to identify where and how proof is presented, but also which objectives are set behind proving tasks.

Level one. Primary school

1- There is first an introductory part to the lesson, in which the teacher asks pupils to make a mental calculation in board: e.g. 19 + 5, or 7 x 6, telling them a short story related to the lesson in order to attract their attention to the goal of the lesson, or giving students a problem and stimulate their curiosity to help them understanding the lesson and solve the problem in question.

According to primary school teachers, only few pupils can do the activity of the first section in a short period of time (5 minutes or less) due to the difficulty of the questions or the long time required for understanding it.

2- The second part of the lesson is the discovery part, in which the teacher presents two activities and explains them to the pupils one by one. After that, they present their results or answers and discuss them with their teacher and their classmates. Finally, they correct them collectively on the blackboard then individually on their books or copybooks.

Sometimes, this section (the discovery section) includes a set of pictures or an experiment about everyday life phenomena, the teacher allows the pupils to notice, wonder (asking why), think and give an explanation and sharing their ideas, in order to increase their enthusiasm to discover and learn.

A primary school teacher reported encountering obstacles to apply the lesson smoothly: sometimes the activity does not help pupils approaching the objective of the lesson, in this case, the teacher gives pupils a substitute activity. Besides, the individual differences (intelligence of pupils) make difficult the completion of the activity and the difficulty of reading the exercises, due to the weak mastery of Arabic language, especially for pupils of the first grade. Moreover their inability of understanding and interpreting the activities data, do not help to go further.

Primary school teachers mentioned also, when parents prepare lessons (from the textbook) with their children, they give them the solutions of the activities, but this behavior affects the pupils negatively as they do not know where does the solutions comes from, except that their parents gave it to them. This would hinder their reasoning, their thinking, and their proper understanding of the lesson, and particularly their interaction in class. In fact, they become inactive in class, having their parents' solution impedes them to try to find it by themselves by discovering all the interactive and cognitive paths. And this does not help, not only teachers but mostly the argumentation learning process of children.

3- In the third section, pupils with the help of the teacher deduce the rule or the definition of the new concept learned during the lesson, e.g. surface = length x width.

4 -The last section contains an activity (direct question or a short exercise) to make pupils solving a problem effortlessly (without using their mathematical skills) by directly applying what they have already learned during the course. It is rare where the teacher presents to his pupils an activity containing wrong statements asking to correct them.

Conclusion

Our further study of mathematics textbook of the primary school and other textbooks (of primary level and of other courses) do not show signs of argumentation, which would prepare children to proof and proving. Children in the primary school seem to not be in touch with argumentation and not be stimulated to be introduced to it later, in the next school level.

Level two. Middle school

The mathematical activities in middle school's mathematical textbooks are divided into two types: numerical activities and geometrical activities.

The course is divided into four sections:

1. The first section is a short reminder to verify the prerequisites of the students (regarding related concepts to the present lesson) and remind them about what they had learned during the past years regarding the present lesson. It is presented in the form of direct simple questions with lower difficulty, which does not require time and a long reasoning, or cleverness, only direct application of the previous related concepts. Furthermore, this reminder

serves to determine the learning level of students, and their mathematical background. The following figures are an example of this part:

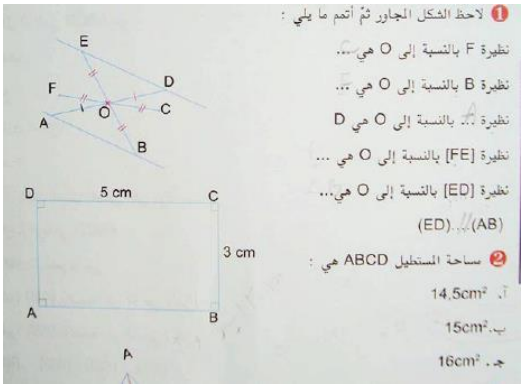


Figure 1

A reminder example of the parallelogram lesson from the geometrical activities

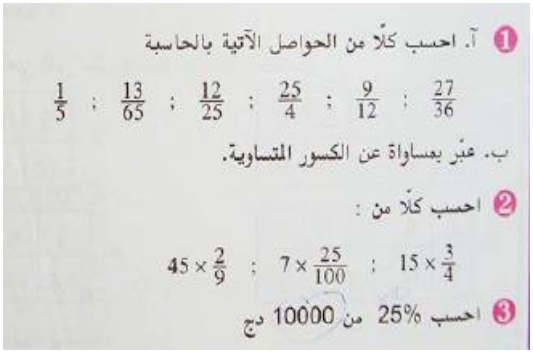
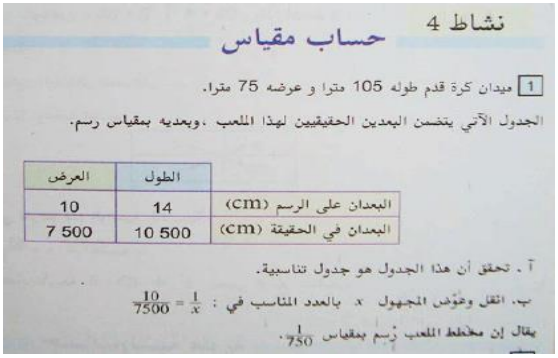


Figure 2

A reminder example of the proportional lesson from the numerical activities

2. The second section is the discovery part, which contains a set of activities, each activity aims to solve a problem of one concept among the concepts that we want the student to learn or discover through this activity.



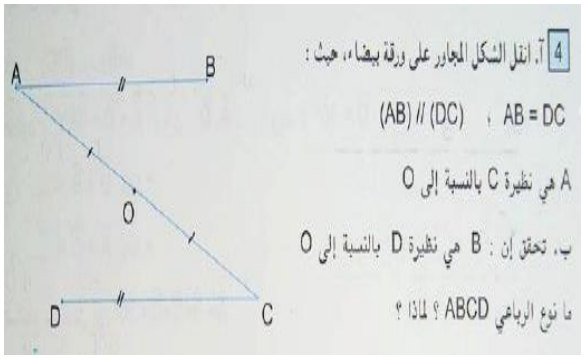


Figure 3

Geometrical activity (right) and numerical activity (left)

3. The third section contains definitions, properties and examples for each concept they had learned before in the second section. These properties and definitions stand for the tools to solve the proposed problems (coming in the next section) concerning the lesson.

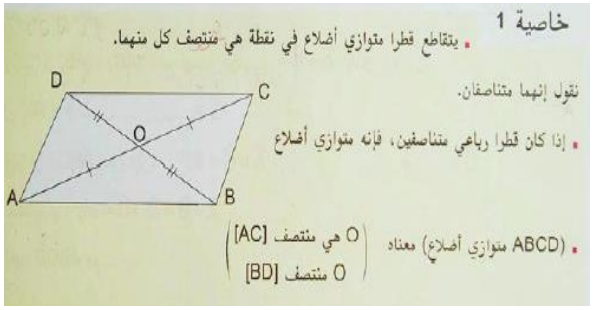


Figure 4
Property of parallelogram

4. The last section is the last part of the course; it is about a set of exercises and problems concerning the lesson's concepts.

