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ON CLASS OF FRACTIONAL ELLIPTIC PROBLEMS

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DEDICATION

To the one whose name I carry with pride,

the one who harvested the thorns from my path to pave the way for me to my ***dear father****.*

To the invisible hand, the compassionate heart, and the owner of sincere prayers, my ***great mother***

To my support in life, to my only brother ***Radwan***

To the right hand and the rib that does not lean towards my sisters ***Nabila,Chahla,Aya***

To the sugar of home ***mayar*** *and* ***racim***

To my friends who taught me that life without connection, love and cooperation is worth nothing.

ABSTRACT

The work presented in this dissertation is devoted to study a class of fractional elliptic problems involving the fractional Laplacian operator $(-\Delta)^s$ for $s \in (0,1)$.We recall the essential definitions and notions which we will use in the later chapters.Then,we prove the existence of the unique solution by variational method, and We'll also prove some regularity estimate for the fractional laplacian problems .Finly, we using the Mountain Pass Theorem to prove the existence of solutions for equations driven by a non-local integro-differential operator with homogeneous Dirichlet boundary conditions.

NOTATIONS

$\mathbb{R}^N$	euclidean space of dimension, n ≥ 1.
$\mathbb{R}^{m\times N}$	the space of real matrices.
Ω	an open bounded set of $\mathbb{R}^N$.
$\partial\Omega$	topological boundary of Ω.
$\|\Omega\|$	Lebesgue measure of the set Ω.
$x=(x_1,\cdots,x_N)$	generic point of $\mathbb{R}^N$.
$\mathrm{d}x=\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_N$	Lebesgue measure on Ω.
$\overline{\Omega}$	the closure of the set Ω.
$\rightharpoonup$	weak convergence.
$\to$	strong convergence.
$\hookrightarrow$	Compact injection.
V	a Banach space.
$\langle\cdot,\cdot\rangle_{V'\times V}$	the duality bracket between the space V and its topological dual V'.
$\|.\|_V$	the norm on the Banach space V.
$[\cdot]$	designates the semi-norm.
$(\cdot,\cdot)$	the inner product on the Hilbert space H.
$\|.\|$	the norm on the Hilbert space H.
$\mathscr{C}^0(\mathbb{R})$	the space of continuous functions.
∂_i	The i-th partial derivative; $\partial_i=\frac{\partial}{\partial x_i}$.
∂^α	The derivative of order α; $\partial^\alpha=\frac{\partial^{\|\alpha\|}}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}$.
$\nabla u=\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\cdots,\frac{\partial u}{\partial x_N}\right)$	gradient of $v:\Omega\longrightarrow\mathbb{R}$.
Δu	The Laplacian of u.
div v	The divergence of the vector $v=(v_1,...,v_N)$. div $v=\frac{\partial v_1}{\partial x_1}+...+\frac{\partial v_N}{\partial x_N}$.

$[s]$	denotes the integer part of s.
a.e.	almost everywhere.
p'	The conjugate exponent of p, That is to say $p' = \frac{p}{p-1}$.

INTRODUCTION

The topic of differential fractional equations is one of the most important subjects since the concept of fractional calculus appeared in the correspondence of Leibniz and Lopital in 1695. this topic has several important applications in the real world. Physics, financial, mechanics, chemestry and other various phenomena in diverse fields that can be studied as a fractional equation such as nuclear reactor dynamics, thermoelasticity, mecanical vebrations, bilogical tissues, fractional entropy, fractional diffusion , phase transitions, materials science, water waves, conservation laws . For more details on this subject, we refer to [6] and [1].

