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Results on Appell Sequences

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Abstract

This work, based on the article "Recurrence Relation for the Appell Sequenes", introduces explicit formulas and a recurrence relation for the computation of Appell polynomials. This recurrence relation can be used to efficiently compute the polynomials for large values of their parameters. Furthermore, we present several examples of Appell-type polynomials and their corresponding formulas provided in the article [20].

Keywords: Generating functions, explicit formulas, recurrence relations, Appell polynomials, Bernoulli polynomials, Bell polynomials, Euler polynomials, Genocchi polynomials, derangement polynomials.

Notations

$\mathbb{C}$	The field of complex numbers.
$\mathbb{R}$	The field of real numbers.
$\mathbb{Z}$	The set of entire numbers.
$\mathbb{N}$	The set of natural numbers.
$[\mathrm{n}]$	The set of $\{1, 2, \ldots, n\}$.
$\langle x \rangle_n$	The rising factorial.
$(x)_n$	The falling factorial.
$\delta_{n,k}$	The Kronecker delta symbol.
$\mathfrak{S}_n$	The set of permutations .
σ	The permutation.
$\delta_{i,j}$	The Kronecker delta symbol.
$s(n,k)$	Signed Stirling numbers of the first kind.
$\binom{n}{k}$	The binomial coefficient.
$\begin{bmatrix} n \\ k \end{bmatrix}$	Unsigned Stirling numbers of the first kind .
$\left\{ {n \atop k} \right\}$	Stirling numbers of the second kind.
$f'(t) = \dfrac{df}{dt}$	The first order derivative of a function.
Δ	The hypergeometric function.
$\left\{ {n \atop k} \right\}_r$	The r-Stirling numbers of the second kind.
$S_n^k(x)$	The weighted Stirling numbers of the second kind.
$W_m(n,k)$	The Whitney numbers.
$w_{m,r}(n,k)$	The r-Whitney numbers of first kind.
$W_{m,r}(n,k)$	The r-Whitney numbers of the second kind.

${}_pF_q\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};x\right)$	The hypergeometric function.
$B_{n,k}\left(x_1,x_2,\ldots,x_n\right)$	The partial Bell partition polynomials.
$B_n\left(x_1,x_2,\ldots,x_n\right)$	The complete Bell partition polynomials.
$\left(P_n\left(x\right)\right)_{n\geq 0}$	The Appell sequence.
$B_n(x)$	The Bernoulli polynomials.
$E_n(x)$	The Euler polynomials.
$G_n(x)$	The Genocchi polynomials.
$\mathcal{B}_n(x)$	The Bell polynomials.
$\mathcal{H}_n(x)$	The Hermite polynomials.
$B_{n,p}(x)$	p-Bernoulli polynomials.
$B_n^{(\alpha)}(x)$	Bernoulli Polynomials of order α.
$E_n^{(\alpha)}(x)$	The generalized Euler polynomials.
$H_n(x\|\lambda)$	The Frobenius-Euler polynomials.
$H_n^{(\alpha)}(x\|\lambda)$	The Frobenius–Euler polynomials of order α.
$A_n(x\|\lambda)$	The generalized Eulerian polynomial.
$A_n^{(\alpha)}(x\|\lambda)$	The Frobenius–type Eulerian polynomials of order α.
$G_n^F(x,\lambda)$	The Frobenius-Genocchi polynomials.
$\mathcal{B}_{n,p}(x)$	The p-Bell Polynomials.
$d_n^{(r)}(x)$	Derangement polynomials of order r.
$(h_n)_n$	The Sequence related to Appell polynomails.

Introduction

Introduction

Appell polynomials have always held a central place in mathematics due to their vital role in function approximation, differential equations, and combinatorial analysis. Among the important families that have emerged in this context are the Appell polynomials, a special class of polynomial sequences characterized by a distinctive derivative property expressed as:

$$\frac{d}{dx}P_n(x) = nP_{n-1}(x), \quad \forall n \in \mathbb{N}^*.$$

This relation clearly shows the recursive and structural nature of these sequences, allowing each term to be directly connected to its predecessor. This makes Appell polynomials highly significant in various fields such as analysis, generating functions, and differential equations.

This type of polynomial sequence is attributed to the French mathematician Paul Appell (1855–1930), who was the first to systematically study these sequences at the end of the 19th century. Born in Strasbourg, France, Appell was renowned for his work in complex analysis and differential equations. He held several prestigious academic positions and was a member of the French Academy of Sciences. In one of his studies, Appell classified polynomial sequences that satisfy the derivative property above, showing that they form a general framework that includes many well-known families such as: Bernoulli polynomials, Euler polynomials, Bell polynomials, Eulerian polynomials, and others.

Appell's generalization opened the door for researchers to use unified tools in the study of polynomials, facilitating the discovery of recurrence relations, explicit formulas, and common generating functions among these families.

Among the recent key results in this area is the general recurrence relation for Appell polynomials [20] established by researchers G. Guettai, D. Laissaoui, M. Rahmani, and M. Sebaoui. This relation is given by:

$$\mathcal{P}_{n+1,m}(x) = \frac{h_{m+1}}{h_m}\mathcal{P}_{n,m+1}(x) + (m+x)\mathcal{P}_{n,m}(x),$$

with the initial condition

$$\mathcal{P}_{0,m}(x) = 1.$$

This recurrence formula represents a novel generalization that was previously unknown, marking an important step towards a deeper understanding of the structure of Appell polynomials. The present thesis largely builds upon the referenced article Recurrence relation for the Appell Sequences [20] authored by these researchers, where this recurrence relation is analyzed and applied to some special

polynomial families.

This thesis is divided into three chapters:

The first chapter introduces the fundamental concepts that prepare the ground for studying polynomials, including the rising and falling factorial, binomial coefficients, Stirling numbers, Whitney and r-Whitney Numbers, Hypergeometric functions, along with an initial introduction to Appell polynomials.

The second chapter is dedicated to studying some special polynomial families, including Bernoulli, Euler, Genocchi, Bell, Hermite and some of their generalizations, discussing their properties, generating functions, and recurrence relations.

The third chapter focuses on the Appell sequence as a general framework, where we derive explicit formulas and establish recurrence relations encompassing all previously studied sequences, allowing for a deeper understanding of their unified structure.

In conclusion, this work aims to contribute to the theoretical understanding of Appell polynomials and to enhance their applicability across various mathematical fields.

1

Basics Notions

In this chapter, we introduce some fundamental concepts that will be essential for our study. We begin with binomial coefficients, permutations, cycles, and partitions, followed by a discussion on generating functions and their importance in handling polynomials. Next, we present the Kronecker symbol and the difference operator before moving on to the rising and falling factorials, Stirling numbers of both kinds, and some of their generalizations. We also study Whitney numbers and Bell numbers and their relevance to these topics. Additionally, we introduce the Faà di Bruno formula, which plays a key role in polynomial-related computations. This chapter is supported by several detailed scientific references, which are listed at the end of the thesis [3, 4, 6, 7, 8, 9, 14, 19, 20, 23, 24, 32].

Preliminairies

1.1 The Rising Factorial and the Falling Factorial

Definition 1.1 *For, $x \in \mathbb{C}$, the rising factorial $\langle x \rangle_n$ is defined as:*

$$\langle x \rangle_n = x(x+1)(x+2)\dots(x+n-1), \quad \textit{with} \quad \langle x \rangle_0 = 1,$$

and the falling factorial, denoted by $(x)_n$, is defined by:

$$(x)_n = x\,(x-1)\,(x-2)\dots(x-n+1)\,, \quad \textit{with} \quad (x)_0 = 1.$$

1.2 Binomial Coefficients

Definition 1.2 *The binomial coefficient, denoted $\binom{n}{k}$ is the number of ways to choose a subset of k elements from a set of n elements.*

Definition 1.3 *We define the binomial coefficient for all non-negative n and k as follows:*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n)_k}{k!} \quad (0 \le k \le n), \tag{1.1}$$

where $\binom{n}{k} = 0$ for $k > n$.

Example 1.1 *In the set $[3]=\{1,2,3\}$, there are three ways to choose two elements:*

$$\{1,2\}\,;\{1,3\}\,;\{2,3\}\,.$$

As a result, $\binom{3}{2} = 3$.

Substituting $n = 3$ and $k = 2$ into equation (1.1) gives:

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3.$$

These coefficients obey the following recurrence relation:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad \text{for every} \quad n \geq 1. \tag{1.2}$$

Using the recurrence relation (1.2), one can construct Pascal's triangle:

$n \diagdown k$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Table 1.1: Pascal's triangle numbers

The binomial coefficients are involved in numerous identities, including the following:

1. The absorption identity

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}. \tag{1.3}$$

2. The symmetry

$$\binom{n}{k} = \binom{n}{n-k}, \text{with } n \geqslant k \geqslant 0. \tag{1.4}$$

1.3 The Kronecker Symbol

The Kronecker Symbol is given by:

$$\delta_{n,k} = \begin{cases} 1 & \text{if} \quad n = k \\ 0 & \text{otherwise.} \end{cases}$$

1.4 The Difference Operator Δ

For any function f from $\mathbb{R}$ to $\mathbb{R}$ (or from $\mathbb{C}$ to $\mathbb{C}$), the difference operator Δf of a function f at a point x is defined by the rule:

$$\Delta f(x) = f(x+1) - f(x). \tag{1.5}$$

The explicit formula for the n-th difference operator Δ^n is given by:

$$\Delta^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k). \tag{1.6}$$

1.5 Permutations, Cycles, and Partitions

1.5.1 Permutations

Definition 1.4 *A permutation is defined as a bijective function σ from the finite set $[n] = \{1, 2, \ldots, n\}$ onto itself. The total number of permutations of the set $[n]$ is given by $n!$, the factorial of n.*

A permutation can be represented in matrix form as follows:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \ldots & \sigma(n) \end{pmatrix}. \tag{1.7}$$

For $n \geq 3$, the set $\mathfrak{S}_n$ consisting of all permutations of the set $[n]$, together with the operation of composition (denoted $\sigma\mu$ for two permutations σ and μ), forms a group known as the symmetric group on $[n]$. This group is non-commutative, meaning the order in which the permutations are applied matters. The identity element of this group is the identity permutation, denoted by ε, which leaves every element unchanged. A fixed point of $\sigma \in \mathfrak{S}_n$, is a point i such that $\sigma(i) = i$. A mobile element, a non-fixed element.

1.5.2 Cycles

Definition 1.5 *In a permutation of a finite set, a cycle is a way of describing how a subset of elements are permuted among themselves, while all other elements remain fixed.*

More precisely, a cycle of length k (with $k \leq n$) is a permutation σ of the set $[n] = \{1, 2, \ldots, n\}$ such that:

$$\sigma(i_1) = i_2, \quad \sigma(i_2) = i_3, \quad \ldots, \quad \sigma(i_{k-1}) = i_k, \quad \sigma(i_k) = i_1,$$

and for all other elements $j \notin \{i_1, \ldots, i_k\}$, we have $\sigma(j) = j$. We usually write this cycle as:

$$\sigma = (i_1\ i_2\ \ldots\ i_k).$$

Example 1.2 *In the set $[8]$, the following permutation is given by:*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 1 & 7 & 6 & 3 & 4 & 8 \end{pmatrix}.$$

The element 2 is a fixed point because $\sigma(2) = 2$, and similarly, the element 8 is a fixed point since $\sigma(8) = 8$. This means that for all $k \in \mathbb{N}$, we have $\sigma^k(2) = 2$ and $\sigma^k(8) = 8$.
Now, observe that:

$$\sigma(1) = 5, \quad \sigma(5) = 6, \quad \sigma(6) = 3, \quad \sigma(3) = 1,$$

so after four applications, we return to the starting point:

$$\sigma^4(1) = \sigma^4(3) = 1.$$

The elements (1, 5, 6, 3), taken in this order, form a cycle of length 4. Therefore, we can express the permutation σ as a product of cycles. In this example:

$$\sigma = (1\ 5\ 6\ 3)(2)(8).$$

1.5.3 Partitions

Definition 1.6 *Let E be a non-empty set. A partition of E is a collection of subsets $\{A_1, A_2, \ldots, A_k\}$ such that, each subset A_i is called a block of the partition. This means:*

1. *Each block is non-empty.*
2. *The blocks are disjoint.*
3. *All elements of E are included exactly once.*

Example 1.3 *Let's consider the following cases:*

1. *If $n = 2$, $[n] = \{1, 2\}$, there are two possible partitions: $\{\{1\}, \{2\}\}$ and $\{1, 2\}$*
2. *If $n = 3$, $[n] = \{1, 2, 3\}$, there are five possible partitions:*

$$\{\{1\}, \{2\}\{3\}\}; \quad \{\{1\}, \{2, 3\}\}; \quad \{\{2\}, \{1, 3\}\}; \quad \{\{3\}, \{1, 2\}\}; \quad \{1, 2, 3\}.$$

1.6 Generating Functions

Generating are functions used to represent sequences as power series. They simplify combinatorial problems by turning them into algebraic ones. This makes it easier to solve recurrence relations, find formulas, and count complex structures.

Throughout what follows, we denote by $\mathbb{K}$ a commutative ring with a unit element.

1.6.1 Ordinary Generating Function

Let n be a natural number, and let $(a_n)_{n\geq 0}$ be a sequence of complex numbers. The ordinary generating function of the sequence is defined as:

$$A(x) = \sum_{n\geq 0} a_n x^n,$$

where the coefficients a_n belong to a ring $\mathbb{K}$, and is an indeterminate x. The addition and multiplication of two ordinary generating functions are defined as follows:

$$\sum_{n\geq 0} a_n x^n + \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0} (a_n + b_n) x^n$$

and

$$\left(\sum_{n\geq 0} a_n x_n\right)\left(\sum_{n\geq 0} b_n x_n\right) = \sum_{n\geq 0} C_n x_n,$$

with

$$C_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

1.6.2 Exponential Generating Function

The exponential generating function of the sequence $(a_n)_{n\geq 0}$, where n is natural number and $(a_n)_{n\geq 0}$ is a sequence of complex or real number, is defined as:

$$B(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}.$$

The addition and multiplication of two exponential generating functions are defined as follows:

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} + \sum_{n\geq 0} b_n \frac{x^n}{n!} = \sum_{n\geq 0} (a_n + b_n)\frac{x^n}{n!}$$

and

$$\left(\sum_{n\geq 0} a_n \frac{x^n}{n!}\right)\left(\sum_{n\geq 0} b_n \frac{x^n}{n!}\right) = \sum_{n\geq 0} c_n \frac{x^n}{n!},$$

with

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

1.7 Hypergeometric Functions

The hypergeometric function, denoted by ${}_pF_q$ is defined by the following generating function:

$${}_pF_q\left(\begin{array}{c} a_1, a_2, a_3, \ldots, a_p \\ b_1, b_2, b_3, \ldots, b_q \end{array}; x\right) = \sum_{n=0}^{\infty} \frac{\langle a\rangle_1 \langle a\rangle_2 \ldots \langle a\rangle_p}{\langle b\rangle_1 \langle b\rangle_2 \ldots \langle b\rangle_q} \frac{x^n}{n!}.$$

In particular, when $p = 0$ and $q = 0$, with no parameters other than x, we obtain the exponential series:

$$\begin{aligned} {}_0F_0\left(\begin{array}{c} \\ \end{array}; x\right) &= \sum_{n\geq 0} \frac{x^n}{n!} \\ &= e^x. \end{aligned}$$

If $p = 1$ and $q = 1$, we obtain Kummer's confluent hypergeometric function is defined by:

$${}_1F_1\left(\begin{array}{c} a \\ b \end{array}; x\right) = \sum_{n\geq 0} \frac{\langle a\rangle_n}{\langle b\rangle_n} \frac{x^n}{n!}.$$

For $p = 2$ and $q = 1$, we obtain Gauss's hypergeometric function defined by:

$${}_2F_1\left(\begin{array}{c} a, b \\ c \end{array}; x\right) = \sum_{n\geq 0} \frac{\langle a\rangle_n \langle b\rangle_n}{\langle c\rangle_n} \frac{x^n}{n!}.$$

1.8 The Stirling Numbers

Stirling numbers are special numbers in mathematics that arise when we count ways to group or arrange elements. There are two kinds: Stirling numbers of the first kind and Stirling numbers of the second kind.