The course is given according to a constructivist approach.

A-priori Analysis

Exercise 1

The following exercise is an example of the last section of the course of the parallelogram lesson from the geometrical activities of the second grade of the secondary school. This type of exercises aims to apply directly definitions and properties of the present lesson. In fact, applying definitions is a first step to be introduced to proof and proving

In this exercise, in order to find the solution students should use the definition of the parallelogram (Parallelogram is a quadrilateral in which every two opposite sides are parallel), and two other properties: two opposite angles are equal and every two opposite parallel sides are equal. This exercise is one of twenty-four among the forty-seven exercises that proposes 'a direct application of definition and properties, and do not require a particular mode of argumentation'.

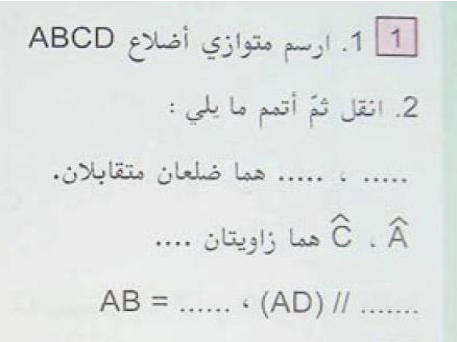


Figure 5

Exercise of direct application

The solution

1. After drawing a parallelogram ABCD
2. AB, CD (or AD, BC) are two opposite sides.
3. $\hat{A}$, $\hat{C}$ are two equal angles.
4. AB = CD, (AD) // (BC).

After doing this exercise, students would memorize the definition of the parallelogram and the used properties.

Exercise 2

The next exercise is one of the twenty-three among the forty-seven exercises, which requires making a proof or to give a justification with increasing difficulty in every exercise. In this exercise, students should choose the appropriate properties and definition on their own to prove what should be proved.

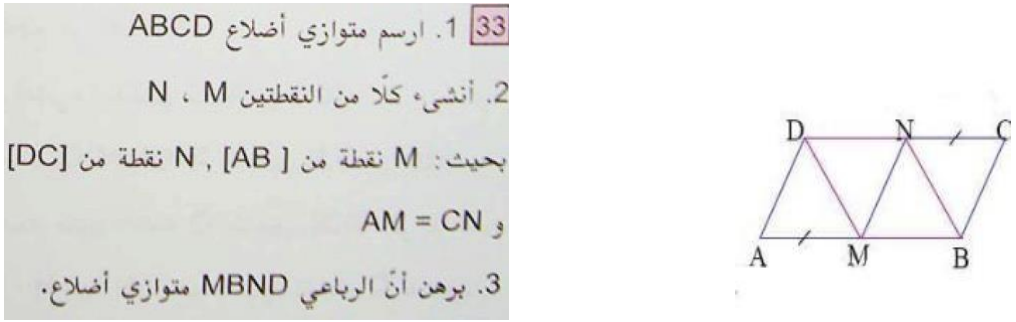


Figure 6

Exercise requiring proof

Proof

We have ABCD is a parallelogram;

So: (AB) // (CD), (definition of a parallelogram), and AB = CD (property)

Also, we have AM = CN (by hypothesis);

As the point M belongs to [AB] and the point N belongs to [CD];

This means: DN = MB, and as (MB) // (DN); so: (MD) // (BN)

Then, we conclude that the quadrilateral MBND is a parallelogram.

First of all, students should use the definition of the parallelogram in order to extract the property of the definition as a tool for later, after this, beside the exercise data, they will try to link between the definition and the property extracted in the beginning. So, the method of identifying by their own the right choices of definitions and properties would help them later in learning constructing proofs and would develop their mathematical skills in proving.

Exercise 3

The next exercise is an example of the proportional lesson from the numerical activities of the second year.

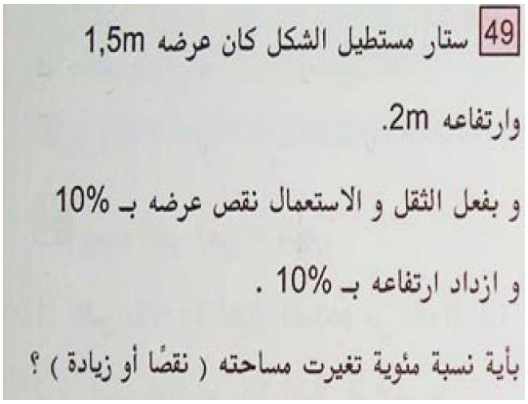

49 ستار مستطيل الشكل كان عرضه 1,5m
وارتفاعه 2m.
و بفعل الثقل و الاستعمال نقص عرضه بـ 10%
و ازداد ارتفاعه بـ 10% .
بأية نسبة مئوية تغيرت مساحته (نقصًا أو زيادة) ؟

Figure 7

Exercise with simple calculation

Proof

The curtain area is 3 m². The new curtain width is 1.35 m.

The new curtain length is 2.2 m.

The new curtain area is 2.97 m².

Then: the curtain area decreased with 1% as 0.03 x (100÷3) = 1.

This exercise is a set of successive calculations, the purpose of which is to learn how to calculate a percentage in a different way.

Conclusion:

Students in the secondary level are introduced to argumentation and have many opportunities to make proof and proving, by many exercises, particularly with geometrical activities. But, according to most teachers, for reasons of time, they cannot cover all the exercises dealing with proof, moreover, the difficulties of most students dealing with these exercises, make them not being very successful. So, unfortunately, students do not reach the objectives of these activities and leave the secondary school without the conception about proof and the skills required to follow up later on proving.

Level three. High school

The lesson in high school is based on three sections they are as follows:

1 - In the first part of the lesson, the teacher quickly reviews the contents of the previous lessons, which should be remembered by students. These reviews are in form of an activity, which contains prerequisites and does not require a particular reasoning, higher intelligence or a long thinking. This part is necessary to ensure the interaction of all students; it makes them fully concentrated and prepares them to receive the new information.

2 - The second part is the constructive part, it consists of some definitions, or properties followed by an example and an activity for each, these activities are simple and are a direct application to the given concepts and routine problems that do not require particular mathematical skills or any form of cleverness.

3 - The last section is an overall exercise (containing most of the lesson concepts and properties). The exercise is very similar to the second section examples: direct questions are given with no particular reasoning skills or new methods required from students for solving the tasks.

This method of teaching does not help teachers to evaluate whether students are able to prove and make proofs. In fact teachers receive from them only what they gave them.

A-Priory analysis

The following exercise is an activity presented in the last section of the course, students are asked to prove using recurrence proof type.

Exercise

لتكن المتتالية: U_n المعرفة على N بالعبارة التالي $U_n = 2n + 1$

برهن بالتراجع ان $U_n > 0$.