Non-local equations have many important applications in different relevant subjects; such as mathematical nance with the price of assets which can be modelled by Levy processes with jump, nonlocal minimal surfaces, ultra-relativistic limits of quantum mechanic , nonlocal electrostatics, the thin obstacle problem etc. Non-local equations, in particular fractional differential equations can be elliptic, parabolic or hyperbolic. This is a vague classification since each type of these equations has its specific characterizations. The elliptic equations are the one that doesn't depends on time, they are called the time-independent equations. We gave many general statements, now to fix some ideas, we consider the following elliptic equation called Poisson's equation,

$$-\Delta u = f \tag{1}$$

where $f \in L^2(\mathbb{R}^n)$ is the nonhomogenous term or the source function and $u \in H^2(\mathbb{R}^n)$. Physically, $u(x)$ can be the density or the concentration of some quantity at a position x. $(-\Delta)$ is the dissusion term in an isotropic medium. the last Equation is a second-order differential equation since the operator that has the highest order is the Laplace's operator $-\Delta = -\sum_j \delta_j^2$. We consider the similar equation of order $2s$ that involves the so-called

fractional Laplacian $(-\Delta)^s$ with $s > 0$,

$$(-\Delta)^s = f \tag{2}$$

The first concern face to a PDE is to give it a sense in a suitable functional space. That is classically to check eventual existence and unicity of a solution in that space. So, in that spirit, does the equation (2) make sense in some spaces? Maybe some kind of Sobolev spaces. And can even the so-called fractional Laplacian be explicitly defined? By definition,

$$(-\Delta)^s u(x) = c(n,s) \lim_{\varepsilon \longrightarrow 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n-2s}} dy, x \in \mathbb{R}^n$$

Those operators naturally incorporate nonlocality,meaning that the value of a solution at a given point depends on the values across the entire domain, not just in the immediate neighborhood, such equation arise in various applications including anomalous diffusion, quantum mechanics, image processing, viscoelasticity,and financial modeling. nonlinear fractional elliptic problems stand out as a key class of equations that combine the effects of nonlocality-arising from fractional derivatives with nonlinear behaviors frequently observed in physical, biological, and engineering systems.

These models have been successfully applied to describe anomalous diffusion in heterogeneous media, heat conduction in materials with memory or fractal structure, and mechanical systems with long-range interactions. Moreover, they appear in diverse areas such as fluid dynamics, image processing, viscoelasticity, and financial mathematics, where classical diffusion operators fail to capture nonlocal dependencies and spatial memory effects.

Mathematically, these problems often involve the fractional Laplacian operator $(-\Delta)^s$ where $s \in (0,1)$, which introduces significant analytical challenges due to its inherently nonlocal nature. The analysis requires the use of fractional Sobolev spaces $H^s(R^N)$, which provide the appropriate functional framework to handle such operators.For more details we study the probelm

$$\begin{cases} (-\Delta)^s u &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3}$$

Here, $s \in (0,1)$ is fixed and $(-\Delta)^s$ is the fractional Laplace operator, which (up to normaliza-

tion factors) may be defined as

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy, \quad x \in \mathbb{R}^n. \tag{4}$$

The dissertation comprises three chapters, each of which is succinctly outlined below.

Chapter 1:(Preliminaries) we start by giving some basic notions, that concern the functional framework necessary to obtain the results of the existence of solutions for the considered problem refers to [5, 9, 12]

Chapter 2: ,will be devoted to the study of fractional elliptic problem of the form

$$\begin{cases} (-\Delta)^s u = f, & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5}$$

We will discuss the well-posedness prove the the existence of the unique solution by variational method. We'll also prove some regularity estimate up to the boundary for the distributional solution of the last equation , see [3],[4],[2]

Chapter 3: We study the existence of solutions for non-local elliptic operators of type

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

using the Mountain Pass Theorem for them.[8][13],[14],[16].

CHAPTER 1

Preliminaries

In this chapter, we present some definitions and recall the necessary results for the remainder of this work. After reviewing some basic results on function spaces and integration, we also provide an introduction to critical point theory, see [5],[7],[**?**].

1.1 Lebesgue spaces $L^p(\Omega)$

Let Ω be an open subset of $\mathbb{R}^N$.