1.8.1 Stirling Number of the First Kind

Definition 1.7 *The unsigned Stirling numbers of the first kind, denoted by* $\left[{n \atop k}\right]$*, count the number of permutations of* n *elements that have exactly* k *cycles. These numbers appear as the coefficients when the falling factorial polynomial* $\langle x\rangle_n$ *is expressed in terms of the standard monomial basis* x^n*, that is:*

$$\langle x\rangle_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

Example 1.4 *In the set* $[3] = \{1, 2, 3\}$*, there are* 3 *permutations consisting of two cycles, which matches the Stirling number* $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3$*. These permutations are:*

$$(1\ 2)(3), \quad (1\ 3)(2), \quad (2\ 3)(1).$$

Definition 1.8 *The signed Stirling numbers of the first kind, denoted* $s(n, k)$*, are the coefficients in the expansion of the falling factorial* $(x)_n$ *when expressed in terms of powers of* x*. This expansion is given by:*

$$(x)_n = \sum_{k=0}^{n} s(n, k)\, x^k.$$

These signed Stirling numbers are related to the unsigned Stirling numbers of the first kind by the identity:

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k}\, s(n, k).$$

Example 1.5 *Write* $(x)_n$ *as a polynomial*

$$\begin{aligned} (x)_2 &= x(x-1) \\ &= x^2 - x. \end{aligned}$$

We can say from this example that:
$s(2, 0) = 0.$
$s(2, 1) = -1.$
$s(2, 2) = 1.$

Properties

The Stirling numbers of the first kind have several important properties:

1. **Generating function**: For all $k \in \mathbb{N}$, we have the following exponential generating function of the signed Stirling numbers of the first kind is given by:

$$\sum_{n=k}^{\infty} s\,(n, k)\, \frac{z^n}{n!} = \frac{(\ln(1+z))^k}{k!}. \tag{1.8}$$

2. **Explicit formulas**: The explicit formula for the unsigned Stirling numbers of the first kind:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j,h=0}^{n-k} (-1)^{j+h} \binom{h}{j} \binom{n-1+h}{n-k+h}.$$

   The explicit formula for the signed Stirling numbers of the first kind:

$$s(n, k) = \frac{n!}{k!} \sum_{i_1 + \cdots + i_k = n} \frac{1}{i_1 \dots i_k}.$$

3. **Recurrence relation**: The Stirling numbers of the first kind satisfy the following recurrence relations:

$$\begin{cases} s(n, 0) = s(0, k) = \delta_{n,k}, \quad s(0, 0) = \delta_{n,k}, \\ s(n+1, k) = s(n, k-1) - n s(n, k), \quad (n, k \geqslant 1), \end{cases} \tag{1.9}$$

Table of the Stirling Numbres of the First Kind

Using the triangular recurrence relation (1.9), we can compute the Stirling numbers of the first kind $s(n,k)$ for values of n and k ranging from 0 to 7.

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	−1	1					
3	0	2	−3	1				
4	0	−6	11	−6	1			
5	0	24	−50	35	−10	1		
6	0	−120	274	−225	85	−15	1	
7	0	720	−1764	1624	−735	175	−21	1

Table 1.2: Stirling numbres of the first kind

1.8.2 Stirling Numbers of the Second Kind

Definition 1.9 *The Stirling numbers of the second kind, denoted* $\left\{ {n \atop k} \right\}$*, count the number of ways to partition a set of n distinct elements into k non-empty, disjoint subsets. These numbers also appear as the coefficients when the monomial* x^n *is expressed in terms of the falling factorial basis* $(x)_n$*, as follows:*

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k \,. \tag{1.10}$$

Example 1.6 *The set* $[3] = \{1,2,3\}$ *can be partitioned into 2 blocks (i.e., two non-empty disjoint subsets) as follows:*

$$\{\{1,2\},\{3\}\}\,, \quad \{\{1,3\},\{2\}\}\,, \quad \{\{2,3\},\{1\}\}\,.$$

Thus, $\left\{ {3 \atop 2} \right\} = 3$.

Example 1.7 *The set* $[4] = \{1,2,3,4\}$*, is partitioned into 2 blocks (into two non-empty subsets)*

$$\begin{aligned} &\{\{1,2,3\}\,\{4\}\}\,, \quad \{\{1,2,4\}\,\{3\}\}\,, \quad \{\{1,3,4\}\,\{2\}\}\,, \\ &\{\{2,3,4\}\,\{1\}\}\,, \quad \{\{1,2\}\,\{3,4\}\}\,, \quad \{\{1,3\}\,\{2,4\}\}\,, \\ &\qquad\qquad\quad \{\{1,4\}\,\{2,3\}\}\,. \end{aligned}$$

Thus, $\left\{ {4 \atop 2} \right\} = 7$.

Properties

1. **Generating functions**: For all $k \in \mathbb{N}$, we have the following exponential generating function:

$$\sum_{n \geq k} \left\{ {n \atop k} \right\} \frac{z^n}{n!} = \frac{(e^z - 1)^k}{k!}, \tag{1.11}$$

and the ordinary generating function:

$$\sum_{n \geq 0} \left\{ {n \atop k} \right\} x^n = \frac{x^k}{(1-x)\,(1-2x)\ldots(1-kx)}.$$

2. **Explicit formulas**: For $\left\{ {n \atop k} \right\}$, the explicit formula is given by:

$$\begin{aligned} \left\{ {n \atop k} \right\} &= \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n \\ &= \frac{1}{k!} \sum_{j=0}^{k} (-1)^{j} \binom{k}{j} (k-j)^n . \end{aligned}$$

3. **Reccurrence relation**: The Stirling numbers of the second kind satisfy the following recurrence:

$$\begin{cases} \left\{ {n \atop 0} \right\} = \left\{ {0 \atop k} \right\} = \delta_{n,k}, \quad \left\{ {0 \atop 0} \right\} = \delta_{n,k}, \\ \\ \left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}, \ n \geq k \geq 1. \end{cases} \tag{1.12}$$

Table of Stirling Numbers of Second Kind

Using the triangular recurrence relation (1.12), we can compute the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ for values of n and k ranging from 1 to 7.

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	1					
3	0	1	3	1				
4	0	1	7	6	1			
5	0	1	15	25	10	1		
6	0	1	31	90	65	15	1	
7	0	1	63	301	350	140	21	1

Table 1.3: Stirling numbers of the second kind

1.9 The Generalization of Stirling Numbers

The generalization of Stirling numbers extends the classical Stirling numbers of the first and second kinds to broader families, such as the r-Stirling numbers, q-Stirling numbers, and others. These generalized forms preserve key combinatorial properties while introducing new parameters that allow for deeper interpretations and applications in combinatorics, number theory,

1.9.1 The r-Stirling Numbers of the First Kind

Broder [3] introduced the r-Stirling numbers and studied their identities, recurrence relations, generating functions, and combinatorial meanings.

Definition 1.10 *The r-Stirling numbers of the first kind denoted $|s(n+r, k+r)|$, represent the number of ways to arrange the elements of the set n into k distinct cycles. These numbers are described by the generating function:*

$$\sum_{n \geq 0} s(n+r, k+r) \frac{x^n}{n!} = \frac{(\ln(x+1))^k}{k!(1+x)^r}, \tag{1.13}$$

when $r = 0$, we get the Stirling numbers of the first kind $s(n,k)$ (1.8). These numbers are defined using the base change formula:

$$(x+r)_n = \sum_{k=0}^{n} s(n+r, k+r) x^k. \tag{1.14}$$

1.9.2 The r-Stirling Numbers of the Second Kind

Definition 1.11 *The r-Stirling numbers of the second kind, denoted $\left\{ {n \atop k} \right\}_r$, count the number of ways to partition a set of n elements into k non-empty subsets (blocks), with the condition that the first r elements must each belong to a distinct block. These numbers are defined by the exponential generating function:*

$$\sum_{n \geq k} \left\{ {n+r \atop k+r} \right\}_r \frac{x^n}{n!} = e^{rx} \frac{(e^x - 1)^k}{k!}.$$

For $r = 0$ or $r = 1$, we retrieve the classical Stirling numbers of the second kind (1.10). They also appear as the coefficients in the expansion of the monomial $(x + r)^n$, that is,

$$(x + r)^n = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r (x)_k .$$

Example 1.8 *The set $[4] = 1, 2, 3, 4$ has 5 partitions into three blocks, such that the first two elements 1 and 2 are in distinct blocks. The partitions are*

$$\{1\}\,\{2,3\}\,\{4\}$$

$$\{1\}\,\{3,4\}\,\{2\}$$

$$\{1\}\,\{2,4\}\,\{3\}$$

$$\{2\}\,\{1,3\}\,\{4\}$$

$$\{2\}\,\{1,4\}\,\{3\}$$

As a result: $\left\{ {4 \atop 3} \right\}_2 = 5$.

Properties

1. **Generating function**: The ordinary generating functions for the sequence of r-Stirling numbers of the second kind are given by:

$$\sum_{n \geq k} \left\{ {n+r \atop k+r} \right\}_r x^n = \frac{x^k}{(1 - rx)(1 - (r+1)x) \ldots (1 - (r+k)x)}, \quad k \geq r \geq 0$$

2. **Explicit formula**: For $\left\{ {n \atop k} \right\}_r$, the explicit formula is given by:

$$\left\{ {n+r \atop k+r} \right\}_r = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (r + j)^n .$$

3. **Reccurrence relations**: The r-Stirling numbers of the second kind satisfy the following recurrence relation :

$$\left\{ {n \atop k} \right\}_r = \delta_{k,r}, \quad n = r$$
$$\left\{ {n \atop k} \right\}_r = 0, \quad n < r, ,$$
$$\left\{ {n \atop k} \right\}_r = k \left\{ {n-1 \atop k} \right\}_r + \left\{ {n-1 \atop k-1} \right\}_r, \quad n > r.$$

Table of r-Stirling Numbers of Second Kind

We present in the following tables the initial values of these numbers for $r = 2$.

$n \backslash k$	2	3	4	5	6	7	8	9
2	1							
3	2	1						
4	4	5	1					
5	8	19	9	1				
6	16	65	55	14	1			
7	32	211	285	125	20	1		
8	64	655	1351	910	245	27	1	
9	128	2059	6069	5901	2380	434	35	1

Table 1.4: The first values of the 2-Stirling numbers of the second kind

1.9.3 The Weighted Stirling Numbers of the Second Kind

Definition 1.12 *The weighted Stirling numbers of the second kind [7, 8], denoted $S_n^k(x)$, are given by the following generating function:*

$$\sum_{n\geq 0} S_n^k(x)\frac{z^n}{n!} = e^{xz}\frac{(e^z-1)^k}{k!}. \tag{1.15}$$

***Explicit Formula**: For $S_n^k(x)$, the explicit formula is given by*

$$S_n^k(x) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(x+j)^n,$$

where

$$S_n^k(x) = \frac{1}{k!}\triangle^k x^n,$$

with

$$\frac{1}{k!}\triangle^k x^n|_{x=0} = \begin{Bmatrix} n \\ k \end{Bmatrix} \quad \text{and} \quad S_n^k(r) = \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r.$$

1.9.4 Whitney and r-Whitney Numbers

Whitney numbers and r-Whitney numbers are generalizations of Stirling numbers. They are used in combinatorics to count special types of arrangements or partitions under additional conditions.

Whitney Numbers of the Second Kind

Definition 1.13 *The Whitney numbers, denoted by $W_m(n,k)$, are introduced in [4] and defined by the exponential generating function:*

$$\sum_{n\geq k} W_m(n,k)\frac{z^n}{n!} = \frac{e^z}{m^k}\cdot\frac{(e^{mz}-1)^k}{k!}.$$

Explicit formula: These numbers can also be expressed through explicit formula:

$$W_m(n,k) = \frac{1}{m^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(mj+1)^n.$$

Recurrence relation: The Whitney numbers are defined by the recurrence relation:

$$W_m(n,k) = (1+mk)W_m(n-1,k) + W_m(n-1,k-1), \quad n \geq k \geq 1.$$

The r-Whitney Numbers of the First and the Second Kinds

Definition 1.14 *The r-Whitney numbers of the first kind $w_{m,r}(n,k)$ and of the second kind $W_{m,r}(n,k)$ were introduced by Mező and Ramírez [23, 24] and are defined via the following generating functions:*

$$\sum_{n=k}^{\infty} w_{m,r}(n,k)\frac{z^n}{n!} = (1+mz)^{\frac{-r}{m}}\frac{\ln^k(1+mz)}{m^k k!},$$

and

$$\sum_{n=k}^{\infty} W_{m,r}(n,k)\frac{z^n}{n!} = \frac{e^{rz}}{m^k}\frac{(e^{mz}-1)^k}{k!}, \tag{1.16}$$

respectively. They are defined by the change of basis:

$$m^n\,(x)_k = \sum_{k=0}^{n} w_{m,r}(n,k)\,(mx+r)^k,$$

and

$$(mx+r)^n = \sum_{k=0}^{n} W_{m,r}(n,k)m^k\,(x)_k\,.$$

For the r-Whitney numbers of the second kind, we have the following **explicit form**:

$$W_{m,r}(n,k) = \frac{1}{m^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(mj+r)^n.$$

En particular, we have

$$W_{1,0}(n,k) = \binom{n}{k}, \quad W_{1,r}(n,k) = \binom{n+r}{k+r}_r, \quad \textit{and} \quad W_{m,1}(n,k) = W_m(n,k).$$

Recurrence relation: The r-Whitney numbers of the first kind and the second kind satisfy the following recurrence relations respectively:

$$w_{m,r}(n,k) = w_{m,r}(n-1,k-1) + (m-nm-r)w_{m,r}(n-1,k),$$

and

$$W_{m,r}(n,k) = W_{m,r}(n-1,k-1) + (km+r)W_{m,r}(n-1,k).$$

The relationship between the r-Whitney numbers and the Stirling weighted numbers of the second kind is given by:

$$W_{m,r}(n,k) = m^{n-k}S_n^k\left(\frac{r}{m}\right). \tag{1.17}$$

1.10 Degenerate Stirling Numbers of the First Kind

The degenerate Stirling numbers of the first kind are a generalization of the classical Stirling numbers of the first kind, obtained by introducing a degeneracy parameter λ. **Carlitz** [14] introduced the concept of degenerate Stirling numbers of the first kind.

Definition 1.15 *The degenerate exponential function $e_\lambda^x(z)$ is defined as:*

$$e_\lambda^x(z) = (1+\lambda z)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{z^n}{n!}, \tag{1.18}$$

where $(x)_{n,\lambda} = x(x-\lambda)\ldots(x-(n-1)\lambda)$ for for $n \geq 1$, $(x)_{0,\lambda} = 1$ and $\lim e_\lambda^x(z) = e^{xz}$ if $\lambda \longmapsto o$.

For brevity, we denote $e_\lambda^x(z)$ by $e_\lambda(z)$. From (1.18), we can derive the inversion function of $e_\lambda(z)$ given by:

$$\log_\lambda(z) = \frac{1}{\lambda}(z^\lambda - 1).$$

Definition 1.16 *Let λ be a real number. Then the degenerate Stirling numbers of the first kind are defined by the generating function:*

$$\sum_{n=k}^{\infty} s_{\lambda}(n,k)\frac{z^n}{n!} = \frac{1}{k!}\left(\frac{(1+z)^{\lambda}-1}{\lambda}\right)^k. \tag{1.19}$$

The degenerate Stirling numbers of the first kind are defined either by

$$(x)_n = \sum_{k=0}^{n} s_{\lambda}(n,k)(x)_{k,\lambda}.$$

1.11 Partial and Complete Bell Polynomials

Partial and complete Bell polynomials are important tools in combinatorics and analysis. They describe the ways to partition a set and appear in formulas involving derivatives, moments, and series expansions. According to [20].