The proof

1 - Starting step

For n=0

U_0 = 2(0) + 1

U_0 = 1, so $U_0 > 0$

So, $P(0)$ is true for n = 0

2- We suppose that P_n is valid, which means $U_n > 0$, i.e. 2n +1 > 0

3- The proof part: let us prove the validity of U_{n+1}, i.e. we prove that $U_{n+1} > 0$

We have: U_{n+1}= 2 (n+1) +1 = 2n +2 +1 = U_n + 2

We have already supposed that $U_n > 0$ and 2 > 0, so $U_{n+1} > 0$, which means P_{n+1} is valid

Then we conclude that P_n is valid, which means $U_n > 0$, for all naturals integers.

Conclusion

We can conclude that making proofs in high school curriculum is just applying a known type of proof and making calculations. Unfortunately, it is not constructing proof. It does not follow up on what has already been done before in the secondary level. If there were continuity, in the process of teaching proof and proving, between secondary and high school, there will be more exercises on proof where students would build their proofs and choose their proving method. Moreover, this only proof type that is well memorized and used by students, when they arrive to the university, seems to be not clearly understood. High school teachers declare that most students do not understand this technique, but they know how to use it perfectly.

Finally, our study of proof and proving in the three school levels revealed that students are not taught proving and how to make proofs properly, more importantly, they are not presented what is proof and what it means to construct it. The process of learning proving by applying definitions starts in secondary schools, but does not move further. It does not provide opportunities for students to make proofs using the concept of proof, deductive reasoning, modes of

argumentation, later in high school. Only one type of proof is introduced to students who use it by memorizing it without understanding the essence of this technique used only with natural integers. The teaching is focused more on presenting concepts, with their definitions and property than teaching proof and proving. This explains clearly why students arrive to the university without a clear conception of what is proving, why do we prove and what is supposed to be done in a proof. Which is incoherent with the behaviors of university teachers (who treat student as already having been in touch with proof) and with the university curriculum at the first and the second year, whose courses do not introduce students to what proof and proving are.

METHODOLOGY

Our experiment is based on interviews with undergraduates' students from different levels (years of study), in order to understand how the teaching of proof and proving happened with them from school to the university, according to them. This could indicate to us the impact on their learning proof but also the nature of their difficulties related to proof and proving. The interviews were semi-structured, they have been performed during three months (May, June and July 2020) in Medea, by the means of social media (due to the COVID-19 pandemic' measurements) and each interview took around two and half hour. To examine the teaching of proof and proving through the different levels of learning, in the university, it was important to choose students from different levels, and all levels of the discipline of mathematics. In the Algerian university, the discipline of mathematics is about two degrees: bachelor degree, which takes three years and master degree, which takes two years. We will call the different years of study as levels and categorize bachelor and master students as students of five levels, for the three levels of the bachelor degree, we will use L and the number of the level: example a student of L2 stands for a student of a second year (of bachelor). For students of master level, we will use M1 and M2.

In this experiment, we interviewed twenty-one students from different Algerian universities: three students of L1 from the university of Annaba, Oran and Ain Defla, four students of L2 from the university of Mascara, Medea, USTHB Algiers and Oran, three students of L3 from the university of Skikda, Oran and Chlef, four students of M1 from the university of Medea, and seven students of M2 also from the university of Medea.

We have designed ten questions. We avoided as much as possible, asking embarrassing questions directly in order to reach clear information, which could describe and reveal the real situation about teaching proof. The questions were asked in the Algerian dialect and the responses were also given in the dialect. Students' responses have been recorded, written and then translated into English. The questions and their motivations were as follow:

1- Do you remember when did you encounter proof for the first time?

The first question was posed to know when students encountered the proof for the first time and how long they have been studying the proof.

2- How do you evaluate what you have learned during that time? Was it efficient, did it help you learning proof later?

This question allows students to question and compare their learning about proof in the different school levels and in the university. Being at the university, we think that students have enough distance to comment their learning regarding: their difficulties with proof, the way they have learned proof and proving, and their beliefs about what is proof, which could tell us more clearly the impact of how proof is taught through the different levels. We need to know which learning and beliefs, regarding proof and proving students have when they reach the university because this would greatly affect their learning of proof in university.

3- Why do some students feel anxious or uncomfortable when we ask them to prove?

This question is about the difficulties of students with proof and proving, according to them, and the possible origins of these difficulties. Students are required to interpret, link and explain the original reasons of their negative feelings.

4- According to you, what are the most important things to do when students are asked to prove?

This question is about how students define a proof and how do they think it is structured. In fact, it is known that what teachers ask students to do in a proof might be different from what students understand they should do, these different conceptions might be the origin of different problems for students that would be

still unknown if we do not understand clearly how students think what they are asked to do when they are required to make a proof.

5- What do students think about the proofs presented to them by their teachers?

By this question, we wanted to find out what students think about the proofs provided by their teachers; whether they understand them, and understand the way they are written, and whether they are aware about their structure, and all the different choices that teachers make in these proofs.

6- Some students think that proofs written in the textbooks are different at some extend from proofs as they are presented by their teachers. What do you think about their opinion?

We aimed, by this question to know the difference, for students, between the proofs written in the textbooks and the proofs that are presented by their teachers. This would tell us how often students read proofs in the mathematics textbooks, and what are their preferences proofs of their teachers or proofs of textbooks and why.

7- According to you, why do several students have difficulties with proving?

8- According to you, why do several students have difficulties with writing proofs?

As we suppose that making proof is different from writing proof, questions Q7 and Q8 have been asked to understand clearly the problems of students with making and writing proofs, and whether they are aware about this particular distinction.

9- During the exams, why do several students write unsatisfactory proofs?

Making and writing a proof during an exam are important activities under some particular conditions: students are alone, they make all their decisions by their own, they try to use what they have learned before, according to their comprehension, and have a limit of time. The responses to this question might reveal other type of difficulties about making and writing proofs.

10- According to you, how should we improve teaching proof and proving to students at the university?

The responses to this question, which are the suggestions of students for teaching proof at the university, would identify students' difficulties and how do

they conceive and analyze them, how did they deal with their past and actual teaching of proof and proving, and more importantly what could help them, regarding teaching proof, to improve their difficulties, according to them.

To analyze students' responses, we have gathered the responses having the same ideas together. We meant also to compare between the responses of students of different levels and extract what changes and what is constant through the levels, which could indicate how does the teaching of proof develop through the school and the university curriculum.

RESULTS

In this part, we will report the responses of students for each question and for each level.

Responses for the first question

Q1. Do you remember when did you encounter proof for the first time?

R1. All interviewed students (of the five levels) said they did encounter proof for the first time at middle school, except one student of M2 who said he did encounter proof in the second year of university exactly in topology course.

Responses for the second question

Q2. How do you evaluate what you have learned during that time? Was it efficient, did it help you learning proof later?

R2. L1. One student said: *'in the university, the method of proving is completely different'*. Another said: *'the fact that the proofs at the university use known concepts was not difficult for us'*. One student responded that he does not know if it helped him in learning proof or not.