Definition 1.1 *Let $p \in \mathbb{R}$. Then:*

- *For $1 \leq p < +\infty$, we define:*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}^N \text{ measurable} \,\middle|\, \int_\Omega |f(x)|^p \, dx < +\infty \right\}.$$

This is a vector space over $\mathbb{R}^N$, equipped with the norm:

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}.$$

- *For $p = +\infty$, we define:*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R}^N \text{ measurable} \,\middle|\, \exists c > 0 \text{ such that } |f(x)| \leq c \text{ a.e. on } \Omega \right\}.$$

This is a vector space over $\mathbb{R}^N$, equipped with the norm:

$$\|f\|_{L^\infty(\Omega)} = \inf \left\{ c > 0 \,\middle|\, |f(x)| \leq c \text{ a.e. on } \Omega \right\}.$$

- *For $1 \leq p \leq \infty$, we define:*

$$L^p_{loc}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable} \,\middle|\, f \cdot \chi_K \in L^p(\Omega), \, \forall K \text{ compact} \subset \Omega \right\},$$

where χ_K is the indicator function of K.

Notation

For $1 \le p \le +\infty$, we denote by p' the conjugate exponent of p, i.e.:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 1.1 ***(Hölder's Inequality)*** *Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \le p \le +\infty$. Then $f \cdot g \in L^1(\Omega)$, and:*

$$\int_\Omega |f(x)g(x)|\, dx \le \|f\|_{L^p(\Omega)} \cdot \|g\|_{L^{p'}(\Omega)}.$$

Proposition 1.1 *If Ω has finite measure and $1 \le p_1 \le p_2 \le +\infty$, then:*

$$L^{p_2}(\Omega) \hookrightarrow L^{p_1}(\Omega).$$

In particular:

$$L^\infty(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^1(\Omega), \quad \forall 1 \le p \le +\infty.$$

Theorem 1.2 ***(Interpolation Inequality):*** *Let $1 \le p \le q \le +\infty$. If $f \in L^p(\Omega) \cap L^q(\Omega)$, then $f \in L^r(\Omega)$ for all $r \in [p,q]$, and we have:*

$$\|f\|_{L^r(\Omega)} \le \|f\|^{\alpha}_{L^p(\Omega)} \cdot \|f\|^{1-\alpha}_{L^q(\Omega)},$$

where:

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \quad \alpha \in [0,1].$$

Theorem 1.3 .

- *$L^p(\Omega)$ is a Banach space for all $1 \le p \le +\infty$.*
- *$L^p(\Omega)$ is a separable space for all $1 \le p < +\infty$.*
- *$L^p(\Omega)$ is a reflexive space for all $1 < p < +\infty$.*
- *$L^p(\Omega)$ is an uniformly convex space for all $1 < p < +\infty$.*

1.2 Sobolev spaces

1.2.1 Classical Sobolev spaces

Let Ω be an open subset of $\mathbb{R}^N$ $(N>1)$, and let $m\in\mathbb{N}$ and $p\in[1,+\infty]$.

Definition 1.2 *We define the classical Sobolev space $W^{m,p}(\Omega)$ by:*

$$W^{m,p}(\Omega)=\left\{u\in L^p(\Omega)\;\middle|\; D^\alpha u\in L^p(\Omega),\ \forall|\alpha|\le m\right\}$$

where $\alpha=(\alpha_1,\dots,\alpha_N)\in\mathbb{N}^N$ is a multi-index, $D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}$, and $|\alpha|=\sum_{i=1}^N\alpha_i$. Here $D^\alpha u$ denotes the derivative in the distributional sense, i.e.,

$$\langle D^\alpha u,\varphi\rangle=(-1)^{|\alpha|}\langle u,D^\alpha\varphi\rangle,\quad\forall\varphi\in\mathscr{D}(\Omega)$$

$W^{m,p}(\Omega)$ is a vector space over $\mathbb{R}$, equipped with the norm:

- *For $p\in[1,+\infty)$:*
$$\|u\|_{m,p}=\left(\sum_{|\alpha|\le m}\|D^\alpha u\|_p^p\right)^{1/p}$$
- *For $p=+\infty$:*
$$\|u\|_{m,\infty}=\max_{|\alpha|\le m}\|D^\alpha u\|_\infty$$

Theorem 1.4 .