1.11.1 Partial Bell Polynomials

The partial Bell partition polynomials $B_{n,k}(x_1, x_2, \ldots, x_n)$ in the variables $x_n, (n \geq 1)$ of degree k, were introduced by Bell as a mathematical tool for representing the nth derivative of a composite function. They are defined by their generating function:

$$\sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_n)\frac{z^n}{n!} = \frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right)^k, \tag{1.20}$$

and are given explicitly by the formula:

$$B_{n,k}(x_1, x_2, \ldots, x_n) = \sum_{\pi(n,k)} \frac{n!}{k_1!\ldots k_n!}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}\cdots\left(\frac{x_n}{n!}\right)^{k_n},$$

where

$$\pi(n,k) = \left\{\ k_1, \ldots, k_n \in \mathbb{N}^n : \textstyle\sum_{i=1}^{n} k_i = k, \sum_{i=1}^{n} i k_i = n\ \right\}.$$

It is easy to have the following expression:

$$B_{n,k}(ax_1, ax_2, ax_3, \ldots) = a^k B_{n,k}(x_1, x_2, x_3, \ldots). \tag{1.21}$$
$$B_{n,k}\left(ax_1, a^2x_2, a^3x_3, \ldots\right) = a^n B_{n,k}(x_1, x_2, x_3, \ldots).$$

It is well-known that for appropriate choices of the variables x_n, the exponential (partial) Bell partition polynomials can be reduced to some special combinatorial sequences. We will mention the following special cases:

$$s(n,k) = B_{n,k}\left(0!, -1!, 2!, -3!, \ldots, (-1)^{n+1}(n-1)!\right). \tag{1.22}$$

$$\left\{ n \atop k \right\} = B_{n,k}(1, 1, 1, \ldots, 1). \tag{1.23}$$

1.11.2 Complete Bell Polynomials

The (exponential) complete Bell partition polynomials are defined by their generating function as follows

$$\sum_{n=0}^{\infty} B_n(x_1, x_2, \ldots, x_n)\frac{z^n}{n!} = exp\left(\sum_{m=1}^{\infty} x_m \frac{z^m}{m!}\right),$$

from which, it follows that:

$$B_n(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{n} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}). \tag{1.24}$$

1.12 Faà di Bruno's Formula

Faà di Bruno's formula gives the coefficients of the Taylor expansion of the composition of two serie, can be found in [6].

Theorem 1.1 *Let f and g be two Taylor series:*

$$f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}, \quad g(t) = \sum_{n\geq 1} g_n \frac{t^n}{n!}.$$

Let

$$h(t) = f(g(t)) = \sum_{n\geq 0} h_n \frac{t^n}{n!}.$$

Then the coefficients h_n are given by:

$$\begin{cases} h_0 = f_0 \\ \\ h_n = \sum_{k=1}^{n} f_k \, B_{n,k}(g_1, g_2, \dots), \quad n \geq 1 \end{cases}$$

Example 1.9

$$\begin{aligned} g(x) = \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \\ &= \sum_{n\geq 1} (-1)^{n+1}(n-1)! \frac{x^n}{n}, \end{aligned}$$

and

$$\begin{aligned} f(x) = \frac{1}{1-x} &= 1 + x + x^2 + \cdots \\ &= \sum_{n\geq 0} n! \frac{x^n}{n!}. \end{aligned}$$

Then,

$$f(g(x)) = \frac{1}{1 - \ln(1+x)}$$

According to Faà di Bruno's formula, we obtain the coefficients h_n

$$\begin{cases} h_0 = 1 \\ \\ h_n = \sum_{k=1}^{n} k! \, B_{n,k}(1, -1, 2, -6, \dots, (-1)^{n-k+2}(n-k)). \end{cases}$$

2

Appell-Type Polynomials

After establishing in chapter one the fundamental concepts related to polynomials, generating functions, and recurrence relations, this chapter is devoted to the study of some Appell-type polynomials, a special class of polynomials characterized by a fundamental differential relation that facilitates both algebraic and analytical treatment.

We examine in particular the Bernoulli, Euler, Genocchi, Bell, and Hermite polynomials, focusing on their essential properties, recurrence relations, and generating functions. We also consider certain generalizations proposed in the literature, which provide a broader understanding of the structure and potential applications of these mathematical objects.

This chapter is based on several references that have addressed the topic from various perspectives, including the works of [5, 10, 11, 12, 13, 17, 18, 19, 20, 21, 22, 25, 27, 29, 31, 32].

2.1 Appell Polynomials

According to [20], Appell polynomials are a special class of polynomial sequences characterized by a simple differential relation, making them highly significant in mathematical analysis, algebra, and probability theory, these sequences were first introduced by Paul Émile Appell in 1880.

Recall that a sequence of polynomials $(P_n(x))_{n\geq 0}$ is called an Appell sequence if it satisfies:

$$\frac{d}{dx}P_n(x) = nP_{n-1}(x) \quad and \quad P_0(x) = 1.$$

This means that differentiating any term in the sequence returns the previous term multiplied by its degree. This property is shared by well-known polynomial families such as Hermite, Bernoulli, and Euler polynomials.

Appell polynomials can often be represented by a generating function of the form:

$$A(z)e^{xz} = \sum_{n=0}^{\infty} P_n(x)\frac{z^n}{n!}, \tag{2.1}$$

where $A(z)$ is an analytic function that determines the type of the sequence

$$A(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \quad with \quad a_0 = 1. \tag{2.2}$$

After defining the generating function of Appell polynomials, the explicit form $(P_n(x))_{n\geq 0}$ is defined as follows

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k}. \tag{2.3}$$

2.2 Bernoulli Polynomials

Bernoulli polynomials are a sequence of polynomials related to Bernoulli numbers. They are important in analysis, number theory, and have a well-known generating function and recurrence relation [10, 19].

Definition 2.1 *The exponential generating function of the sequence of Bernoulli polynomials, denoted $B_n(x)$, is given by:*

$$\frac{ze^{zx}}{e^z-1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}, \quad |z| < 2\pi. \tag{2.4}$$

Thus, the first four such polynomials, respectively, are

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \qquad B_2(x) = x^2 - x + \frac{1}{6}, \qquad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

We recall some fundamental properties of the Bernoulli polynomials that will be useful in our study:

Theorem 2.1 *The Bernoulli polynomials $B_n(x)$ satisfy the following identities:*

1. $\forall n \geq 1,\ B_n'(x) = nB_{n-1}(x)$.
2. $\forall n \geq 1, \int_0^1 B_n(x)\mathrm{d}x = 0$.

Proof. Differentiating the left-hand side of (2.4) with respect to x, we get

$$\begin{aligned}
\frac{d}{dx}\left(\frac{ze^{zx}}{e^z-1}\right) &= \frac{z}{e^z-1}\frac{de^{zx}}{dx} \\
&= z\frac{ze^{zx}}{e^x-1} \\
&= z\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}z^n \\
&= \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}z^{n+1} \\
&= \sum_{n=1}^{\infty} B_{n-1}(x)\frac{z^n}{(n-1)!} \\
&= \sum_{n=1}^{\infty} B_{n-1}(x)\frac{z^n}{(n-1)!}.
\end{aligned}$$

On the other hand, it follows from (2.4) that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{ze^{zx}}{e^z-1}\right) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}\right) \\
&= \sum_{n=1}^{\infty}\left(\frac{dB_n(x)}{dx}\right)\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} B_n'(x)\frac{z^n}{n!}.
\end{aligned}$$

Equating the two series yields

$$B_0'(x) + \sum_{n=1}^{\infty} B_n'(x)\frac{z^n}{n!} = \sum_{n=1}^{\infty} B_{n-1}(x)\frac{z^n}{(n-1)!},$$

and since $B_0'(x) = 0$ ($B_0(x) = 1$), thus for $n \geq 1$, we have

$$\sum_{n=1}^{\infty} B_n'(x)\frac{z^n}{n!} = \sum_{n=1}^{\infty} nB_{n-1}(x)\frac{z^n}{n!}.$$

Thus, the first part is proved.

Now, we integrate the left-hand side of (2.4) with respect to x, we obtain

$$\begin{aligned}\int_0^1 \frac{ze^{zx}}{e^z-1}dx &= \frac{z}{e^z-1}\int_0^1 e^{zx}dx \\ &= \frac{z}{e^z-1}\left[\frac{e^{zx}}{z}\right]_0^1 = 1.\end{aligned}$$

On the other hand, integrating the right-hand side of (2.4) with respect to x, we find

$$\int_0^1 \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}dx = \sum_{n=0}^{\infty}\left(\int_0^1 B_n(x)dx\right)\frac{z^n}{n!}.$$

Therefore, we obtain the identity

$$\sum_{n=0}^{\infty}\left(\int_0^1 B_n(x)dx\right)\frac{z^n}{n!} = 1,$$

which is equivalent to

$$\int_0^1 B_0(x)dx + \sum_{n=1}^{\infty}\left(\int_0^1 B_n(x)dx\right)\frac{t^n}{n!} = 1,$$

and also

$$1 + \sum_{n=1}^{\infty}\left(\int_0^1 B_n(x)dx\right)\frac{t^n}{n!} = 1.$$

This completes the proof of 2. ∎

Here is an important theorem about Bernoulli polynomials:

Theorem 2.2 *For any integer $n \geq 0$, we have*

$$B_n(x+1) = \sum_{k=0}^{n}\binom{n}{k}B_k(x). \tag{2.5}$$

Proof. Using the generating function of the Bernoulli polynomials in the Definition 2.1, we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(x+1)\frac{z^n}{n!} &= \frac{ze^{z(x+1)}}{e^z-1} \\ &= \frac{ze^{zx}}{e^z-1}e^z \\ &= \left(\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}B_k(x)\right)\frac{z^n}{n!}.\end{aligned}$$

By identification the first and the last formulas, we obtain:

$$B_n(x+1) = \sum_{k=0}^{n}\binom{n}{k}B_k(x).$$

The explicit formula for Bernoulli polynomials is given by the following equation:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \tag{2.6}$$

where $B_k := B_k(0)$ is the Bernoulli number for each $k = 0, 1, ..., n$.

Special Case

According to [19]. In this special case, we consider the Bernoulli polynomial $B_n(x)$ evaluated at $x = 0$. By definition, the exponential generating function for Bernoulli numbers is given by:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \tag{2.7}$$

and the explicit formula is defined as follows:

Theorem 2.3 *For $n \geq 0$, we have*

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} \left\{ n \atop k \right\}.$$

Proof. We assume that

$$z = \ln(1 + (e^z - 1)),$$

such that

$$\begin{aligned} \ln(1 + (e^z - 1)) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(e^z - 1)^n}{n} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(e^z - 1)^{n+1}}{n+1}. \end{aligned}$$

On the other hand, using the generating function (2.7), we find

$$\begin{aligned} \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} &= \frac{z}{e^z - 1} \\ &= \frac{1}{e^z - 1} \sum_{k=0}^{\infty} (-1)^k \frac{(e^z - 1)^{k+1}}{k+1} \\ &= (e^z - 1)^{-1} \sum_{k=0}^{\infty} (-1)^k \frac{(e^z - 1)^{k+1}}{k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(e^z - 1)^k}{k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{k!}{k+1} \frac{(e^z - 1)^k}{k!} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{k!}{k+1} \sum_{n=0}^{\infty} \left\{ n \atop k \right\} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} (-1)^k \frac{k!}{k+1} \left\{ n \atop k \right\} \right) \frac{z^n}{n!}. \end{aligned}$$

By comparing coefficients, we get:

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} \left\{ n \atop k \right\}.$$

This completes the proof of the desired result.

2.3 Euler Polynomials

Euler polynomials are polynomial functions defined by a specific generating function, and they play an important role in mathematical analysis and number theory [5, 10].

Definition 2.2 *The exponential generating function of the sequence of Euler polynomials, denoted $E_n(x)$, is given by:*

$$\frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}, \quad |z| < \pi. \tag{2.8}$$

Thus, the first four Euler polynomials are given by:

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x, \quad E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}.$$

The explicit formula for Euler polynomials is given by the following theorem:

Theorem 2.4 *For $n \geq 0$, we have*

$$E_n(x) = \sum_{k=0}^{n} \frac{1}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x+j)^n. \tag{2.9}$$

Proof. We consider the exponential generating function of the Euler polynomials (2.8)

$$\frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!}.$$

Let us define:

$$\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x+j)^n \right) \frac{z^n}{n!}.$$

We interchange the order of summation:

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x+j)^n \right) \frac{z^n}{n!} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\sum_{n=0}^{\infty} \frac{(x+j)^n z^n}{n!} \right) \\
&= \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} e^{(x+j)z} \\
&= e^{xz} \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} e^{jz} \\
&= e^{xz} \sum_{k=0}^{\infty} \frac{1}{2^k} (1 - e^z)^k, \\
&= e^{xz} \sum_{k=0}^{\infty} \left(\frac{1 - e^z}{2} \right)^k \\
&= e^{xz} \cdot \frac{1}{1 - \frac{1-e^z}{2}} = \frac{2e^{xz}}{e^z + 1}.
\end{aligned}$$

Hence, we recover the generating function of the Euler polynomials. This proves the identity (2.9).

■

The Euler polynomials satisfy the following shift relation:

Theorem 2.5 *For any integer $n \geq 0$, we have*

$$E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x).$$

Proof. We have

$$\frac{2e^{(x+1)z}}{e^z+1} = \sum_{n=0}^{\infty} E_n(x+1)\frac{z^n}{n!},$$

then

$$\begin{aligned}
\frac{2e^{(x+1)z}}{e^z+1} &= \frac{2e^{xz}}{e^z+1}e^z \\
&= \sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!}\sum_{n=0}^{\infty}\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{E_k(x)}{k!}\frac{n!}{(n-k)!}\right)\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_k(x)\right)\frac{z^n}{n!}.
\end{aligned}$$

By identification, we obtain

$$E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x).$$

Hence the result.

■

Special Case

In this special case, we consider the Euler polynomial $E_n(x)$ evaluated at $x = 0$. By Definition 2.2, we have:

$$E_n(0) = E_n.$$

This means that Euler numbers E_n are the values of Euler polynomials at zero.
The exponential generating function for Euler numbers E_n is:

$$\frac{2}{e^z+1} = \sum_{n=0}^{+\infty} E_n\frac{z^n}{n!}.$$

An explicit formula for E_n is given by:

$$E_n = \sum_{k=0}^{n}\frac{1}{2^k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(j)^n.$$

2.4 Genocchi Polynomials

Genocchi polynomials are a part of the Appell polynomials family. They are used to study the properties of Genocchi numbers and appear in areas like mathematical analysis and number theory. They are characterized by recursive relations and generating functions, which make their study easier. According to [15, 26].

Definition 2.3 *The Genocchi polynomials $G_n(x)$ are defined by the generating function:*

$$\frac{2z}{e^z+1}e^{xz} = \sum_{n=0}^{\infty} G_n(x)\frac{z^n}{n!}, \quad |z| < \pi. \tag{2.10}$$

The Genocchi polynomials are also defined by the explicit formula:

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} G_k,$$

where $G_k := G_k(0)$ is the Genocchi numbers.

Special Case

When we substitute $x = 0$ into the expression of Genocchi polynomials (2.10), we get the exponential generating function for the classical Genocchi numbers given by:

$$\frac{2z}{e^z+1} = \sum_{n=0}^{\infty} G_n\frac{z^n}{n!}, \quad |z| < \pi. \tag{2.11}$$

The relationship between the Genocchi numbers G_n and the Bernoulli numbers B_n and the Euler polynomials $E_n(x)$ is

$$G_n = 2\left(1-2^n\right)B_n = nE_{n-1}(0), \quad for \quad n \geq 1.$$

2.5 Bell Polynomials

Bell polynomials are used in analysis and combinatorics. They help compute composite derivatives and count set partitions. In this work, we study them for their useful properties and links to special number sequences [32].