L2. One student said that it was useful regarding the concepts, according to the student the concepts known at the university are complementary to the concepts already learned in the previous levels (middle and high school): *'we cannot prove a concept if we do not have any idea about it'*. One student said: *'in the past, we were memorizing the method of proving then applying it in exercises of the same type, but at the university; we go deeper regarding the concepts and what we are required to prove'*. Another one said that he just had studied mathematic and he has no idea what was it. One student said: *'no, it was not helpful, and it did not help me, because in the past we were doing only calculations'*.

L3. Three students said they have had only an idea about proof: *'it was not helpful, because our middle and high schools' teachers focused only on the application of definitions, I have had only an idea about proof; I knew that there is an activity called 'proof' that existed in mathematic'*. *'Yes, it was helpful, because it gave me an idea about proof'*.

M1. One student said: *'in the past, we were only applying definitions in exercises of the same type'*. Two students said that learning how to make a proof with a deductive reasoning, at middle school and high school was helpful to learn the proof at university: *'I do not remember the concepts used in the proofs, but I do remember how we were proving deductively'*. One student said: *'there is a big change in the way of proving, in the past the required result was clear to me, but now the required result is not clear'.*

M2. Two students said they have had only an idea about proof it did not really help them in learning proof later: *'I thought that making a proofs is only applying of definitions; but I realized that proving at university; is more than that*'. One student responded: *'no, it didn't help me in learning proof later, I didn't even know it was proof'*. Four students said it was helpful: *'what I have learned helped me to be able at least to understand a proof by myself in a mathematic textbook, especially that at the third year of the high school; we have learned the proofs types (induction and contradiction) and proving skills'*.

Responses for the third question

Q3. Why do some students feel anxious or uncomfortable when we ask them to prove?

R3. L1. Two students said they are not sure about the correctness of their proofs and they think they would never prove something correctly: *'I'm not sure about the correctness of my proof because I never made a correct proof before, this is why I'm always afraid of making a proof in the university'*. The other student said that those who can prove and make proofs are only those who solved a similar problem: *'you cannot respond if you never see it before'*.

L2. One student said students think they are not competent at proving and at making proofs for many reason: *'a student cannot make proofs because he does not understand his lesson, or he can have the idea of the proof but does not know how to express it and write it, and sometimes he cannot even find his mistakes'*. One student said his proofs have frequently been rejected, by his teachers, that makes him unsure that he could make it: *'unlike with my high*

school's teachers, I'm afraid that my proof will not be accepted by my teachers in the university', he added: *'because our teachers expect a typical answer from us, even though they haven't given us much'*. Another response: *'we do not know what we should use as an argument to construct our proof, due to lack of proving-exercises'*. One student said he does not know which proof-type he should use to build up his proof, he added: '*this situation makes me being confused'*.

L3. One of the students declared that those who feel anxious about proving are those who have not a mathematical reasoning, he added: *'they must be in the wrong field'*. One student said: *'the difficulties with proving are caused by lack of students' engagement with proofs and proving-exercises, and if they make proofs, it is just with problems of the same type which do not require a different technique or a schema'*. Another student admitted that students are always afraid of their teachers' judgment on their proofs: *'students are afraid of sharing their ideas due to their fear from their teachers' bad judgment on their personal image as a student'*.

M1. One student responded that students do not understand clearly what to prove and how to begin a proof: *'because they do not know what are required to do, and often they do not even know how to begin a proof or how to reach the required result'*. Another student mentioned four reasons: *'because they do not know how to express their ideas and how proof is actually constructed, because of lack of proving-exercise and because they do not understand what is required to be proved as they are not good at French language'*. One student admitted that indeed she is one among those who feel anxious when they are asked to prove: '*because I do not know how to begin a proof, and it is not been a long time since I knew there are many types of proving'*. One student said she believes that her proofs will be rejected by her teachers: *'I'm sure that my proofs will not be accepted by my teachers, they do not support us and this makes me uncertain of my ability to build a proof*'.

M2. One student declared: *'they might not understand how a proof is effectively constructed, they do not know how to begin a proof and because of the lack of prerequisites'*. Two students believe that it is impossible to construct a proof correctly by a student, and are never sure about the correctness of their proofs: *'a student is never able to do exactly what he is asked to do, and he is always unsure when making a proof*', *'constructing proofs is so difficult'*. One student said: *'the first students' encounter with proof in the university (either in the*

course, in their first attempt of making a proof for an exercise or in the exam), was a shock to them, because they found out that the proof is extremely difficult so that student are never able to make such proofs, so this bad feeling will be with them every time they encounter a proof. They must learn how to make proofs from a young age'. Another student said students do not know if the proof-type they used to build up their proof is correct or not: *'a student is often not sure about the correctness of his method, because the proof is not like calculation where the same steps might be repeated, but also there is more than one method to prove a mathematical statement'*. One student reported two reasons, which are lack of prerequisites and not knowing to make hypotheses: *'students do not know to choose which hypotheses to build up their proofs on, and lack of prerequisites'*. Another student responded that: *'students are not sure of the correctness of their responses and their capacities in building a proof, and at the university, we get deeper on the concept and this is not what we used to do in middle and high school'*.

Responses for the fourth question

Q4. According to you, what are the most important things to do when students are asked to prove?

R4. L1. Two students said that based on the exercise data, and their prerequisites (regarding the concept in question) they try to reach the required result. One student said he would use one of the proof types he learned before: *'I prove by contradiction, by contraposition or by induction'*.

L2. One student said: *'based on the information provided by the exercise and my own information about the concept in question, I construct my proof'*. Another student said that a proof needs a set of steps: *'first of all, I have to choose which method is appropriate for this problem and then I try to go through by a set of steps to reach the required result'*. One student declared that: *'if I'm sure of my proof, I will respond, if I'm not sure of my reasoning, I will not take a risk and write a proof, because I know my teacher will reject it'*. The forth student said: *'if it is a new problem, I try to use a theorem appropriate to the concept, if it does not work I try another theorem'*.

L3. Two students said that first of all, they try to understand the question, and then determine the exercise data using their prerequisites and their information, with these last, which would make it possible to reach the result. The third

student said: *'I do whatever it takes to reach the final result, using any method, even if this method was from other courses'*.

M1. One student declared that "prove that" is the biggest problem in mathematics: *'I have to reach the conclusion, but I mostly I do not know how'*. Another student said: *'firstly, we have to choose the convenient proof-type (contradiction, contraposition or induction), and then, determining the required result, after that, we choose the appropriate information (definition, properties) to build this proof'*. One student said: '*in case I have never seen a similar problem before; I try to remember everything related to this problem and then I build the proof based on that information, but if I see it before, I will write what I have memorized'*. One student responded: *'students must know two things the required result and how to begin a proof'*.

M2. One student said that she analyses the exercise data, and extract the properties of that concept indicating an important thing, which is: *'I have to justify every step, passage or a theorem (mentioning its name) in my proof'*. One student responded: *'I find out what I'm going to prove, analyze the exercise data, and then use definitions and properties of the concept'*. Another one declared: *'I do what I'm required to do, not more not less, and I try to prove it with my own method; according to my understanding and my information of that concept*'. One student said: *'the proof is a set of logically sequenced steps, to prove the structure and functionality of a theorem'*.