- *$W^{m,p}(\Omega)$ is a Banach space for all $1\le p\le+\infty$*
- *$W^{m,p}(\Omega)$ is separable for all $1\le p<+\infty$*
- *$W^{m,p}(\Omega)$ is reflexive for all $1<p<+\infty$*
- *$W^{m,p}(\Omega)$ is uniformly convex for all $1<p<+\infty$*
- *$W^{m,p}(\Omega)$ is a Hilbert space when $p=2$*

1.2.2 Fractional Sobolev Spaces $W^{s,p}(\Omega)$

Let Ω be an open set in $\mathbb{R}^n$. For any real $s > 0$ and any $p \in [1,\infty)$, we define the fractional Sobolev spaces $W^{s,p}(\Omega)$. They are also called Aronszajn, Gagliardo, or Slobodeckij spaces.

Definition 1.3 *Let $s \in (0,1)$. For any $p \in [1,+\infty)$, we define $W^{s,p}(\Omega)$ as follows:*

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{p}+s}} \in L^p(\Omega\times\Omega) \right\}, \tag{1.1}$$

thus, $W^{s,p}(\Omega)$ is an intermediary Banach space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_\Omega |u|^p \, dx + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx\, dy \right)^{\frac{1}{p}}.$$

Here, the term

$$[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx\, dy \right)^{\frac{1}{p}}$$

is the so-called Gagliardo (semi)norm of u.

It is worth noticing that, as in the classical case with s being an integer, the space $W^{s_0,p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$ when $s \le s_0$. The same happens also when $s_0 = 1$, provided $\partial\Omega$ is $C^{0,1}$ with bounded boundary.

Remark 1 *If $s \ge 1$, the definition 1.1 doesn't hold. Suppose Ω is connected; any measurable function $u : \Omega \to \mathbb{R}$ such that*

$$\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx\, dy < +\infty$$

is actually constant . This is strictly related to the fact that

$$\lim_{s\to 1^-} (s-1)^{\frac{1}{p}} \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \, dx\, dy = C \int_\Omega |\nabla u|^p \, dx$$

Theorem 1.5 *For any $s \in (0,1)$, the space $C_c^\infty(\mathbb{R}^n)$ of smooth functions with compact support is dense in $W^{s,p}(\mathbb{R}^n)$. That is, any function in $W^{s,p}(\mathbb{R}^n)$ can be approximated by a sequence of smooth functions compactly supported in $\mathbb{R}^n$. For more general $\Omega \subset \mathbb{R}^n$, one can have this*

result with the aid of an extension operator that can extend functions from Ω *to* $\mathbb{R}^n$*, provided* Ω *is smooth enough .*

Let $D^{s,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{W^{s,p}}}$ *,we have*

$$D^{s,p}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n),$$

which is not true for a general set $\Omega \subset \mathbb{R}^n$*.*

Special Case ($p = 2$)

$W^{s,2}(\Omega)$ (also denoted $H^s(\Omega)$) is a pre-Hilbert space with inner product:

$$\langle u, v\rangle_{H^s} = \sum_{|\alpha|\le s} \int_\Omega (D^\alpha u)(x)(D^\alpha v)(x)\, dx$$

We have $\langle u, u\rangle_{H^s} = \|u\|^2_{H^s}$.

1.2.3 Espace $H^s(\Omega)$

In this section, we focus on the particular case where $p = 2$.