Definition 2.4 *The exponential generating function of the Bell polynomials is defined by*

$$\sum_{n=0}^{\infty} \mathcal{B}_n(x)\frac{z^n}{n!} = e^{x(e^z-1)}. \tag{2.12}$$

Theorem 2.6 *For $n \in \mathbb{N}$, the Bell polynomials, they are defined by*

$$\mathcal{B}_n(x) = \sum_{k=0}^{n} \left\{ n \atop k \right\} x^k. \tag{2.13}$$

Proof. By using the exponential generating function (2.12)

$$\sum_{n=0}^{\infty} \mathcal{B}_n(x)\frac{z^n}{n!} = e^{x(e^z-1)},$$

and the exponential generating function of the Stirling numbers of the second kind:

$$\frac{(e^z-1)^k}{k!} = \sum_{n=k}^{\infty} \left\{ n \atop k \right\} \frac{z^n}{n!},$$

we obtain:

$$\begin{aligned} e^{x(e^z-1)} &= \sum_{k=0}^{\infty}(e^z-1)^k\frac{x^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(k!\sum_{n=k}^{\infty}\left\{{n \atop k}\right\}\frac{z^n}{n!}\right)\frac{x^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(k!\sum_{n=0}^{n}\left\{{n \atop k}\right\}\frac{x^k}{k!}\right)\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\{{n \atop k}\right\}x^k\right)\frac{z^n}{n!}. \end{aligned}$$

By matching the coefficients of $\frac{z^n}{n!}$ on both sides, we obtain the desired relation. ■

The first Bell polynomials are

$$\begin{aligned} &\mathcal{B}_0(x) = 1, \\ &\mathcal{B}_1(x) = x, \\ &\mathcal{B}_2(x) = x + x^2, \\ &\mathcal{B}_3(x) = x + 3x^2 + x^3, \\ &\mathcal{B}_4(x) = x + 7x^2 + 6x^3 + x^4. \end{aligned}$$

We now give a formula for the Bell numbers, known as **Dobinski's Formula**:

Theorem 2.7 *The Bell polynomials $\mathcal{B}_n(x)$ are given by:*

$$\mathcal{B}_n(x) = e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k.$$

Proof. By means of the generating function (2.12)

$$\sum_{n=0}^{\infty}\mathcal{B}_n(x)\frac{z^n}{n!} = e^{x(e^z-1)},$$

we have

$$\begin{aligned} e^{x(e^z-1)} &= e^{-x}e^{xe^z} \\ &= e^{-x}\sum_{k=0}^{\infty}\frac{x^k}{k!}(e^z)^k \\ &= e^{-x}\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{n=0}^{\infty}k^n\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(e^{-x}\sum_{k=0}^{\infty}\frac{k^n}{k!}x^k\right)\frac{z^n}{n!}. \end{aligned}$$

By matching the coefficients of $\frac{z^n}{n!}$ on both sides, we obtain the desired relation. ■

The Bell polynomials can be calculated using a recurrence relation given in the following theorem:

Theorem 2.8 *The recurrence relation of the Bell polynomials is given by:*

$$\mathcal{B}_0(x) = 1 \quad \textit{and} \quad \mathcal{B}_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k(x).$$

Proof. We start from the exponential generating function of the Bell polynomials (2.12):

$$\sum_{n=0}^{\infty} \mathcal{B}_n(x) \frac{z^n}{n!} = \exp\left(x(e^z - 1)\right).$$

Differentiating both sides with respect to z, we get:

$$\sum_{n=1}^{\infty} \mathcal{B}_n(x) \frac{z^{n-1}}{(n-1)!} = \exp\left(x(e^z - 1)\right) x e^z.$$

We set $n - 1 = n$, we find

$$\sum_{n=0}^{\infty} \mathcal{B}_{n+1}(x) \frac{z^n}{n!} = \exp\left(x(e^z - 1)\right) x e^z.$$

Then

$$\begin{aligned} \exp\left(x(e^z - 1)\right) x e^z &= x \left(\sum_{n=0}^{\infty} \mathcal{B}_n(x) \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(x \sum_{k=0}^{n} \binom{n}{k} B_k(x) \right) \frac{z^n}{n!}. \end{aligned}$$

By comparing the coefficients of $\frac{z^n}{n!}$, we obtain:

$$\mathcal{B}_{n+1}(x) = x \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k(x).$$

■

Special Case

When we substitute $x = 0$ in the Bell polynomials (2.12), we obtain the Bell numbers. These numbers appear as the coefficients in the exponential generating function.

$$\sum_{k=0}^{n} \mathcal{B}_n \frac{z^n}{n!} = e^{(e^z - 1)}. \tag{2.14}$$

The recurrence relation is given by:

Theorem 2.9 *For $n \geq 0$, we have*

$$\mathcal{B}_0 = 1 \quad \textit{and} \quad \mathcal{B}_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k.$$

Proof. Using the generating function (2.14):

$$\sum_{k=0}^{n} \mathcal{B}_n \frac{z^n}{n!} = e^{(e^z - 1)}.$$

We have

$$\begin{aligned}\frac{d}{dz}\left(e^{(e^z-1)}\right) &= e^z e^{(e^z-1)} \\ &= \left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{k=0}^{n}\mathcal{B}_n\frac{z^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\right)\frac{z^n}{n!}.\end{aligned}$$

On the other hand, we have

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty}\mathcal{B}_n\frac{z^n}{n!}\right) = \sum_{n=0}^{\infty}\mathcal{B}_{n+1}\frac{z^n}{n!}.$$

By identification, we find:

$$\mathcal{B}_{n+1} = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k.$$

■

2.6 Hermite Polynomials

Similar to Appell polynomials, Hermite polynomials represent an important example of polynomial sequences characterized by explicit generating functions and recurrence relations [18].

Definition 2.5 *The Hermite polynomials $\mathcal{H}_n(x)$ are defined by the generating function:*

$$e^{2xz-z^2} = \sum_{n=0}^{\infty}\mathcal{H}_n(x)\frac{z^n}{n!}. \tag{2.15}$$

Alternatively, they can be expressed using the Rodrigue's formula:

$$\mathcal{H}_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right). \tag{2.16}$$

Moreover, the Hermite polynomials satisfy the following recurrence relation:

$$n\mathcal{H}_n(x) = 2x\mathcal{H}_{n-1}(x) - 2\mathcal{H}_{n-2}(x), \quad n \geq 2. \tag{2.17}$$

Special Case

Whene $x = 0$, Hermite polynomials are reduced to the Hermite numbers, which are given by the generating function:

$$e^{-z^2} = \sum_{n=0}^{\infty}\mathcal{H}_n\frac{z^n}{n!}. \tag{2.18}$$

2.7 Generalizations of Classical Special Polynomials

2.7.1 p-Bernoulli Polynomials

In on p-Bernoulli numbers and polynomials [27], Rahmani defines the p-Bernoulli numbers and polynomials and discusses their generating functions and recurrence relations.

Definition 2.6 *The p-Bernoulli polynomials are denoted by $B_{n,p}(x)$, and their generating function is defined by the following expression:*

$$ {}_2F_1\left(\begin{array}{c}1,1\\ p+2\end{array};1-e^z\right)e^{xz}\sum_{n=0}^{\infty}B_{n,p}(x)\frac{z^n}{n!}. \tag{2.19}$$

Since the generating function is written in the form

$$A(z)e^{xz}=\sum_{n=0}^{\infty}\frac{B_{n,p}(x)}{n!}z^n,$$

the p-Bernoulli polynomials belong to the class of Appell polynomials. The first few p-Bernoulli polynomials are:

$$\begin{aligned}
&B_{0,p}(x)=1,\\
&B_{1,p}(x)=x-\frac{1}{p+2},\\
&B_{2,p}(x)=x^2-\frac{2x}{p+2},\\
&B_{3,p}(x)=x^3-\frac{3x^2}{p+2}-\frac{3(p-1)x}{(p+2)(p+3)}-\frac{p(p-5)}{(p+2)(p+3)(p+4)}.
\end{aligned}$$

The following results are correct.

Theorem 2.10 *For $n\geq 0$ and $p\geq 0$, we have:*

1. $\dfrac{d}{dx}B_{n,p}(x)=nB_{n-1,p}(x).$
2. $\displaystyle\int_0^x B_{n,p}(t)dt=\frac{B_{n+1,p}(x)-B_{n+1,p}}{n+1}.$

Proof.

1. *We differentiate the right-hand side of (2.19) with respect to x, and we otain:*

$$\begin{aligned}
\frac{d}{dx}\left({}_2F_1\left(\begin{array}{c}1,1\\ p+2\end{array};1-e^z\right)e^{xz}\right)&=z\sum_{n=0}^{\infty}B_{n,p}(x)\frac{z^n}{n!}=\sum_{n=0}^{\infty}B_{n,p}(x)\frac{z^{n+1}}{n!}\\
&=\sum_{n=1}^{\infty}B_{n-1,p}(x)\frac{z^n}{(n-1)!}.
\end{aligned}$$

Similarly, we differentiate the left-hand side of (2.19)

$$\begin{aligned}
\frac{d}{dx}\left({}_2F_1\left(\begin{array}{c}1,1\\ p+2\end{array};1-e^z\right)e^{x}z\right)&=\frac{d}{dx}\left(\sum_{n=0}^{\infty}B_{n,p}(x)\frac{z^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\frac{d}{dx}B_{n,p}(x)\frac{z^n}{n!}.
\end{aligned}$$

Then the equality of two series gives us

$$\frac{d}{dx}B_{0,p}(x)+\sum_{n\geq 1}\frac{d}{dx}B_{n,p}(x)\frac{z^n}{n!}=\sum_{n=1}^{\infty}\frac{d}{dx}B_{n-1,p}(x)\frac{z^n}{(n-1)!},$$

and since $\dfrac{d}{dx}B_{0,p}(x)=0$, then

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{d}{dx}B_{n,p}(x)\frac{z^n}{(n-1)!}=\sum_{n=1}^{\infty}B_{n-1,p}(x)\frac{z^n}{(n-1)!}.$$

By comparing the coefficients of $\frac{z^n}{(n-1)!}$, *we obtain*

$$\text{for all } n\geq 1,\quad \frac{d}{dx}B_{n,p}(x)=nB_{n-1,p}(x).$$

2. *According to 1. we have* $B_{n,p}(t)=\dfrac{1}{(n+1)}\dfrac{d}{dt}B_{n+1,p}(t)$. *Now, we integrate this identity with respect to* t *from* 0 *to* x, *and we obtain*

$$\begin{aligned}\int_0^x B_{n,p}(t)dt&=\frac{1}{n+1}\int_0^x\frac{d}{dt}B_{n+1,p}(t)dt\\&=\left[\frac{B_{n+1,p}(t)}{n+1}\right]_0^x\\&=\frac{B_{n+1,p}(x)-B_{n+1,p}(0)}{n+1}\\&=\frac{B_{n+1,p}(x)-B_{n+1,p}}{n+1}.\end{aligned}$$

Hence the result.

■

The three-term recurrence relation for computing the p-Bernoulli polynomials is given in the following theorem:

Theorem 2.11 *For* $p\geq 0$, *we have*

$$B_{0,p}(x)=1;B_{n+1,p}(x)=(x+p)B_{n,p}(x)-\frac{(p+1)^2}{p+2}B_{n,p+1}(x).$$

Theorem 2.12 *One has*

$$B_{n,p}(x)=\frac{p+1}{p!}\sum_{k=0}^{n}(-1)^k\frac{(k+p)!}{k+p+1}S_n^k(x+p),\tag{2.20}$$

where

$$S_n^k(x)=\frac{1}{k!}\Delta^k x^n=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^n,\tag{2.21}$$

Δ *being the forward difference operator.*

Special Case

In the special case where $p=0$, we obtain the classical Bernoulli polynomials $B_n(x)$ (2.4). By setting $x=0$, we get the p-Bernoulli numbers $B_{n,p}$, which are defined by the generating function:

$$\sum_{n=0}^{\infty}B_{n,p}\frac{z^n}{n!}={}_2F_1\left(\begin{matrix}1,1\\p+2\end{matrix};1-e^z\right).$$

The explicit form of $B_{n,p}$ is given by the following theorem:

Theorem 2.13 *For* $p\geq -1$, *we have*

$$B_{n,p}=\sum_{k=0}^{n}(-1)^k\binom{k+p+1}{k}^{-1}k!\left\{\begin{matrix}n\\k\end{matrix}\right\}.$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{k+p+1}{k}^{-1}k!\left\{{n \atop k}\right\}\frac{z^n}{n!} &= \sum_{k=0}^{\infty}(-1)^k k!\binom{k+p+1}{k}^{-1}\sum_{n=k}^{\infty}\left\{{n \atop k}\right\}\frac{t^n}{n!} \\
&= \sum_{k=0}^{\infty}\binom{k+p+1}{k}^{-1}(1-e^z)^k \\
&= \sum_{k=0}^{\infty}\frac{(p+1)!k!k!}{(k+p+1)!}\frac{(1-e^z)^k}{k!} \\
&= \sum_{k=0}^{\infty}\frac{(1)^k(1)^k}{(p+2)^k}\frac{(1-e^z)^k}{k!} \\
&= {}_2F_1\left(\begin{matrix}1,1\\ p+2\end{matrix};1-e^z\right) \\
&= \sum_{n=0}^{\infty}B_{n,p}\frac{z^n}{n!}.
\end{aligned}$$

By matching the coefficients of $\frac{t^n}{n!}$ on both sides, we obtain the desired relation. ∎

The recurrence relation for the Bernoulli numbers is given in the following theorem:

Theorem 2.14 *For $n \geq 0$, we get*

$$B_{0,p} = 1, \quad B_{n+1,p} = pB_{n,p} - \frac{(p+1)^2}{p+2}B_{n,p+1}.$$

Proof. We begin with the recurrence relation of the Stirling numbers of the first kind:

$$s(n+1,k) = s(n,k-1) - ns(p,k),$$

we have

$$\begin{aligned}
B_{n,p+1} &= (-1)^{p+1}\frac{p+2}{(p+1)!}\sum_{k=1}^{p+1}\left(s(p,k-1) - ps(p,k)\right)B_{n+k} \\
&= (-1)^{p+1}\frac{p+2}{(p+1)!}\sum_{k=1}^{p+1}s(p,k-1) - p(-1)^{p+1}\frac{p+2}{(p+1)!}\sum_{k=1}^{p+1}s(p,k)B_{n+k}.
\end{aligned}$$

By the change of variable. $k-1=k$, we get

$$\begin{aligned}
B_{n,p+1} &= -\frac{p+2}{(p+1)^2}(-1)^p\frac{p+1}{p!}\sum_{k=0}^{p}s(p,k)B_{n+k+1} + \frac{(p+2)p}{(p+1)^2}(-1)^p\frac{p+1}{p!}\sum_{k=1}^{p}s(p,k)B_{n+k} \\
&= -\frac{p+2}{(p+1)^2}B_{n+1,p} + \frac{(p+2)p}{(p+1)^2}B_{n,p}.
\end{aligned}$$

Then

$$B_{n+1,p} = pB_{n,p} - \frac{(p+1)^2}{p+2}B_{n,p+1}.$$

Hence the result. ∎

2.7.2 Bernoulli Polynomials of Order α

In this part, the authors [5, 17], generalized the Bernoulli polynomials to order α and studied them using a special generating function.