Two students said when they are asked to prove they start from the statement or the claim required to be proved and they have to reach the final result: *'I use all what I know (definition, properties, etc.) about that concept, and try to use them correctly'*. Another student said: *'the most important is that I have to know the definition of the concept, of course, beside my prerequisites and the exercise data'*.

Responses for the fifth question

Q5. What do students think about the proofs presented to them by their teachers?

R5. L1. The first student said that he could not understand it by himself: *'I only understand the proof when the teacher explains it, but I cannot construct it by myself. In the exam, if my teacher changes something, as it used to be, for a required proof, regardless whether the change is noticeable or not, I cannot do anything, because I have just memorized the past proofs without understanding*

them'. The second student said that the proofs of their teachers are clear and comprehensible, but sometimes it happens that they cannot understand them fully. The third one responded that his understanding of the proof presented by his teachers is depending on his own efforts: *'if I revise my lesson before my teacher's explanation, I will understand them better, but if I miss the information that I should have to make the proof, then I will not be able to keep up with my teacher's explanation'*.

L2. Three students of the second year said it is depending on the teacher's explanation and the difficulty of the proof: *'the presentation and the explanation of the proof are different from one teacher to another and from one proof to another, but I think that students are responsible of understanding the proof; they must make an effort'*. *'With some teachers, I immediately understand the proof when they explain it, but sometimes, I do not even know what is necessary to know about proof needs, until the end of the semester'*. One student said: *'it always seems difficult for me for the first time, but it becomes understandable after seeing it in a few examples'*.

L3. Two students said: *'it depends on the teacher; some teachers' proofs are comprehensible and easy to understand and some teachers' proofs are so mysterious. And sometimes, the only thing would be clear is the mathematical skills (integral calculus, derivation, etc.) or the passage from one step to another, but I do wonder why doing all that'*. One student said most of proofs given by his teachers are so complicated: *'hypothesis, type of proving; why do we use this type and not the other, I'm always wondering about them'*.

M1. Two students said that it depends on the difficulty of the proof: *'there is no problem with a simple proof; it is clear and comprehensible, but some proofs are overlong (two or three pages); contain properties and theorems that we should know to understand the proof. Moreover, I'm always wondering how teachers could build a proof'*. Another response: *'more there are details in a proof, the more obvious it will be, however, those steps and details of proving (how to begin, what should I prove and what should I suppose, why the teacher uses or supposes this particular one, etc.) are an extremely difficult things to do on my own'*. Another student said the undergraduate proof is complicated, and seems difficult to understand the first time: *'the proof at the university is more complicated and seems ambiguous at the beginning, because it contains a variety of concepts, so we need time to understand it, but I do not know how actually teachers build proofs'*. One student declared that it depends on the

teacher's way of explanation, she added: *'sometimes, I do not even know what my teacher is doing or what he exactly would decide to do to prove'*.

M2. Three students responded that it highly depends on the teacher's way of explanation: *'some teachers simplify the proof, and make it clear and lucid, while some of them make me feel completely impotent, I cannot even understand how the proof is being constructed and what is the point of all of that'*, *'if teachers master the proof properly and understand it, they would simplify it, but if they do not understand it as it should be, they will not be able to simplify it for us'*. Two students said that it's very clear: *'it is easy to understand, the teacher's proof, it is simple and it contains details'*, *'it is obvious, if I find some difficulty, I ask my teacher to make it clear for me'*. One student said: *'some teachers explain proofs during the lecture, and some teachers explain the proof by solving problems, to understand it better'*. Another student responded: *'if I have the necessary information and prerequisite then the proof would not be difficult for me. But, some teachers' knowledge level is much higher than ours, therefore they think that some concepts are evident for us, but they are difficult for us'*.

Responses for the sixth question

Q6. Some students think that proofs written in the textbooks are different at some extend from proofs as they are presented by their teachers. What do you think about their opinion?

R6. L1. One student declared that he never used a book to learn a proof at the university. Another one said that the proofs of their teachers are simpler than the textbooks' proofs: *'because the teacher simplifies it to make it clearer for students'*. One student said: *'sometimes the proofs presented by our teachers contain more details than the textbooks' proofs, and sometimes some textbook's proofs helped me to understand better the proof that presented my teachers'*.

L2. All interviewed students of second level responded that teachers' proofs are more detailed, and are explained in a different way: *'because the teacher presents the proof based on students' level and teachers' proofs contain more details than textbook's proofs'*.

L3. Two students responded similarly as students of the second level as teachers' proofs contain more details than textbooks' proofs. One student said: *'there are some details given by the teacher, which are not presented on textbooks. However, sometimes the teacher misses the details that are showed in textbooks and focus on the idea of the concept to avoid misleading the student'*.

M1. Two students responded that teachers simplify the proof for students: *'teachers' proofs are clearer than the textbooks' proofs'*. *'The textbooks' proofs are complicated and contain tricks we cannot understand'*. One student said: *'the textbooks' proofs are a little mysterious; they contain concepts we do not know, and our teachers illustrate the proof based on our level of mastery of the concepts, which is why teachers' proofs are clearer'*. Another student declared that the difference is in terms of difficulty: *'sometimes teachers' proof are very complicated compared to textbooks proof, and sometimes we find the opposite; some details are hidden on the textbooks but they are presented in teachers' proofs'*.

M2. One student said that they are the same, and thinks that students who believe being different are those who are used to understand only the proofs with their teachers: *'they are not used to understand proofs by themselves'*. Another one said that the teacher's method of explanation is simpler for students: *'when I read a proof from a mathematics textbook, it would look so complicated, but it becomes clearer when my teacher explains it'*. The rest of students responded in line with the same idea that teachers' proofs are simpler and contain more details than textbooks proofs.

Responses for the seventh question

Q7. According to you, why do several students have difficulties with proving?

R7. L1. Two students said that the reason is that these students make no effort: *'mathematics need a hardworking student, a mathematician student must work hard every day'*. One student said*: 'I would certainly find difficulties to solve a new problem, if I did not solve a similar problem type before'*.

L2. One student complained from having many courses and according to him, this would lead to a weak mastery of the concepts and thus also of the involved proofs. He added: *'not having a mathematical reasoning and lack of prerequisites might be other reasons'*. A student said: *'without understanding the concepts as it should be (definitions, theorems or properties regarding that concept), we cannot construct a proof, that is why students have difficulties with proving'*. Another student said: *'there are three reasons; the first one is the language; students are bad in French language, thus, they have difficulties in learning mathematics in French, the second is to justify the passage from one step to another while building a proof, because this justification might require*

another proof, and the last reason is the choice of the appropriate type of proving'. The forth student said: *'I do not know how to prove a definition'*.