Definition 1.4

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) \text{ such that } (x,y) \mapsto \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\Omega\times\Omega) \right\},$$

equipped with the scalar product:

$$\langle u, v\rangle_{H^s(\Omega)} = \int_\Omega u(x)v(x)\, dx + \int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dx\, dy.$$

We can also define this space using the Fourier transform as follows:

Definition 1.5 *For* $s \in (0,1)$*, we have:*

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : \int_\Omega (1+|\xi|^{2s})|\mathcal{F}u(\xi)|^2\, d\xi < +\infty \right\},$$

where $\mathcal{F}u(\xi)$ *denotes the Fourier transform of* u*.*

Proposition 1.2 *Let $s \in]0,1[$. For any $u \in H^s(\Omega)$, then:*

$$[u]^2_{H^s(\Omega)} = \frac{2}{C}\int_\Omega |\xi|^{2s} |\mathscr{F}u(\xi)|^2 \, d\xi,$$

that is to say

$$[u]_{H^s(\Omega)} = C\|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\Omega)},$$

where $C = C(N,s)$ is a positive constant.

Definition 1.6 ***(Fourier Transform)*** *Let $\phi \in \mathscr{S}(\mathbb{R}^n)$. The Fourier transform of ϕ is defined by*

$$\mathscr{F}\phi(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-i\xi\cdot x}\phi(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

Note that one can extend $\mathscr{F}$ from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}'(\mathbb{R}^n)$.

The following definition also holds by a simple scaling argument:

$$\mathscr{F}\phi(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\phi(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

1.3 Fractional laplacian and main proporties

In this paragraph, we will give some equivalent definitions of the fractional Laplacian in $\mathbb{R}^N$ and its main properties.

Definition 1.7 *Let $u \in \mathscr{S}(\mathbb{R}^N)$. We have that $\mathscr{F}u \in \mathscr{S}(\mathbb{R}^N)$, but the function $\xi \mapsto \|\xi\|^{2s}\mathscr{F}u(\xi) \notin \mathscr{S}(\mathbb{R}^N)$ because $\|\xi\|^{2s}$ creates a singularity at $\xi = 0$. On the other hand, $\|\xi\|^{2s}\mathscr{F}u(\xi) \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, so we can use the inverse Fourier transform. Therefore, we have the following definition:*

For any $u \in \mathscr{S}(\mathbb{R}^N)$, we define the operator $(-\Delta)^s : \mathscr{S}(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ by

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}(\|\xi\|^{2s}(\mathscr{F}u)(\xi))(x) \quad \forall x \in \mathbb{R}^N,$$

This operator is called the fractional Laplacian of order s.

Definition 1.8 *For any $u \in \mathscr{S}(\mathbb{R}^N)$,*

$$(-\Delta)^s u(x) := C(N,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{\|x-y\|^{N+2s}} \, dy \quad \forall x \in \mathbb{R}^N,$$

where

$$C(N,s) := \frac{s 4^s \pi^{\frac{N}{2}} \Gamma\left(s+\frac{N}{2}\right)}{\Gamma(1-s)}.$$

Remark 2.1

Let us give some light on the normalizing constant $C(N,s)$.

(i) $C(N,s)$ is chosen so that the four definitions (1.7), (1.8), (1.9) and (1.10) are equivalent and the following two identities:

$$\lim_{s \to 0^+} (-\Delta)^s u = u \quad \text{and} \quad \lim_{s \to 1^-} (-\Delta)^s u = -\Delta u$$

(ii) Moreover,

$$C(N,s) \sim s(1-s) \quad \text{as} \quad s \to 0^+ \quad \text{and} \quad s \to 1^-. \tag{2.4}$$

For more details on the asymptotic behavior of $C(N,s)$, We recall that by definition:

$$\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \, dy := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \, dy.$$

In other words, P.V. stands for the "Cauchy principal value". Since then, we can write

$$(-\Delta)^s u(x) := C(N,s) \, \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \, dy \quad \forall x \in \mathbb{R}^N. \tag{1.2}$$

Theorem 1.6 *The following theorem shows that we can do without using the "Cauchy principal value" in (1.2) if $s \in (0, \frac{1}{2})$.*

Let $u \in \mathscr{S}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. The integral

$$\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \, dy,$$

is absolutely convergent if and only if $s \in (0, \frac{1}{2})$.