Definition 2.7 *The Bernoulli polynomials of order α, $B_n^{(\alpha)}(x)$, are a generalization of the classical Bernoulli polynomials. They are represented by a generating function of the form:*

$$\frac{z^\alpha}{(e^z-1)^\alpha}e^{xz}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{z^n}{n!},\quad |z|\leq 2\pi. \tag{2.22}$$

Setting $x=0$ in (2.22), we get $B_n^{(\alpha)}(0):=B_n^{(\alpha)}$ known as the Bernoulli numbers of order α. We also note that when $\alpha=1$ in (2.22), the polynomials $B_n^{(\alpha)}(x)$ and numbers $B_n^{(\alpha)}$ reduce to the classical Bernoulli polynomials $B_n(x)$ and numbers B_n.

When applying the same explicit form to Bernoulli polynomials of order α, we would have:

$$B_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}x^{n-k}. \tag{2.23}$$

2.7.3 The Generalized Euler Polynomials

Euler polynomials of order α are a generalization of the classical Euler polynomials [5].

Definition 2.8 *The generalization of the classical Euler polynomials, defined by means of the following generating function:*

$$\left(\frac{2}{e^z+1}\right)^\alpha e^{xz}=\sum_{n=0}^{\infty}E_n^{(\alpha)}(x)\frac{z^n}{n!},\quad |z|\leq 2\pi. \tag{2.24}$$

An explicit formula for the generalized Euler polynomials is given by the following theorem.

Theorem 2.15 *The following relationship holds true:*

$$E_n^{(\alpha)}(x)=\sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k S_n^k(x).$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k S_n^k(x)\right)\frac{z^n}{n!} &= \sum_{k=0}^{n}\frac{(-1)^k}{2^k}(\alpha)_k\sum_{n=k}^{\infty}S_n^k(x)\frac{z^n}{n!}\\
&= e^{xz}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}(\alpha)_k\frac{1}{k!}(e^x-1)^k\\
&= e^{xz}\sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}\left(\frac{1-e^x}{2}\right)^k\\
&= e^{xz}\left(1-\frac{1-e^x}{2}\right)^{-\alpha}\\
&= \left(\frac{2}{e^z+1}\right)^\alpha e^{xz}\\
&= \sum_{n=0}^{\infty}E_n^{(\alpha)}(x)\frac{z^n}{n!}.
\end{aligned}$$

By comparing the coefficients of $\frac{z^n}{n!}$. The proof of Theorem 2.15 is thus completed.

Special Case

In special case, we observe that

$$E_n^{(\alpha)}(0) = E_n^{(\alpha)} \tag{2.25}$$

and, therefore,

$$\left(\frac{2}{e^z+1}\right)^{\alpha} = \sum_{n=0}^{\infty} E_n^{(\alpha)} \frac{z^n}{n!},$$

where $E_n^{(\alpha)}$ are the generalized Euler numbers.

2.7.4 The Frobenius-Euler Polynomials

Among the various generalizations of classical special polynomials, the Frobenius–Euler polynomials (Eulerian polynomials) arise as a natural extension of Euler polynomials, incorporating an additional parameter that enriches their structure and applications. According to [11, 20].

Definition 2.9 *For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius–Euler polynomials $H_n(x|\lambda)$, are defined by the following generating function:*

$$\frac{1-\lambda}{e^z-\lambda} e^{xz} = \sum_{n=0}^{\infty} H_n(x|\lambda) \frac{z^n}{n!}. \tag{2.26}$$

The explicit formula of the Frobenius–Euler polynomials is given in the following theorem:

Theorem 2.16 *For $n \geq 0$, we have*

$$H_n(\lambda, x) = \sum_{k=0}^{n} \binom{n}{k} H_k(\lambda) x^{n-k}, \tag{2.27}$$

such that $H_k(\lambda)$, the nth Frobenius–Euler numbers, is defined by the following generating function:

$$\frac{1-\lambda}{e^z-\lambda} = \sum_{n=0}^{\infty} H_n(\lambda) \frac{z^n}{n!}. \tag{2.28}$$

Proof. We have the following generating function (2.26)

$$\frac{1-\lambda}{e^z-\lambda} e^{xz} = \sum_{n=0}^{\infty} H_n(x|\lambda) \frac{z^n}{n!},$$

then

$$\begin{aligned} \frac{1-\lambda}{e^z-\lambda} e^{xz} &= \left(\sum_{n=0}^{\infty} H_n(\lambda) \frac{z^n}{n!}\right) \left(\sum_{n=0}^{\infty} x^n \frac{z^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} H_k(\lambda) x^{n-k}\right) \frac{z^n}{n!}. \end{aligned}$$

By comparing both sides, we find that

$$H_n(\lambda, x) = \sum_{k=0}^{n} \binom{n}{k} H_k(\lambda) x^{n-k}.$$

Thus, we obtain the desired result.

Special Case

For $x = 0$ in (2.26), we easily get $H_n(x|-1) = E_n(x)$ defined in (2.8).

2.7.5 The Frobenius–Euler Polynomials of Order α

The Frobenius–Euler polynomials of order α are a generalization of the classical Euler polynomials, involving two parameters α and λ. They are defined via an exponential generating function and are useful in exploring various analytic and combinatorial properties [12].

Definition 2.10 *For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius–Euler polynomials $H_n^{(\alpha)}(x|\lambda)$ of order α, are defined by the following generating function:*

$$\left(\frac{1-\lambda}{e^z-\lambda}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x|\lambda)\frac{z^n}{n!}. \tag{2.29}$$

The explicit formula for the Frobenius–Euler polynomials of order α is given in the following theorem:

Theorem 2.17 *For $n \geq 0$, we have*

$$H_n^{(\alpha)}(x|\lambda) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}^{(\alpha)}(\lambda)x^k. \tag{2.30}$$

Proof. By means of the generating function (2.29), we obtain

$$\left(\frac{1-\lambda}{e^z-\lambda}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x|\lambda)\frac{z^n}{n!}.$$

We have

$$\begin{aligned}\left(\frac{1-\lambda}{e^z-\lambda}\right)^{\alpha} e^{xz} &= \sum_{n=0}^{\infty} H_n^{(\alpha)}(\lambda)\frac{z^n}{n!}\sum_{n=0}^{\infty} x^n\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}H_{n-k}^{(\alpha)}(\lambda)x^k\right)\frac{z^n}{n!}.\end{aligned}$$

By identification, we obtain

$$H_n^{(\alpha)}(x|\lambda) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}^{(\alpha)}(\lambda)x^k.$$

■

Special Case

In the special case when $\alpha = 1$, the Frobenius–Euler polynomials of order α are reduced to the Frobenius–Euler polynomials $H_n(x|\lambda)$ defined in (2.26).

By substituting $x = 0$, we get $H_n^{(\alpha)}(\lambda)$ are called the Frobenius-Euler numbers of order α is given by the generating functuion

$$\left(\frac{1-\lambda}{e^z-\lambda}\right)^{\alpha} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(\lambda)\frac{z^n}{n!},$$

and when $\lambda = -1$,

$$H_n^{(\alpha)}(x \mid -1) := E_n^{(\alpha)}(x),$$

are the Euler polynomials of order α in (2.24).

2.7.6 The Generalized Eulerian Polynomial

The generalized Eulerian polynomials extend classical Eulerian polynomials. They help study permutations more precisely and appear in combinatorics [31].

Definition 2.11 *The generalized Eulerian polynomials $A_n(x|\lambda)$ are usually defined by the following generating function:*

$$\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}e^{xz}=\sum_{n=0}^{\infty}A_n(x|\lambda)\frac{z^n}{n!}, \tag{2.31}$$

Special Cases

In the special case when $x=0$ the generalized Eulerian polynomials are reduced to the generalized Eulerian numbers $A_n(\lambda)$ defined by:

$$\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}=\sum_{n=0}^{\infty}A_n(\lambda)\frac{z^n}{n!}. \tag{2.32}$$

The polynomials $A_n(\lambda)$ can be computed using the following recurrence relation:

Theorem 2.18 *For $n\geq 1$, we have*

$$A_n(\lambda)=\sum_{k=0}^{n-1}\binom{n}{k}A_k(\lambda)(\lambda-1)^{n-k-1},\quad n\geq 1.$$

Proof. By using the generating function (2.32), we obtain

$$\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}=\sum_{n=0}^{\infty}A_n(\lambda)\frac{z^n}{n!}.$$

We multiply both sides of the generating function by $e^{z(\lambda-1)}-\lambda$

$$\begin{aligned}(1-\lambda)&=(e^{z(\lambda-1)}-\lambda)\sum_{n=0}^{\infty}A_n(\lambda)\frac{z^n}{n!}\\&=e^{z(\lambda-1)}\sum_{n=0}^{\infty}A_n(\lambda)\frac{z^n}{n!}-\lambda\sum_{n=0}^{\infty}A_n(\lambda)\frac{z^n}{n!}\\&=\sum_{n=0}^{\infty}(1-\lambda)^n\frac{z^n}{n!}\sum_{n=0}^{\infty}A_n(\lambda)\frac{z^n}{n!}-\lambda\sum_{n=0}^{\infty}A_n(\lambda)\frac{z^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}A_k(\lambda)(1-\lambda)^{n-k}-\lambda A_n(\lambda)\right)\frac{z^n}{n!}.\end{aligned}$$

By comparing both sides, we observe that the left-hand side is a constant equal to $1-\lambda$, which means that the coefficient of z^0 must be $1-\lambda$, and all the remaining coefficients must be zero. Therefore, we obtain

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}A_k(\lambda)(1-\lambda)^{n-k}-\lambda A_n(\lambda)&=0\\\binom{n}{n}A_n(\lambda)(1-\lambda)^{n-n}+\sum_{k=0}^{n-1}\binom{n}{k}A_k(\lambda)(1-\lambda)^{n-k}&=\lambda A_n(\lambda)\\A_n(\lambda)+\sum_{k=0}^{n-1}\binom{n}{k}A_k(\lambda)(1-\lambda)^{n-k}&=\lambda A_n(\lambda)\\\sum_{k=0}^{n-1}\binom{n}{k}A_k(\lambda)(1-\lambda)^{n-k}&=A_n(\lambda)(\lambda-1).\end{aligned}$$

By dividing both sides by $(\lambda-1)$, we get

$$A_n(\lambda)=\sum_{k=0}^{n-1}\binom{n}{k}A_k(\lambda)(\lambda-1)^{n-k-1}.$$

This is the desired result.

2.7.7 The Frobenius-type Eulerian Polynomial of Order α

In this study, the definition of the Frobenius-type Eulerian polynomial of order α has been considered, as presented in [11, 12, 13, 21, 29, 31].

Definition 2.12 *The Frobenius-type Eulerian polynomials $A_n(x|\lambda)$ of order $\alpha\in\mathbb{C}$ are usually defined by means of the following generating function:*

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{xz}=\sum_{n=0}^{\infty}A_n^{(\alpha)}(x|\lambda)\frac{z^n}{n!},\tag{2.33}$$

where λ is a complex number with $\lambda\neq 1$.

We now present an explicit expression for the Frobenius-type Eulerian polynomials of order α in the following theorem:

Theorem 2.19 *For $n\geq 0$, we have*

$$A_n^{(\alpha)}(x|\lambda)=\sum_{k=0}^{n}\binom{n}{k}A_k^{(\alpha)}(\lambda)\,x^{n-k}.\tag{2.34}$$

Proof. By means of the generating function (2.33)

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{xz}=\sum_{n=0}^{\infty}A_n^{(\alpha)}(x|\lambda)\frac{z^n}{n!},$$

we obtain

$$\begin{aligned}\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}e^{xz}&=\left(\sum_{n=0}^{\infty}A_n^{(\alpha)}(\lambda)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}x^n\frac{z^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{n=0}^{k}\binom{n}{k}A_k^{(\alpha)}(\lambda)x^{n-k}\right)\frac{z^n}{n!}.\end{aligned}$$

By comparing coefficients, we get

$$A_n^{(\alpha)}(x|\lambda)=\sum_{n=0}^{k}\binom{n}{k}A_k^{(\alpha)}(\lambda)x^{n-k}.\tag{2.35}$$

■

In the following theorem, we give an explicit formula for the Frobenius-type Eulerian polynomials in terms of the weighted Stirling numbers of the second kind:

Theorem 2.20 *The following relationship holds:*

$$A_n^{(\alpha)}(x|\lambda)=\sum_{k=0}^{n}(\alpha)_k\,(\lambda-1)^{n-k}\,S_n^k\left(\frac{x}{\lambda-1}\right).\tag{2.36}$$

Proof. From the weighted Stirling numbers of the second kind (1.15), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(\alpha)_k\,(\lambda-1)^{n-k}\,S_n^k\left(\frac{x}{\lambda-1}\right)\right)\frac{z^n}{n!}\\
&=\sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\lambda-1)^k}\left(\sum_{n=k}^{\infty}S_n^k\left(\frac{x}{\lambda-1}\right)\frac{((\lambda-1)z)^n}{n!}\right)\\
&=e^{xz}\sum_{k=0}^{\infty}(\alpha)_k\frac{1}{k!}\left(\frac{e^{(\lambda-1)z}-1}{\lambda-1}\right)^k.
\end{aligned}$$

Since

$$\sum_{n=0}^{\infty}(a)_n\,\frac{z^n}{n!}=(1-z)^{-\alpha},$$

we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(\alpha)_k\,(\lambda-1)^{n-k}\,S_n^k\left(\frac{x}{\lambda-1}\right)\right)\frac{z^n}{n!}\\
&=e^{xz}\left(1-\frac{e^{(\lambda-1)z}-1}{\lambda-1}\right)^{-\alpha}\\
&=\sum_{n=0}^{\infty}A_n^{(\alpha)}(x|\lambda)\,\frac{z^n}{n!},
\end{aligned}$$

which gives, by identification, the desired result.

Special Cases

In the special case when $\alpha=1$, the Frobenius–type Eulerian polynomials of order α are reduced to the generalized Eulerian polynomials $A_n(x|\lambda)$. By substituting $x=0$, we recover the Frobenius-type Eulerian numbers of order α $A_n^{(\alpha)}(\lambda)$ as follows:

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha}=\sum_{n=0}^{\infty}A_n^{(\alpha)}(\lambda)\frac{z^n}{n!}.$$

2.7.8 The Frobenius-Genocchi Polynomials

Frobenius-Genocchi polynomials are a special type of Appell polynomials. They come from number theory and are used to solve certain differential equations and study special number sequences.

Definition 2.13 *Frobenius-Genocchi polynomials are defined by means of the following generating function:*

$$\frac{(1-\lambda)z}{e^z-\lambda}e^{xt}=\sum_{n=0}^{\infty}G_n^F(x,\lambda)\frac{z^n}{n!}. \tag{2.37}$$

The relations between Frobenius-Euler and Frobenius-Genocchi polynomials:

$$G_n^F(x;\lambda)=nE_{n-1}^F(x;\lambda).$$

$$G_n^F(x;\lambda)=n\sum_{k=0}^{n-1}\binom{n-1}{k}E_{n-k-1}^F(\lambda)x^k.$$

$$2^{n-1}G_n^F(x+y;\lambda^2)=\sum_{k=0}^{n}\binom{n}{k}G_{n-k}^F(2x;\lambda)E_k^F(2y;-\lambda).$$

Special Case

In the special case $\lambda = -1$ in (2.37), we obtain the Genocchi polynomials (2.10).

2.7.9 The p-Bell Polynomials

The p-Bell polynomials are a generalization of the classical Bell polynomials. They are used in the study of ordered partitions and combinatorial structures, and are characterized by a specific generating function. According to [25].