L3. One student said that lack of proving-exercises makes students being afraid of proving: *'we did not have many opportunities at the university to encounter enough proofs and to engage with them in different problem-solving activity, also at high school, most of the time we were not required to prove'*. Another student responded: *'lack of prerequisites and the necessary mathematical concepts makes proving something so hard'*. One student declared that the absence of logic in their reasoning (logical connection of information) is the reason for these difficulties, he added: *'because mathematical logic course (a course of the previous (second) year) was insignificant to students, that is why it has been marginalize, they did not know how important it can be now'*.

M1. One student said: *'students do not have the proof techniques, and do not know what they should prove'.* Another respond: *'students are not used to proving in their school life not only in mathematics, but in all areas, they just accept without any interaction (discussion, criticism) even when they do not agree with their teachers' claims'*. One student said: *'several students have not the ideas and information to build their proofs. However, in case of they have what the proof requires (definitions, any idea concerning the concept) they do not know what they are required to prove'*. Another one responded: *'I don't know how to begin a proof, when I ever do, I do have difficulties with the next steps. Moreover, sometimes the question is unclear; I do not know what is clearly required. In general, I cannot make a proof I have not seen before'*.

M2. All interviewed students declared that several students have difficulties with proving because of lack of prerequisites: *'students do not know which concept or axiom to use exactly and how exactly to construct a proof'*. *'Students do not understand their lessons, they do not have the right method to construct the proof, and the language; is so important to understand the question; unfortunately, the low French level leads to many difficulties, students have never constructed proofs by themselves'*. *'They believe that constructing a proof is so hard and it is even impossible to write a proof'*. *'Students do not know how to make hypotheses or which things to suppose in their proof, or how to interpret information from the exercise data'*. *'Proving is something new for them, often they are not able to deal with a complex concept deeply. Besides, it takes time to understand how a proof is actually constructed and to acquire some particular proving skills'*. *'They have difficulties with proving because of*

lack of proving-exercises and lack of logical reasoning'. 'We do not know how to begin a proof especially a new one that has never been seen before. And mostly, building a proof takes time, and we have never been able to build it completely'.

Responses for the eighth question

Q8. According to you, why do several students have difficulties with writing proofs?

R8. L1. One student said that students cannot write a full proof, they just give the main idea to their teacher: *'a proof needs a lot of unknown rules, that is why we are not able of writing a proof completely'*. Another said that there is a variety of methods to prove a mathematical task, he added: *'sometimes I feel lost'*. A student said he does not know how to begin a proof: *'I often do not know how to begin a proof, what matter is that I have to write something, whether it is correct or not'.*

L2. One student said that written proofs depend on the organization of steps and ideas: *'a student constructs his proof randomly on the draft, but he must know the method to organize the steps to present it to his teacher'*. Three students said the reason behind these difficulties is the lack of constructing proofs: *'proving skills are acquired through practice and solving proving-exercises, the more the proving tasks made by students, the more ideas and proving skills are learned'*.

L3. One student said: *'because students have had very few opportunities of constructing proofs by themselves with their instructor, or maybe not even once, and another reason is that students cannot logically connect their ideas'*. Another student responded: *'students did not solve enough of proving problems to be able to express their ideas on the paper correctly and logically'*. One said: *'writing proofs need logical thinking, and we do not use the mathematical logic in the way we think, therefore we have difficulties in writing proofs'*.

M1. One student pointed out that it depends highly on the French language: *'I have been studying in Arabic since primary level, and suddenly in the undergraduate level, everything becomes in French. In case of the question was clear (linguistically) I do not know how to express my ideas, in addition, I do not know how to begin a proof and how to reach the conclusion'*. Another student said: *'the most difficult part in proving is the writing; students do not master the verbal mathematical expressions, I do not know why, maybe because we always focused on mathematical formulas'.*

M2. One student said: *'they have not been taught how proofs should be written and how it is actually constructed, they cannot learn proving skills and proof techniques by themselves'*. Another student declared that it is due to the way of explanation and writing of the proofs by their teachers on the board: *'when the teacher explains the ideas of the proof, he presents it randomly and irregularly on the board, but the final proof text he writes it down without clarification. So it is better to share students with him in this process. We never wrote a proof by ourselves'*.

One student responded: *'we used to write with symbols mostly, that is why we have lost writing skills (written expression) with time'*. One student declared: *'we do not know how to express our ideas. In addition, we use three languages (Arabic, French and dialect), we just put some French words as the teacher do (alors, donc, on suppose, etc.)'*. Another student said: *'students must practice and solve proving-problems, and should learn from their teachers and from solved exercises and so on until they develop their proving skills'*. One student declared that she does not know the reason: *'I always believe that I'm doing well while I'm building my proof, but then, my teachers reject it, I do not know where the problem is'*. Another one said: *'the main problem is that students are not used to prove by themselves, they depend on their teacher, sometimes, when they do not know what they should do; they just put all together what they have as given information, in a random way whether was it correct or not'*.

Responses for the ninth question

Q9. During the exams, why do several students write unsatisfactory proofs?

R9. L1. One student said: *'we do not have enough time to write an organized response'*. Another said: *'during the exam, we have two obstacles: the short time and making teacher satisfied'*. One said: *'I think that the revision of the last day before the exam makes me confused and unsure of my answers, and so, I would put anything to get the mark'*.

L2. A student said it depends on the way of the teacher's explanation and the student's efforts: *'if the teacher explains well the lesson and the student prepares properly for the exam, I think student will write a satisfactory proof'*. One student responded: *'students write unsatisfactory proofs because of their worries about the correctness of their answers due to the lack of proving exercises and also the feeling of stress of the exam'*. Another student said: *'there are two reasons: the fear of the exam and I do not know which proof type to*

use'. A student mentioned two reasons: *'students cannot make a proof if they did not solve a similar problem type before, due to the shortage of time, and the stress feeling is another reason of writing unsatisfactory proof'*.

L3. One student said that students need more practice in writing proofs by themselves. He added: *'they do not understand the definition of the concept as it should be, and they have not enough ideas to construct proofs'*. Two students said: *'there are two reasons: the first is the fear of the exam, students are afraid of the difficulty of exam questions, the second is that students do not know how to use their information correctly'*. Another response: '*the accumulation of information and concepts in the student's mind due to the revision of the last moments before the exam is a source of confusion'*.

M1. One student responded: *'because we did not understand the proof. In case we memorized it, we could reach the conclusion, but if we did not, we cannot build a proof based on what we have'*. One student said: *'because several students have difficulties with proving and writing proofs (the responses of the two last questions), that is why they write unsatisfactory proofs'*. Another said: *'the exam questions are totally new and different from the lessons and exercises that we solved in the classroom, a difference regarding the difficulty, and sometimes the question is unclear due of weak French language mastery'*.

M2. Two students said that because of the stress and the fear of passing the exam: *'the stress and fear affect students negatively, and confuse them'*. Three students responded: *'we have not enough time to build a satisfactory proof'*. '*The shortage of time in the exam is an obstacle for students to prove a mathematical task never seen before'*. Another respond: *'all the proofs are so similar, I do not know which one is the intended'*. One admitted that the problem is in the beginning of her proofs: *'I do not know exactly why, but I think that I do not know how to build a correct beginning'*.

Responses of the tenth question

Q10. According to you, how should we improve teaching proof and proving to students at the university?

R10. L1. One student said: *'our teachers should help us to learn the proof, we have to learn how it is written'*. Another student indicated that: *'the high school curriculum should contain an Algebra course'*. The third student declared that difficulties with proving are a problem without solution, he mentioned the

reason of this belief, by saying: '*because Algerian students are too lazy to study mathematics, even though teaching proof would be good'*.

L2. One student said: *'presence of students must be compulsory; because without understanding the lessons, they will not be able to do any activity (solving exercises or passing the exam)'*. Another student said: *'if the teacher presents the lesson with a detailed explanation and simplifies it, I will do my best and be interested because thanks to the teacher; I would like that course'*. A student suggested: *'I prefer that teachers explain the lesson using the drawings, provide students with some external references and teach them how to reason as mathematicians'*.

L3. One student suggested: *'I think they should let students proving by themselves and it is important to know the history of theorems and the reasons to creating them in order to make students excited about the lesson'*. A student said: *'we have to build the proofs by ourselves with the teacher's instructions. In addition, I suggest to the teacher to let the student lead the lesson by providing the student with the title of the subject and some references to do the research'*. One student said: *'we have to understand the types of proof, how we use each one of them, and students should study daily in order to be able to keep up with their teachers and find out their weaknesses'*.

M1. A student suggested three things: *'firstly, we need more examples in the courses to understand better the definitions, rules and properties. Secondly, it is better to construct proofs with our teachers, step by step and give us all possible details in order to make them obvious. Finally, I propose the exam questions to be related to what we learn during the courses'*. Another student said: *'students should master their courses, as it should be, because building proofs often is based on the definitions and also they should learn the mathematical reasoning. In addition, teachers must teach students how to construct a proof'*. One student responded that the program is not helpful, she said: *'we take many fundamental courses during one semester, so we cannot master all of them at once'*. One student suggested one thing: *'students should be taught how to construct a proof'*.

M2. One student proposed a suggestion: *'the instructor should teach students how to select exactly the appropriate definitions and theorems to construct a proof'*. Another student said: *'firstly, a fundamental course should be taught during a year not only during one semester in order to understand it properly,*

secondly; students should build and write proofs by themselves'. One declared that the student should assume the responsibility of improving his education and developing himself: *'there is no problem in the curriculum and the way of teachers' teaching, therefore the student is the one who has to develop himself by using other resources, besides what he learned at the university'*. Another student said: *'we should learn the proof in high school before the undergraduate level, and we have to know the purpose of our courses; why this course, what I'm going do with this in the realty, in addition, the best way to learn how proofs are constructed is building them with our teachers; I should be the one makes it'*. One student proposed three suggestions: *'students should be taught two things; proofs techniques and how to connect logically the different information. Moreover, students should start learning proving and proofs from high school or even from the middle school, in order to get used to proving'*. Another student suggested that: *'students should change their purpose of studying at university: the purpose must be to understand the lessons instead of just obtaining the mark, besides, the middle and high school curriculum should also contain scientific words in French language, to avoid the difficulty of understanding concepts at university in French language'*. One student said: *'teachers should teach us how proofs are actually constructed: how to begin, how to identify the required result and also how to use the information provided in an exercise in building a proof'*.

CONCLUSION

Responses to Q1 indicate that all interviewed students (of the five levels) did encounter proof for the first time at middle school, the exception was for one student of M2 who realized what really a proof is, in the second university year, in the topology course. Responses of students for the second question about the impact of their learning proof in the school were heterogeneous. Most of them mentioned a low impact on learning proof and proving at the university and an abrupt change from proving in school and proving in the university. According to students, proving in the school was about memorizing a given method, making calculations, applying directly a definition, or just having a vague idea about proof as knowing that there is a mathematical activity called proof, which is consistent with our study of how proof and proving were introduced to students during the three school levels. But in the university, students reported that proving and making proof is different, it deals deeper with a concept and with the required result, but more importantly, what students are asked to prove

is no longer clear to them as it was with proofs made in the school. Only few master students, who could have a distance about their learning proof at school, realized that the deductive reasoning learned before in the school helped them to understand some proofs at the university. For the third question, students gave many interesting reasons to why they feel uncomfortable when they make proofs: their doubts about the correctness of their proofs, their belief about the difficulty and even their inability of making a proof, their fear from their teachers' bad judgment on their personal image as a student, and the anxiety that their proofs would not be accepted by their teachers. Master students (M1 and M2) added some other reasons: the fact that students do not understand how proofs are structured and how to decide about what to prove and how to prove it, besides, the inability of expressing ideas and the language difficulties that prevent students from understanding clearly what to prove. The first thing that students do when they are asked to prove is different from a student to another; the fifth-level students (M2), who have manipulated proofs at the university during five years, would analyze and interpret the information from the exercises data, while students of the other levels (L1, L2, L3 and M1) would determine those data in case they are easy to extract and to notice. Common responses to students of all levels are: choosing the appropriate proof type, theorem or definition, and using their prerequisites regarding the concept. However, only master students (M1 and M2) declared that it is important to know two things while building a proof: how to begin a proof and the required result. They added that, in case the required result is clear for students, most of them do not know how to reach it.

As concerns proofs presented by teachers, eighteen students' responses showed that students depend highly on teachers' explanation of the proof, and the understanding of the proof depends on: the way of teachers' explanation and the difficulty of the proof. Three students (of L1, L3 and M2) indicated that understanding the proof depends on students' effort and prerequisites. In addition, twelve students (of eighteen students mentioned before) are taking into-account how the proofs are constructed; they have been always wondering how their teachers are building the proofs: what about the beginning, what about the conjectures made, what about making hypothesis and what do exactly teachers want to prove and why they did make such particular choices to build their proof; these points seem to be very ambiguous for those students. For seventeen students of the five levels, proofs presented by teachers are simpler and clearer than proofs presented in textbooks. Three students (one L2 and two

L3) think that the difference is on the details showed in the proof; either the textbooks' proof or teachers' proof. Moreover, one M2 student thinks that those who are not used to understand proofs by themselves and depend on their teachers' explanation are those who think that there is a difference between textbooks' proofs and teachers' proofs.