Definition 2.14 *The p-Bell polynomials $\mathcal{B}_{n,p}(x)$ are defined by the following generating function:*

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,p}(x)\frac{z^n}{n!} = {}_1F_1\left(\begin{array}{c} 1 \\ p+1 \end{array}; e^z - 1\right) e^{xz}. \tag{2.38}$$

Now, we present the explicit formula for the p-Bell polynomials:

Theorem 2.21 *For $n \geq 0$, we have*

$$\mathcal{B}_{n,p}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_{k,p} x^{n-k}, \tag{2.39}$$

such that $\mathcal{B}_{k,p}$ the p-Bell numbers, is defined by the following generating function:

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,p}\frac{z^n}{n!} = {}_1F_1\left(\begin{array}{c} 1 \\ p+1 \end{array}; e^z - 1\right).$$

Proof. The following identity is obtained using the generating function (2.38)

$$\sum_{n=0}^{\infty} \mathcal{B}_{n,p}(x)\frac{z^n}{n!} = {}_1F_1\left(\begin{array}{c} 1 \\ p+1 \end{array}; e^z - 1\right) e^{xz},$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{B}_{n,p}(x)\frac{z^n}{n!} &= \left(\sum_{n=0}^{\infty} \mathcal{B}_{n,p}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} x^n\frac{z^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{k,p}x^{n-k}\right).\end{aligned}$$

By identification, we have

$$\mathcal{B}_{n,p}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_{k,p} x^{n-k}.$$

■

To compute the first p-Bell polynomials $\mathcal{B}_{k,p}(x)$, I used the explicit formula (2.39). The first few expressions of the p-Bell polynomials are:

$$\mathcal{B}_{0,p}(x) = 1,$$

$$\mathcal{B}_{1,p}(x) = x + \frac{1}{p+1},$$

$$\mathcal{B}_{2,p}(x) = x^2 + \frac{2x}{p+1} + \frac{p+4}{(p+1)(p+2)},$$

$$\mathcal{B}_{3,p}(x) = x^3 + \frac{3x^2}{p+1} + \frac{3(p+4)x}{(p+1)(p+2)} + \frac{p^2+11p+30}{(p+1)(p+2)(p+3)}.$$

Special Case

Substituting $p = 0$, we obtain the classical Bell polynomials (2.12). By setting $p = 0$ and $x = 0$, we recover the Bell numbers (2.14).

2.7.10 Derangement Polynomials of Order r

Derangement polynomials of order r are a natural generalization of the classical derangement polynomials. They introduce an extra parameter r to extend their combinatorial structure and generating function [22].

Definition 2.15 *the derangement polynomials $d_n^{(r)}(x)$ of order r, are defined by the exponential generating function :*

$$\sum_{n=0}^{\infty} d_n^{(r)}(x)\,\frac{z^n}{n!} = \frac{1}{(1-z)^r}e^{xz}. \tag{2.40}$$

The explicit formula for the derangement polynomials of order r is given by the following theorem:

Theorem 2.22 *For $n \geq 0$, we have the explicit formula of the derangement polynomials $d_n^{(r)}(x)$ of order r:*

$$d_n^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} \langle r \rangle_k x^{n-k}.$$

Proof. By using the exponential generating function (2.40) , we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} d_n^{(r)}(x)\frac{z^n}{n!} &= \frac{1}{(1-z)^r}e^{xz} \\
&= \sum_{k=0}^{\infty} \frac{\langle r \rangle_k z^k}{k!} \sum_{n=0}^{\infty} \frac{(xz)^n}{n!} \\
&= \sum_{k=0}^{\infty} \langle r \rangle_k \sum_{n=0}^{\infty} \frac{x^n z^{n+k}}{k!n!} \\
&= \sum_{k=0}^{\infty} \langle r \rangle_k \sum_{n=0}^{\infty} \frac{x^n z^{n+k}}{k!n!} \\
&= \sum_{k=0}^{\infty} \langle r \rangle_k \sum_{n=0}^{\infty} \binom{n+k}{k} x^n \frac{z^{n+k}}{(n+k)!} \\
&= \sum_{k=0}^{\infty} \langle r \rangle_k \sum_{n=0}^{\infty} \binom{n}{k} x^{n-k} \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \langle r \rangle_k x^{n-k} \right) \frac{z^n}{n!}.
\end{aligned}$$

Equating the coefficients of $\frac{z^n}{n!}$ in both sides of the above expression, we get the desired result. ∎

Special Case

When $r = 1$, $d_n^{(r)}(x) = d_n(x)$ are the derangement polynomials whose exponential generating function is given by

$$\sum_{n=0}^{\infty} d_n(x)\frac{z^n}{n!} = \frac{1}{1-z}e^{xz}. \tag{2.41}$$

The explicit formula is defined in the following theorem:

Theorem 2.23 *For $n \geq 0$, we have*

$$d_n(x) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} x^{n-k}. \tag{2.42}$$

Proof. We consider the exponential generating function of the polynomials (2.41):

$$\sum_{n=0}^{\infty} d_n(x) \frac{z^n}{n!} = \frac{1}{1-z} e^{xz}.$$

We have

$$\begin{aligned}
\frac{1}{1-z} e^{xz} &= \frac{1}{1+(-z)} e^{xz} \\
&= \left(\sum_{n=0}^{\infty} (-1)^n z^n \right) \left(\sum_{n=0}^{\infty} x^n \frac{z^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^k \frac{x^{n-k}}{(n-k)!} \right) z^n \\
&= \sum_{n=0}^{\infty} n! \left(\sum_{k=0}^{n} (-1)^k \frac{x^{n-k}}{(n-k)!} \right) \frac{z^n}{n!}.
\end{aligned}$$

Letting $j = n - k$, this becomes:

$$\begin{aligned}
\frac{1}{1-z} e^{xz} &= \sum_{n=0}^{\infty} \left(n! \sum_{j=0}^{n} \frac{(-1)^{n-j}}{j!} x^j \right) \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} x^{n-k} \right) \frac{z^n}{n!}.
\end{aligned}$$

By comparing coefficients, we obtain

$$d_n(x) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} x^{n-k}.$$

This proves the identity. ∎

By substituting $x = 0$ and $r = 0$, we recover the classical derangement numbers d_n, whose exponential generating function is given by

$$\sum_{n=0}^{\infty} d_n \frac{z^n}{n!} = \frac{1}{1-z} e^{-z}.$$

3

Recurrence Relation for the Appell Sequences

Among the most notable Appell sequences are the Bernoulli polynomials, Euler polynomials, Genocchi polynomials, Bell polynomials, Hermite polynomials, and their generalizations [1]. It is well established that the Euler–Seidel matrix method is a valuable tool for studying the properties of certain special polynomials. However, as noted by Dumont in [16], this method fails when applied to Appell sequences. The objective of this chapter is to study an algorithm for computing Appell polynomials introduced by Rahmani et al [20] and based on the recurrence relation

$$\mathcal{P}_{n+1,m}(x) = \frac{h_{m+1}}{h_m}\mathcal{P}_{n,m+1}(x) + (m+x)\mathcal{P}_{n,m}(x), \tag{3.1}$$

with the initial condition

$$\mathcal{P}_{0,m}(x) = 1.$$

The sequence $(h_n)_n$ is defined in [20] as follows.

3.1 The Sequence $(h_n)_n$

We begin by defining this sequence and providing a three-term recurrence relation for computing its terms. Next, we explore the connection between Appell sequences and the Stirling numbers of the first kind through the sequence $(h_n)_n$. Finally, we present several explicit expressions for Appell sequences in terms of the weighted Stirling numbers of the second kind.

Definition 3.1 *The sequence $(h_n)_{n\geq 0}$ is defined by the explicit formula:*

$$h_n = \sum_{j=0}^{n} a_j s\left(n, j\right),$$

where a_n is given by (2.2).

Theorem 3.1 *The exponential generating function for the sequence $(h_n)_n$ is defined by:*

$$A\left(\ln\left(1+z\right)\right) = \sum_{n=0}^{\infty} h_n \frac{z^n}{n!}, \tag{3.2}$$

Thus

$$h_n = \left[z^n\right] A\left(\ln\left(1+z\right)\right), \tag{3.3}$$

where $\left[z^n\right] f\left(z\right) = f_n$ denotes the operation of extracting the coefficient of $\frac{z^n}{n!}$ in the formal power series

$$f(z) = \sum_{n\geq 0} f_n \frac{z^n}{n!}. \tag{3.4}$$

Proof. Since

$$\ln(1+z) = \sum_{m=1}^{\infty} (-1)^{m+1} (m-1)! \frac{z^m}{m!}.$$

From Definition 3.1 and according to partial Bell polynomial (1.22), we get

$$h_0 = 1, \ h_n = \sum_{k=1}^{n} a_k B_{n,k}(0!, -1!, 2!, \ldots).$$

Moreover, by applying Faà di Bruno's formula [6] to the function $A(\ln(1+z))$, we deduce the generating function of h_n. ∎

Now, we present an algorithm that uses a three-term recurrence relation to calculate the sequence $(h_n)_{n\geq 0}$. To make the process easier, we introduce a new two-index sequence $(H_{n,m})_{n,m\geq 0}$ given by the following explicit formula:

Definition 3.2

$$H_{n,m} = \sum_{k=0}^{m} s(m,k)\, a_{n+k}, \tag{3.5}$$

with $H_{n,0} = a_n$ *and* $H_{0,m} = h_m$.

Theorem 3.2 *The sequence* $(H_{n,m})_{n,m\geq 0}$ *satisfies the recurrence relation:*

$$H_{n,m+1} = H_{n+1,m} - mH_{n,m},$$

with the initial sequence given by

$$H_{n,0} = a_n.$$

Proof. It follows immediately from (3.5) and (1.9). ∎

The following lemmas that will be used in the proof of our main results.

Lemma 3.3 *For* $n \geq 1$, *we have*

$$B_n(0!x, -1!x, 2!x, \cdots) = (x)_n.$$

Proof. By using (1.24), (1.21) and (1.23), we obtain

$$\begin{aligned} B_n(0!x, -1!x, 2!x, \ldots) &= \sum_{k=1}^{n} B_{n,k}(0!x, -1!x, 2!x, \ldots) \\ &= \sum_{k=1}^{n} B_{n,k}(0!, -1!, 2!, \ldots) x^k \\ &= \sum_{k=1}^{n} s(n,k) x^k = (x)_n, \qquad (n \geq 1). \end{aligned}$$

∎

Lemma 3.4 *We have*

$$S_n^i(x) = \sum_{k=i}^{n} \left\{ {n \atop k} \right\} \binom{k}{i} (x)_{k-i}.$$

Proof. We start by considering the inner sum and apply the generating function technique. Using identity (1.11), we write:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\{ n\atop k\right\}\binom{k}{i}(x)_{k-i}\right)\frac{z^n}{n!}=\sum_{k=0}^{\infty}\binom{k}{i}(x)_{k-i}\left(\sum_{n=k}^{\infty}\left\{ n\atop k\right\}\frac{z^n}{n!}\right)$$
$$=\sum_{k=0}^{\infty}\binom{k}{i}(x)_{k-i}\frac{(e^z-1)^k}{k!}.$$

We perform a change of index by letting $k=s+i$, so the sum becomes:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\{ n\atop k\right\}\binom{k}{i}(x)_{k-i}\right)\frac{z^n}{n!}=\sum_{s=0}^{\infty}\binom{s+i}{i}(x)_s\frac{(e^z-1)^{s+i}}{(s+i)!}$$
$$=\sum_{k=0}^{\infty}\binom{k}{i}(x)_{k-i}\frac{(e^z-1)^k}{k!}$$
$$=\frac{(e^z-1)^i}{i!}\sum_{s=0}^{\infty}\frac{(x)_s}{s!}(e^z-1)^s.$$

Using the known generating function:

$$\sum_{s=0}^{\infty}\frac{(x)_s}{s!}(e^z-1)^s=\sum_{s=0}^{\infty}\binom{x}{s}(e^z-1)^s(1)^{x-s}$$
$$=((e^z-1)+1)^x=e^{xz},$$

we finally obtain:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\{ n\atop k\right\}\binom{k}{i}(x)_{k-i}\right)\frac{z^n}{n!}=\frac{(e^z-1)^i}{i!}e^{xz},$$

and this is the desired result. ∎

The following identity reveals the relationship between Appell sequences and Stirling numbers of the first kind.

Theorem 3.5 *For $n\geq 0$, we have*

$$\sum_{k=0}^{n}s\left(n,k\right)P_k\left(x\right)=\sum_{k=0}^{n}\binom{n}{k}h_k\left(x\right)_{n-k}.$$

Proof. We begin by applying the generating function technique as shown in identity (1.8). We first write:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}s(n,k)P_k(x)\right)\frac{z^n}{n!}=\sum_{k=0}^{\infty}P_k(x)\left(\sum_{n=k}^{\infty}s(n,k)\frac{z^n}{n!}\right)$$
$$=\sum_{k=0}^{\infty}P_k(x)\frac{1}{k!}\left(\ln(1+z)\right)^k.$$

This expression is simply the generating function for the Appell sequence $P_k(x)$, evaluated at $\ln(1+z)$. Therefore, we can write:

$$\sum_{k=0}^{\infty}P_k(x)\frac{1}{k!}\left(\ln(1+z)\right)^k=A(\ln(1+z))\exp\left(x\ln(1+z)\right).$$

Next, we simplify the exponential term. Using the Faà di Bruno formula [6] as stated in the Theorem 1.1, we know:

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad and \quad g(x) = x\ln(1+z)$$
$$= x\sum_{m=1}^{\infty} (-1)^{m+1}\,(m-1)!\frac{z^m}{m!}.$$

Then

$$\exp\left(x\ln\left(1+z\right)\right) = \sum_{n=0}^{\infty} g_n \frac{z^n}{n!},$$

where

$$\begin{cases} g_0 = 1 \\ \\ g_n = \sum_{k=1}^{n} a_k\, B_{n,k}(x0!, -x1!, x2! \ldots, x(-1)^{n-k+2}(n-k)!), \end{cases}$$

and using Lemma 3.3, we get

$$\exp\left(x\ln\left(1+z\right)\right) = \sum_{n=0}^{\infty} (x)_n \frac{z^n}{n!}. \tag{3.6}$$

By virtue of (3.2) and (3.6), and since the expression is the product of two series, we obtain

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} s\left(n,k\right) P_k\left(x\right)\right)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} h_k\left(x\right)_{n-k}\right)\frac{z^n}{n!}.$$

Equating the coefficients of $\frac{z^n}{n!}$ in both sides of the above equation, we get the result.

■

Corollary 1 *For $n \geq 0$, we have*

$$P_n(x) = \sum_{k=0}^{n} h_k S_n^k(x). \tag{3.7}$$

Remark 1 *Notice that the above formula was obtained more recently by Adell and Lekuona in [2].*

Proof. Starting with the Stirling transform of the second kind (see [28]), and applying the identity from Lemma 3.4, we write:

$$P_n(x) = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\sum_{i=0}^{k}\binom{k}{i} h_i (x)_{k-i}$$
$$= \sum_{i=0}^{n} h_i \sum_{k=i}^{n}\left\{{n \atop k}\right\}\binom{k}{i}(x)_{k-i}.$$

According to Lemma 3.4, we obtain:

$$\sum_{k=i}^{n}\left\{{n \atop k}\right\}\binom{k}{i}(x)_{k-i} = S_n^i(x),$$

which leads to the final result:

$$P_n(x) = \sum_{i=0}^{n} h_i S_n^i(x).$$

■

Now, we present some particular cases of identity (3.7).

In particular, we give an explicit formula for Appell sequences using the r-Whitney numbers (1.16) and the r-Stirling numbers (1.11) of the second kind.

$$P_n\left(\frac{r}{m}\right)=\sum_{k=0}^{n}h_k\frac{1}{m^{n-k}}W_{m,r}\left(n,k\right), \tag{3.8}$$

and

$$P_n\left(r\right)=\sum_{k=0}^{n}h_k\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_r. \tag{3.9}$$

As a consequence of (3.7), we give some explicit formulas arising from Appell type polynomials.