Regarding the difficulties of students with proving, students showed an increasing awareness and clarity about their problems with their increasing practice with proof and proving, more the level of students is higher more the awareness is stronger. Students mentioned a variety of reasons for their difficulties. As common reasons: the inability of making a proof if it is not similar to a proof seen before, lack of students' prerequisites, lack of proving exercises and the absence of mathematics logic in students' reasoning (logical connection of ideas), besides, the weakness of mastering French, which is a big obstacle to understand mathematics. Moreover, the following reasons are stated by students of different levels: mathematic needs a hardworking student, which is not the case for students of nowadays, the lessons (definitions and theorems) are not understood as it should be, also having many courses leads to a weak mastery of the concepts used in proofs. More importantly, students have not learned to have a mathematical reasoning, they do not know which proof type or arguments they should use to prove and they do have difficulties in justifying the passages from one step to another while constructing a proof. Master (M1 and M2) students were more precise about the main reason that cause their inability of building proofs, which is students have never constructed a proof by themselves, therefore they have no idea about how building proof exactly is: they do not know how to begin a proof, how to make hypothesis or how to use definitions, axioms or properties, also how to find the right method to construct a proof and the ability to deal with a complex concept deeply because they did not used to get deeper with concepts in previous levels (middle and high school). In addition, for most master students, the required result is a big problem for them, due to the ambiguity of the question linguistically or conceptually: they have difficulties to identify and understand the required result, but also how to reach it.

With respect to writing proofs, eleven students (seven of them are students of the first two levels) think that writing proofs and proving are the same. That is why they mentioned their difficulties with only proving as: the inability of choosing the appropriate method, in addition to the variety of proving methods

that confuse students, how to begin a proof, lack of proving skills due to the lack of constructing proofs and not having what constructing a proof needs. A student of L2 thinks that writing proofs needs a good organization of ideas. While nine students are aware of the origin of their difficulties with writing proof from the third level, according to them. Three students (of L3, M1 and M2) stated a difficulty in expressing their ideas correctly and logically as a common reason, and according to a student of L3, this difficulty is due to the lack of proving exercises. Additionally, students do not master the verbal mathematical expressions when writing and learning proofs, they just focus on mathematical formulas. Also, one M2 student indicated that students used to write using only symbols and therefore they have lost writing skills. Besides, students are not used to proving by themselves, even when teachers present a proof; they just explain the ideas of the proof but do not explain how to write the final text of the proof, so students require to be taught what teachers do to write proofs. Moreover, writing proofs depends also on French language, which is not mastered by most students, when they are thinking they use three languages; Arabic and French beside to the dialect. One student of M2 admitted that she just put some French words, as a decoration, the way she thinks their teachers do. The absence of mathematical logic, which leads to illogical thinking while writing a proof, is also a reason mentioned by a student of L3.

Concerning the reasons why students write unsatisfactory proofs during the exams, students' responses revealed different reasons that influence making and writing proofs in the exam. The specified duration for the exam at university, mentioned by six students (three of M2, two of L1 and one of L2) as a short and insufficient time to make a satisfactory proof. While eight students (two of M2, four of M1, one of L3 and one of L2) thought that having difficulties with proving is the principle reason. Moreover, other reasons stated by seven students (two of M2, two of L3, two of L2 and on of L1) are: the fear of the difficulty of exam questions, the stress feeling because of the accumulation of information and concepts, also the confusion resulting from a poor preparation for the exam.

Concerning how to improve teaching proof at university, students mentioned a variety of suggestions in order to enhance their learning of proof at university. Regarding the way of teaching mathematics in general, and proofs in particular, students suggested some ways of explanation to understand better their lessons (definitions, rules and properties, etc.): it is better for teachers to give more examples during the courses, explain using drawings, finding a way to relate

what they are teaching to the real world to make students more interested. In addition, an important proposition has been mentioned by (L3, M1 and M2) students, which is students should be taught how the proofs are actually constructed: how to begin, how to select the appropriate definitions and theorems and the appropriate proof type, how to reach the required result and learning how to connect logically the ideas. Also, students should construct proofs by themselves with their teachers during the courses in order to learn the structure of a proof, is a common suggestion mentioned by five students (two of L3, one of M1 and two of M2). Concerning the curriculum, teaching proofs and learning proving, and teaching mathematical terms in French language, should be done in high school before the undergraduate level, according to students. Moreover, students pointed out to an important issue regarding courses which should be taught during all the whole academic year, instead of one semester to reach more a higher level of mastery, and shifting the purpose of students from getting the mark to learning mathematics.

DISCUSSION

In this research, we wanted to examine the teaching of proof and proving at the university. We have examined for that, the teaching of proof and proving at the three school levels, by a study of the textbooks, to understand the background of students regarding proving, making proofs and writing proofs, when they arrive at the university. We have also collected beliefs, difficulties, suggestions and thoughts of students by their responses to our performed interviews. Our findings suggest that:

Even though middle and high school is a sufficient period for teaching proof and proving, students do not know what is a proof, how to make it and the importance of its role in understanding mathematics. Moreover, their teaching of proof and proving, in the school, was very poor, students dealt only with proofs, which had a known and memorized procedure, without understanding how proofs are constructed, and how proofs are written. When students arrive at the university with this proof learning situation, they are presented to university proofs without any introduction, or beginner's guide, they are rather treated at the university by teachers as already knowing what proof is and how to construct it. So, the fact that students are required to do what they are unable to do and what they are not used to do before the undergraduate level, makes them being unconfident, and powerless thinking they are incompetent to make a proof correctly in a way that their teachers would accept it. Some students evoked

having a shock to show how the discontinuity between the two practices is affecting students negatively. As a consequence of this abrupt change, from proving in school to proving in the university, students start to have many difficulties to proving and constructing proofs, but worse, they construct wrong beliefs and practices as: it is not possible to making a proof if it is not similar to one seen before, in other words, those who are able to construct proofs are only those who have made a similar proof before. As a consequence, students learn to memorize some proofs and rewrite them when they are asked to make a proof (with the risk to not be the same proof), they do not construct them, because they seem to never see a proof in construction. This shows that students need to be shown how proofs are really constructed and not only be given ready proofs at the blackboard or in mathematics textbooks, and should be given the opportunity to do the same, by constructing proofs by themselves. Additionally, it takes a long time (might be years) to students to become aware about their problems: more the level of students is higher more their awareness becomes bigger, and students become also aware of the origin of their problems, which is the fact they did not learn the structure of the proof. But unfortunately, becoming aware about the origin of their difficulties with proof and proving does not necessarily mean they are able to overcome them. Master students (M1 and M2) who have practiced proof and proving for at least five years at the university seem to have the same difficulties with proving as students who are newcomers to the university. We can conclude that 'teaching of mathematics at undergraduate level does not introduce students to the awareness of what proving and proof mean' (N. Azrou, 2019, p.15), and does not teach them how a proof is really constructed.

Concerning writing proofs, students think that the final text of a proof and the proving process are the same, because they refer to the proofs presented to them by their teachers and those presented in mathematics textbooks. So, 'how a proof is really constructed and how that results in a written final text is missed, so students learn to do the same: they think and write their first exploratory draft as a final text' (Azrou, 2019, p15). Only few students are aware about their difficulties with writing proofs, and only one M2 student pointed out the necessity to teach the proof-writing phase.

Finally, we can say that students at the university level are not taught how to prove, how to construct a proof and how to write a proof. This lack made students having many difficulties with proof and proving, and developing wrong

practices and beliefs, and this would have impeded them to make and write correct proofs all along their university curriculum.

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