Example 3.1 *The generating function for the Bernoulli polynomials is given by*

$$\frac{z}{e^z-1}e^{xz}=\sum_{n=0}^{\infty}B_n\left(x\right)\frac{z^n}{n!},\quad \left|z\right|<2\pi.$$

From the formula (3.3), the sequence $(h_n)_n$ is given by

$$h_n=[z^n]\frac{\ln\left(1+z\right)}{z}.$$

Since

$$\begin{aligned}\frac{\ln\left(1+z\right)}{z}&=z^{-1}\sum_{n=1}^{\infty}\left(-1\right)^{n+1}\left(n-1\right)!\frac{z^n}{n!}\\&=\sum_{n=1}^{\infty}\left(-1\right)^{n+1}\left(n-1\right)!\frac{z^{n-1}}{n!}.\end{aligned}$$

Let $n-1:=n$, *then*

$$\frac{\ln\left(1+z\right)}{z}=\sum_{n=0}^{\infty}\left(-1\right)^n\frac{n!}{n+1}\frac{z^n}{n!},$$

we get

$$h_n=\left(-1\right)^n\frac{n!}{n+1}. \tag{3.10}$$

Now, from (3.7) we can easily get the well-known explicit formula for the Bernoulli polynomials

$$B_n\left(x\right)=\sum_{k=0}^{n}\left(-1\right)^k\frac{k!}{k+1}S_n^k\left(x\right). \tag{3.11}$$

In particular, from (3.9), we have

$$B_n\left(r\right)=\sum_{k=0}^{n}\left(-1\right)^k\frac{k!}{k+1}\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_r,$$

and from (3.8), we get the following explicit formula:

$$B_n\left(\frac{r}{m}\right)=\sum_{k=0}^{n}\frac{\left(-1\right)^k}{m^{n-k}}\frac{k!}{k+1}W_{m,r}\left(n,k\right).$$

Example 3.2 *The generating function for p−Bernoulli polynomials $B_{n,p}(x)$ is given in [27] by*

$$ {}_2F_1\left(\begin{array}{c} 1,1 \\ p+2 \end{array};1-e^z\right)e^{xz}=\sum_{n=0}^{\infty}B_{n,p}(x)\frac{z^n}{n!}. $$

It follows from (3.3) that

$$ h_{n,p}=[z^n]\ {}_2F_1\left(\begin{array}{c} 1,1 \\ p+2 \end{array};-z\right). $$

Since

$$ \begin{aligned} {}_2F_1\left(\begin{array}{c} 1,1 \\ p+2 \end{array};-z\right) &= \sum_{n=0}^{\infty}\frac{\langle 1\rangle_n\langle 1\rangle_n}{\langle p+2\rangle_n}\frac{(-z)^n}{n!} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(n!)^2(p+1)!}{(p+1+n)!}\frac{z^n}{n!}, \end{aligned} $$

we get

$$ h_{n,p}=(-1)^n\frac{(n!)^2(p+1)!}{(p+1+n)!}. \tag{3.12} $$

From (3.7), we obtain the following explicit formula for $B_{n,p}(x)$

$$ B_{n,p}(x)=\sum_{k=0}^{n}(-1)^k\frac{(k!)^2(p+1)!}{(p+1+k)!}S_n^k(x), \tag{3.13} $$

which is equivalently to the formula in [27] given by

$$ B_{n,p}(x)=\frac{(p+1)}{p!}\sum_{k=0}^{n}(-1)^k\frac{(k+p)!}{k+p+1}S_n^k(x+p). $$

Remark 2 *By setting $p=0$ in the above formula (3.13), we obtain the explicit formula for the classical Bernoulli polynomials in Example 3.1.*

Example 3.3 *For $p\geq 0$, the p−Bell polynomials $\mathcal{B}_{n,p}(x)$ are defined [25] by the following generating function:*

$$ {}_1F_1\left(\begin{array}{c} 1 \\ p+1 \end{array};e^z-1\right)e^{xz}=\sum_{n=0}^{\infty}\mathcal{B}_{n,p}(x)\frac{z^n}{n!}. $$

Using (3.3), we have

$$ h_{n,p}=[z^n]\ {}_1F_1\left(\begin{array}{c} 1 \\ p+1 \end{array};z\right). $$

Since

$$ \begin{aligned} {}_1F_1\left(\begin{array}{c} 1 \\ p+1 \end{array};z\right) &= \sum_{n=0}^{\infty}\frac{\langle 1\rangle_n}{\langle p+1\rangle_n}\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{p!n!}{(p+n)!}\frac{z^n}{n!}=\sum_{n=0}^{\infty}\binom{n+p}{n}^{-1}\frac{z^n}{n!}, \end{aligned} $$

it follows that

$$ h_{n,p}=\binom{n+p}{n}^{-1} \tag{3.14} $$

and from (3.7), we have the well-known explicit formula for p−Bell polynomials

$$ \mathcal{B}_{n,p}(x)=\sum_{k=0}^{n}\binom{k+p}{k}^{-1}S_n^k(x). \tag{3.15} $$

Example 3.4 *The generating function for the Euler polynomials $E_n(x)$ is given by*

$$\frac{2}{e^z+1}e^{xz}=\sum_{n=0}^{\infty}E_n(x)\frac{z^n}{n!}.$$

Using (3.3), we have

$$h_n=[z^n]\frac{2}{z+2}.$$

Since

$$\begin{aligned}\frac{2}{2+z}&=\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n\\&=\sum_{n=0}^{\infty}(-1)^n\frac{n!}{2^n}\frac{z^n}{n!},\end{aligned}$$

we get

$$h_n=(-1)^n\frac{n!}{2^n}, \tag{3.16}$$

and the explicit formula for the Euler polynomials [5] is

$$E_n(x)=\sum_{k=0}^{n}(-1)^k\frac{k!}{2^k}S_n^k(x). \tag{3.17}$$

Example 3.5 *For $\lambda\in\mathbb{C}$ with $\lambda\neq 1$, the Eulerian polynomials $H_n(x|\lambda)$ are defined [12] by the generating function*

$$\left(\frac{1-\lambda}{e^z-\lambda}\right)e^{xz}=\sum_{n=0}^{\infty}H_n(x|\lambda)\frac{z^n}{n!}.$$

From (3.3), we have

$$h_{n,\lambda}=[z^n]\frac{1-\lambda}{z+1-\lambda}.$$

Since

$$\begin{aligned}\frac{1-\lambda}{z+1-\lambda}&=\frac{1}{1-\frac{z}{\lambda-1}}\\&=\sum_{n=0}^{\infty}\left(\frac{z}{\lambda-1}\right)^n\\&=\sum_{n=0}^{\infty}\frac{n!}{(\lambda-1)^n}\frac{z^n}{n!},\end{aligned}$$

we obtain

$$h_{n,\lambda}=\frac{n!}{(\lambda-1)^n}, \tag{3.18}$$

it follows from (3.7) that

$$H_n(x|\lambda)=\sum_{k=0}^{n}\frac{k!}{(\lambda-1)^k}S_n^k(x). \tag{3.19}$$

Remark 3 *By setting $\lambda=-1$, the Eulerian polynomials $H_n(x\mid\lambda)$ reduce to the Euler polynomials $E_n(x)$, as illustrated in Example 3.4.*

Example 3.6 *The Frobenius-Euler polynomials* $H_n^{(\alpha)}(x|\lambda)$ *of order* $\alpha \in \mathbb{C}$, *are defined [11] by the following generating function*

$$\left(\frac{1-\lambda}{e^z-\lambda}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x|\lambda)\frac{z^n}{n!}.$$

From (3.3), *we have*

$$h_n(\lambda,\alpha) = [z^n]\left(\frac{1-\lambda}{1+z-\lambda}\right)^{\alpha}.$$

Since

$$\begin{aligned}\left(\frac{1-\lambda}{1+z-\lambda}\right)^{\alpha} &= \left(\frac{1}{1-\frac{z}{\lambda-1}}\right)^{\alpha}\\ &= \left(1-\frac{z}{\lambda-1}\right)^{-\alpha}\\ &= \sum_{n=0}^{\infty}\frac{\langle\alpha\rangle_n}{n!}\left(\frac{z}{\lambda-1}\right)^n\\ &= \sum_{n=0}^{\infty}\frac{\langle\alpha\rangle_n}{(\lambda-1)^n}\frac{z^n}{n!},\end{aligned}$$

we get

$$h_n(\lambda,\alpha) = \frac{\langle\alpha\rangle_n}{(\lambda-1)^n}. \tag{3.20}$$

Now, the explicit formula [31] for the Frobenius-Euler polynomials $H_n^{(\alpha)}(x|\lambda)$ *of order* $\alpha \in \mathbb{C}$, *is given by the formula* (3.7) *as*

$$H_n^{(\alpha)}(x|\lambda) = \sum_{k=0}^{n}\frac{\langle\alpha\rangle_k}{(\lambda-1)^k}S_n^k(x). \tag{3.21}$$

Example 3.7 *The Frobenius-type Eulerian polynomials* $A_n(x|\lambda)$ *of order* $\alpha \in \mathbb{C}$, *are usually defined in [12, 13, 21, 29, 31] by means of the following generating function*

$$\left(\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} A_n^{(\alpha)}(x|\lambda)\frac{z^n}{n!}.$$

From (3.3), *we have*

$$h_{n,\lambda}(\alpha) = [z^n]\left(\frac{1-\lambda}{(1+z)^{1-\lambda}-\lambda}\right)^{\alpha}.$$

Since

$$\begin{aligned}\left(\frac{1-\lambda}{(1+z)^{\lambda-1}-\lambda}\right)^{\alpha} &= \left(\frac{1}{\frac{(1+z)^{\lambda-1}-1-(\lambda-1)}{\lambda-1}}\right)^{\alpha}\\ &= \left(\frac{1}{\frac{(1+z)^{\lambda-1}-1}{\lambda-1}+1}\right)^{\alpha}\\ &= \left(1-\frac{(1+z)^{\lambda-1}-1}{\lambda-1}\right)^{-\alpha}\\ &= \sum_{k=0}^{\infty}\frac{\langle\alpha\rangle_k}{k!}\left(\frac{(1+z)^{\lambda-1}-1}{\lambda-1}\right)^k\\ &= \sum_{k=0}^{\infty}\langle\alpha\rangle_k\sum_{n=k}^{\infty}s_{\lambda-1}(n,k)\frac{z^n}{n!},\end{aligned}$$

where $s_{\lambda-1}(n,k)$, *are the degenerate Stirling numbers of the first kind defined by Carlitz [14] as*

$$\frac{1}{k!}\left(\frac{(1+z)^{\lambda-1}-1}{\lambda-1}\right)^{k}=\sum_{n=k}^{\infty}s_{\lambda-1}(n,k)\frac{z^{n}}{n!},$$

from which, it follows that

$$A\left(\ln(1+z)\right)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\langle\alpha\rangle_{k}s_{\lambda-1}(n,k)\right)\frac{z^{n}}{n!},$$

and

$$h_{n,\lambda}(\alpha)=\sum_{k=0}^{n}\langle\alpha\rangle_{k}s_{\lambda-1}(n,k). \tag{3.22}$$

Finally, by using the formula (3.7), *we obtain the explicit formula for the Frobenius-type Eulerian polynomials* $A_{n}^{(\alpha)}(x|\lambda)$ *of order* α *in terms of the Stirling numbers of the first kind*

$$A_{n}^{(\alpha)}(x|\lambda)=\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\langle\alpha\rangle_{j}s_{\lambda-1}(k,j)\right)S_{n}^{k}(x). \tag{3.23}$$

The following explicit formulas are readily derived from (3.23)

$$A_{n}^{(\alpha)}(r|\lambda)=\sum_{k=0}^{n}\left(\sum_{j=0}^{k}\langle\alpha\rangle_{j}s_{\lambda-1}(k,j)\right)\left\{\begin{matrix}n+r\\k+r\end{matrix}\right\}_{r}, \tag{3.24}$$

and

$$A_{n}^{(\alpha)}\left(\frac{r}{m}|\lambda\right)=\sum_{k=0}^{n}\frac{1}{m^{n-k}}\left(\sum_{j=0}^{k}\langle\alpha\rangle_{j}s_{\lambda-1}(k,j)\right)W_{m,r}(n,k). \tag{3.25}$$

Remark 4 *By setting* $\alpha=1$, *we obtain the generalized Eulerian polynomials* $A_{n}(x|\lambda)$, *given by*

$$A_{n}(x|\lambda):=A_{n}^{(1)}(x|\lambda),$$

which are defined by the following generating function

$$\frac{1-\lambda}{e^{z(\lambda-1)}-\lambda}e^{xz}=\sum_{n=0}^{\infty}A_{n}(x|\lambda)\frac{z^{n}}{n!},$$

and can be computed by the explicit formula as follows:

$$A_{n}(x|\lambda)=\sum_{k=0}^{n}\left(\sum_{j=0}^{k}j!s_{\lambda-1}(k,j)\right)S_{n}^{k}(x).$$

Example 3.8 *For any* $r\geq1$, *the derangement polynomials* $d_{n}^{(r)}(x)$ *of order* r *are defined by the exponential generating function [22]*

$$\sum_{n=0}^{\infty}d_{n}^{(r)}(x)\frac{z^{n}}{n!}=\frac{1}{(1-z)^{r}}e^{xz}.$$

From the formula (3.3), *we have*

$$h_{n,r}=[z^{n}]\frac{1}{(1-\ln(1+z))^{r}}.$$

Since

$$\begin{aligned}\frac{1}{(1-\ln(1+z))^r} &= \sum_{k=0}^{\infty} \langle r \rangle_k \frac{(\ln(1+z))^k}{k!} \\ &= \sum_{k=0}^{\infty} \langle r \rangle_k \sum_{n=k}^{\infty} s(n,k) \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \langle r \rangle_k s(n,k) \right) \frac{z^n}{n!},\end{aligned}$$

then

$$h_{n,r} = \sum_{k=0}^{n} \langle r \rangle_k s(n,k). \tag{3.26}$$

Now, the explicit formula for the derangement polynomials of order r is:

$$d_n^{(r)}(x) = \sum_{k=0}^{n} \left(\sum_{j=0}^{k} \langle r \rangle_j s(k,j) \right) S_n^k(x). \tag{3.27}$$

3.2 Recurrence Relation for Appell Sequences

In order to provide a simple method for computing Appell polynomials, we introduce a specific double-indexed sequence. By applying the recurrence relation (1.9), we derive a new recurrence relation for Appell sequences. We begin by assuming that $h_m \neq 0$, and define the sequence $(\mathcal{P}_{n,m})_{n,m\geq 0}$ for each integer $m \geq 0$, by $m \geq 0$, by

$$\mathcal{P}_{n,m} = \frac{1}{h_m} \sum_{k=0}^{n} \begin{Bmatrix} n+m \\ k+m \end{Bmatrix}_m h_{k+m}.$$

Theorem 3.6 *The sequence $\mathcal{P}_{n,m}$ may be expressed as*

$$\mathcal{P}_{n,m} = \frac{1}{h_m} \sum_{k=0}^{m} s(m,k) P_{n+k}(0),$$

with $\mathcal{P}_{0,m} = 1$.

Proof. By means of Stirling transform [28], we obtain the desired result.
In the following theorem, we present a recurrence formula for computing Appell polynomials.

Theorem 3.7 *The sequence $\mathcal{P}_{n,m}$ satisfies the following three-term recurrence relation:*

$$\mathcal{P}_{n+1,m} = \frac{h_{m+1}}{h_m} \mathcal{P}_{n,m+1} + m\mathcal{P}_{n,m}, \tag{3.28}$$

with the initial sequence given by $\mathcal{P}_{0,m} = 1$ and $\mathcal{P}_{n,0} = P_n(0)$.

Proof. From Theorem 3.6, the sequence $(\mathcal{P}_{n,m})_{n,m\geq 0}$ is given by

$$\mathcal{P}_{n,m} = \frac{1}{h_m} \sum_{k=0}^{m} s(m,k) P_{n+k}(0).$$

Now, consider $\mathcal{P}_{n,m+1}$:

$$\begin{aligned}\mathcal{P}_{n,m+1} &= \frac{1}{h_{m+1}} \sum_{k=0}^{m+1} s(m+1,k) P_{n+k}(0) \\ &= \frac{1}{h_{m+1}} \sum_{k=0}^{m+1} (s(m,k-1) - m\, s(m,k)) P_{n+k}(0) \\ &= \frac{1}{h_{m+1}} \left[\sum_{k=0}^{m+1} s(m,k-1) P_{n+k}(0) - m \sum_{k=0}^{m+1} s(m,k) P_{n+k}(0) \right]\end{aligned}$$

In the first sum, we perform an index shift: let $j = k - 1$, so when $k = 0$, $j = -1$, and when $k = m + 1$, $j = m$. But since $s(m, -1) = 0$ by convention, we get:

$$\mathcal{P}_{n,m+1} = \frac{1}{h_{m+1}} \left[\sum_{j=0}^{m} s(m,j) P_{n+j+1}(0) - m \sum_{k=0}^{m} s(m,k) P_{n+k}(0) \right].$$

We now recognize these two sums in terms of $\mathcal{P}_{n+1,m}$ and $\mathcal{P}_{n,m}$:

$$\sum_{j=0}^{m} s(m,j) P_{n+j+1}(0) = h_m \mathcal{P}_{n+1,m}, \quad \sum_{k=0}^{m} s(m,k) P_{n+k}(0) = h_m \mathcal{P}_{n,m}.$$

Hence,

$$\begin{aligned} \mathcal{P}_{n,m+1} &= \frac{1}{h_{m+1}} \left(h_m \mathcal{P}_{n+1,m} - m h_m \mathcal{P}_{n,m} \right) \\ &= \frac{h_m}{h_{m+1}} \left(\mathcal{P}_{n+1,m} - m \mathcal{P}_{n,m} \right). \end{aligned}$$

This proves the recurrence relation:

$$\mathcal{P}_{n,m+1} = \frac{h_m}{h_{m+1}} \left(\mathcal{P}_{n+1,m} - m \mathcal{P}_{n,m} \right),$$

which completes the proof of Theorem 3.7. ∎

We now aim to extend Theorem 3.7 to the polynomial case. To this end, we define the polynomials $\mathcal{P}_{n,m}(x)$ by the explicit formula

$$\mathcal{P}_{n,m}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{P}_{k,m}\, x^{n-k}, \tag{3.29}$$

with the initial conditions $\mathcal{P}_{0,m}(x) = 1$ and $\mathcal{P}_{n,0}(x) = P_n(x)$.

Theorem 3.8 *The polynomials $\mathcal{P}_{n,m}(x)$ satisfy the following three-term recurrence relation:*

$$\mathcal{P}_{n+1,m}(x) = \frac{h_{m+1}}{h_m} \mathcal{P}_{n,m+1}(x) + (m + x)\, \mathcal{P}_{n,m}(x), \tag{3.30}$$

with the initial sequence given by

$$\mathcal{P}_{0,m}(x) = 1.$$

Proof. From the Equation (3.29), we have

$$\mathcal{P}_{n,m}(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{P}_{k,m} x^{n-k}.$$

Differentiating both sides with respect to x, we get:

$$\begin{aligned} \frac{d}{dx} \mathcal{P}_{n,m}(x) &= \sum_{k=0}^{n-1} \binom{n}{k} \mathcal{P}_{k,m} (n-k) x^{n-k-1} \\ &= \sum_{k=0}^{n-1} \binom{n}{k} (n-k) \mathcal{P}_{k,m} x^{n-k-1}. \end{aligned}$$

Multiplying both sides by x, we get

$$x\frac{d}{dx}\mathcal{P}_{n,m}(x) = \sum_{k=0}^{n-1}\binom{n}{k}(n-k)\mathcal{P}_{k,m}x^{n-k}$$
$$= n\sum_{k=0}^{n-1}\binom{n-1}{k}\mathcal{P}_{k,m}x^{n-k}.$$

But from the expression of

$$n\mathcal{P}_{n,m}(x) = n\sum_{k=0}^{n}\binom{n}{k}\mathcal{P}_{k,m}x^{n-k}.$$

Now consider the difference

$$x\frac{d}{dx}\mathcal{P}_{n,m}(x) - n\mathcal{P}_{n,m}(x) = n\sum_{k=0}^{n}\left(\binom{n}{k} - \binom{n-1}{k}\right)\mathcal{P}_{k,m}x^{n-k}.$$

Then

$$x\frac{d}{dx}\mathcal{P}_{n,m}(x) = n\mathcal{P}_{n,m}(x) - n\sum_{k=0}^{n}\left(\binom{n}{k} - \binom{n-1}{k}\right)\mathcal{P}_{k,m}x^{n-k}$$
$$= n\mathcal{P}_{n,m}(x) - n\sum_{k=1}^{n}\binom{n-1}{k-1}\mathcal{P}_{k,m}x^{n-k}.$$

Let us set $k-1=j$

$$x\frac{d}{dx}\mathcal{P}_{n,m}(x) = n\mathcal{P}_{n,m}(x) - n\sum_{j=0}^{n-1}\binom{n-1}{j}\mathcal{P}_{j+1,m}x^{n-j-1}.$$

Using the recurrence relation (3.28), we get:

$$x\frac{d}{dx}\mathcal{P}_{n,m}(x) = n\mathcal{P}_{n,m}(x) - n\sum_{j=0}^{n-1}\binom{n-1}{j}x^{n-j-1}\left(\frac{h_{m+1}}{h_m}\mathcal{P}_{j,m+1} + m\mathcal{P}_{j,m}\right)$$
$$= n\mathcal{P}_{n,m}(x) - n\frac{h_{m+1}}{h_m}\sum_{j=0}^{n-1}\binom{n-1}{j}\mathcal{P}_{j,m+1}x^{n-j-1} - nm\sum_{j=0}^{n-1}\binom{n-1}{j}\mathcal{P}_{j,m}x^{n-j-1}$$
$$= n\mathcal{P}_{n,m}(x) - n\frac{h_{m+1}}{h_m}\mathcal{P}_{n-1,m+1}(x) - nm\mathcal{P}_{n-1,m}(x),$$

by dividing by n

$$x\frac{1}{n}\frac{d}{dx}\mathcal{P}_{n,m}(x) = \mathcal{P}_{n,m}(x) - \frac{h_{m+1}}{h_m}\mathcal{P}_{n-1,m+1}(x) - m\mathcal{P}_{n-1,m}(x),$$

we have

$$\frac{d}{dx}\mathcal{P}_{n,m}(x) = n\mathcal{P}_{n-1,m}(x).$$

Then

$$x\mathcal{P}_{n-1,m}(x) = \mathcal{P}_{n,m}(x) - \frac{h_{m+1}}{h_m}\mathcal{P}_{n-1,m+1}(x) - m\mathcal{P}_{n-1,m}(x)$$

$$\mathcal{P}_{n,m}(x) = \frac{h_{m+1}}{h_m}\mathcal{P}_{n-1,m+1}(x) + x\mathcal{P}_{n-1,m}(x) + m\mathcal{P}_{n-1,m}(x)$$
$$= \frac{h_{m+1}}{h_m}\mathcal{P}_{n-1,m+1}(x) + (m+x)\,\mathcal{P}_{n-1,m}(x),$$

which is the desired result (3.30).

■

Next, we give some examples to illustrate our results.

Example 3.9 *According to* (3.30) *and* (3.10), *we get Rahmani's algorithm [27] for computing the classical Bernoulli polynomials given by*

$$B_{n+1,m}(x) = \frac{h_{m+1}}{h_m} B_{n,m+1}(x) + (m+x) B_{n,m}(x).$$

We compute $\frac{h_{m+1}}{h_m}$

$$\begin{aligned}\frac{h_{m+1}}{h_m} &= \frac{(-1)^{m+1}\frac{(m+1)!}{m+2}}{(-1)^m \frac{m!}{m+1}} \\ &= (-1)\frac{m!(m+1)}{m+2} \cdot \frac{(m+1)}{m!} \\ &= -\frac{(m+1)^2}{m+2}.\end{aligned}$$

Then

$$B_{n+1,m}(x) = -\frac{(m+1)^2}{m+2} B_{n,m+1}(x) + (m+x) B_{n,m}(x), \tag{3.31}$$

with the initial condition $B_{0,m}(x) = 1$.

Example 3.10 *Applying* (3.30) *and using* (3.12), *we get a new recurrence relation for calculating* $p-$*Bernoulli polynomials*

$$B_{n+1,m}(x;p) = \frac{h_{m+1}}{h_m} B_{n,m+1}(x;p) + (m+x) B_{n,m}(x;p).$$

We calculate $\frac{h_{m+1}}{h_m}$

$$\begin{aligned}\frac{h_{m+1}}{h_m} &= \frac{(-1)^{m+1}\frac{((m+1)!)^2(p+1)!}{(p+1+m+1)!}}{(-1)^m \frac{((m)!)^2(p+1)!}{(p+1+m)}} \\ &= -\frac{m!(m+1)m!(m+1)}{(p+1+m)!(p+m+2)} \cdot \frac{(p+1+m)!}{m!m!} \\ &= -\frac{(m+1)^2}{p+m+2}.\end{aligned}$$

Then

$$B_{n+1,m}(x;p) = -\frac{(m+1)^2}{p+m+2} B_{n,m+1}(x;p) + (m+x) B_{n,m}(x;p), \tag{3.32}$$

with $B_{0,m}(x;p) = 1$ *and* $B_{n,0}(x;p) = B_{n,p}(x)$.

Example 3.11 *From* (3.14) *and* (3.30), *we obtain the recurrence relation for* $p-$*Bell polynomials defined in [25] as follows:*

$$\mathcal{B}_{n+1,m}(x;p) = \frac{h_{m+1}}{h_m} + (m+x) \mathcal{B}_{n,m}(x;p).$$

We compute $\frac{h_{m+1}}{h_m}$

$$\begin{aligned}\frac{h_{m+1}}{h_m} &= \frac{\binom{m+p}{m}}{\binom{m+1+p}{m+1}} \\ &= \frac{(m+p)!}{m!p!} \cdot \frac{m!(m+1)p!}{(m+p)!(m+1+p)} \\ &= \frac{m+1}{m+1+p}.\end{aligned}$$

So

$$\mathcal{B}_{n+1,m}(x;p)=\frac{m+1}{m+1+p}\mathcal{B}_{n,m+1}(x;p)+(m+x)\mathcal{B}_{n,m}(x;p), \tag{3.33}$$

with the initial condition $\mathcal{B}_{0,m}(x;p)=1$.

In particular, for $x=0$ *we have the recurrence formula for* $p-$*Bell numbers given in [25].*

Example 3.12 *By (3.20) and (3.30), we have an algorithm for the Frobenius-Euler polynomials of order* α *given in [30] by the following three-term recurrence formula:*

$$\mathcal{A}_{n+1,m}(x;\lambda,\alpha)=\frac{h_{m+1}}{h_m}\mathcal{A}_{n,m+1}(x;\lambda,\alpha)+(m+x)\mathcal{A}_{n,m}(x;\lambda,\alpha).$$

We compute $\frac{h_{m+1}}{h_m}$

$$\begin{aligned}\frac{h_{m+1}}{h_m}&=\frac{\langle\alpha\rangle_{m+1}}{(\lambda-1)^{m+1}}\cdot\frac{(\lambda-1)^m}{\langle\alpha\rangle_m}\\&=\frac{\alpha(\alpha+1)+\ldots(\alpha+m-1)(\alpha+m)}{(\lambda-1)^m(\lambda-1)}\cdot\frac{(\lambda-1)^m}{\alpha(\alpha+1)+\ldots(\alpha+m-1)}\\&=\frac{\alpha+m}{\lambda-1}.\end{aligned}$$

Then

$$\mathcal{A}_{n+1,m}(x;\lambda,\alpha)=\frac{\alpha+m}{\lambda-1}\mathcal{A}_{n,m+1}(x;\lambda,\alpha)+(m+x)\mathcal{A}_{n,m}(x;\lambda,\alpha), \tag{3.34}$$

with the initial sequence given by $\mathcal{A}_{0,m}(x;\lambda,\alpha)=1$.

We show that, for $\alpha=1$ *we have the recurrence relation for the classical Eulerian polynomials. Also by taking* $\lambda=-1$, *we obtain an algorithm for the generalized Euler polynomials.*

We have been working under the assumption that $h_m\neq 0$. For the general case, set

$$\mathcal{K}_{n,m}=\sum_{k=0}^{n}\left\{\begin{matrix}n+m\\k+m\end{matrix}\right\}_m h_{k+m}$$

and

$$\mathcal{K}_{n,m}(x)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{K}_{k,m}x^{n-k},$$

with $\mathcal{K}_{0,m}=h_m$ and $\mathcal{K}_{n,0}(x)=P_n(x)$. The following result may be proved in much the same way as Theorem 3.7 and Theorem 3.8.

Theorem 3.9 *The polynomials* $\mathcal{K}_{n,m}(x)$ *satisfy the following three-term recurrence relation:*

$$\mathcal{K}_{n+1,m}(x)=\mathcal{K}_{n,m+1}(x)+(m+x)\mathcal{K}_{n,m}(x), \tag{3.35}$$

with the initial sequence given by

$$\mathcal{K}_{0,m}=h_m.$$

Proof. We have

$$\mathcal{K}_{n,m}(x)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{K}_{k,m}x^{n-k},$$

then

$$\mathcal{K}_{n+1,m}(x)=\sum_{k=0}^{n+1}\binom{n+1}{k}\mathcal{K}_{k,m}x^{n+1-k}.$$

We know that

$$\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}.$$

So

$$\mathcal{K}_{n+1,m}(x) = \sum_{k=0}^{n+1} \binom{n}{k} \mathcal{K}_{k,m} x^{n+1-k} + \sum_{k=0}^{n+1} \binom{n}{k-1} \mathcal{K}_{k,m} x^{n+1-k}.$$

We set $k-1=k$ for the second series:

$$\begin{aligned}\mathcal{K}_{n+1,m}(x) &= x \sum_{k=0}^{n} \binom{n}{k} \mathcal{K}_{k,m} x^{n-k} + \sum_{k=0}^{n} \binom{n}{k} \mathcal{K}_{k+1,m} x^{n-k} \\ &= x\mathcal{K}_{n,m}(x) + \sum_{k=0}^{n} \binom{n}{k} \mathcal{K}_{k+1,m} x^{n-k}.\end{aligned}$$

On the other hand, we have:

$$\mathcal{K}_{k+1,m} = \mathcal{K}_{k,m+1} + m\mathcal{K}_{k,m} x^{n-k},$$

then

$$\begin{aligned}\mathcal{K}_{n+1,m}(x) &= x\mathcal{K}_{n,m}(x) + \sum_{k=0}^{n} \binom{n}{k} \left(\mathcal{K}_{k,m+1} + m\mathcal{K}_{k,m}\right) x^{n-k} \\ &= x\mathcal{K}_{n,m}(x) + \sum_{k=0}^{n} \binom{n}{k} \mathcal{K}_{k,m+1} x^{n-k} + m \sum_{k=0}^{n} \binom{n}{k} \mathcal{K}_{k,m} x^{n-k} \\ &= x\mathcal{K}_{n,m}(x) + \mathcal{K}_{n,m+1}(x) + m\mathcal{K}_{n,m}(x) \\ &= \mathcal{K}_{n,m+1}(x) + (m+x)\,\mathcal{K}_{n,m}(x).\end{aligned}$$

Hence the result.

■

Conclusion

Conclusion

In this work, we studied a specific class of polynomials known as the Appell sequence. We derived their explicit formulas and explored their recurrence relations. Through the three chapters, we showed how the properties of these sequences can be unified within a common mathematical framework.

This study may open new perspectives for further investigation into other types of polynomials, or for applying the obtained results in fields such as analysis or combinatorics.

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