So we have

$$(-\Delta)^s u(x) = C(N,s) \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy,$$

for $s \in (0, \frac{1}{2})$.

Proof. : We write the integral as follows:

$$\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy = \int_{B(x,\epsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy + \int_{\mathbb{R}^N \setminus B(x,\epsilon)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy,$$

where $B(x,\epsilon)$ is a ball of radius ϵ centered at x. Near $y = x$, we use a Taylor expansion of $u(y)$ around x. Since u is smooth, we have:

$$u(y) = u(x) + \nabla u(x) \cdot (y-x) + \mathcal{O}(|y-x|^2).$$

Substituting into the integrand, we obtain:

$$\frac{u(x)-u(y)}{|x-y|^{N+2s}} = \frac{-\nabla u(x)\cdot(y-x) + \mathcal{O}(|y-x|^2)}{|x-y|^{N+2s}}.$$

The dominant term is of order $|x-y|^{-(N+2s-1)}$. For the integral to be absolutely convergent, this singularity must be integrable. In polar coordinates, the integrability condition is:

$$\int_{B(x,\epsilon)} \frac{1}{|x-y|^{N+2s-1}}\, dy = C \int_0^\epsilon r^{-(N+2s-1)} \cdot r^{N-1}\, dr = C \int_0^\epsilon r^{-2s}\, dr.$$

This integral converges if and only if $-2s > -1$, i.e., $s < \frac{1}{2}$. At infinity, $|x-y| \approx |y|$ for large y, and since $u \in \mathscr{S}(\mathbb{R}^N)$, $u(y)$ decays faster than any power of $|y|$. Thus, the integrand behaves like $|y|^{-(N+2s)}$ at infinity. In polar coordinates, the integrability condition is:

$$\int_{\mathbb{R}^N \setminus B(x,R)} \frac{1}{|y|^{N+2s}}\, dy = C \int_R^\infty r^{-(N+2s)} \cdot r^{N-1}\, dr = C \int_R^\infty r^{-1-2s}\, dr.$$

This integral converges if and only if $-1-2s < -1$, i.e., $s > 0$. Combining the two conditions, we find that the integral is absolutely convergent if and only if $s \in \left(0, \frac{1}{2}\right)$.

■

Definition 1.9 *The third definition is given in the form of a well-defined integral on* $\mathbb{R}^N$ *as follows:*

For any $u \in \mathscr{S}(\mathbb{R}^N)$*, we have*

$$(-\Delta)^s u(x) := -\frac{1}{2} C(N,s) \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\, dy, \quad \forall x \in \mathbb{R}^N.$$

Definition 1.10 *Another definition of the fractional Laplacian comes from the theory of semi-groups in* $\mathbb{R}^N \times [0,+\infty)$*. This definition is classical and it is more useful to study partial differential equations.*

For any $u \in \mathscr{S}(\mathbb{R}^N)$*, we have*

$$(-\Delta)^s u(x) = \frac{s}{\Gamma(1-s)} \int_0^{+\infty} \frac{u(x)-w(x,t)}{t^{1+s}}\, dt, \tag{2.8}$$

where $w(x,t)$ *is the solution to the following problem:*

$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w = 0 & on\ \mathbb{R}^N \times [0,+\infty), \\ w(x,0) = u(x). \end{cases}$$

Definition of $(-\Delta)^s$ in the space of tempered distribution $\mathscr{S}'$

Definition 1.11 *Let* Ω *be an open set. Provided* $u \in L^1_s$*, the distribution* $(-\Delta)^s u$ *is defined by*

$$\langle (-\Delta)^s u, \phi \rangle = \int_{\mathbb{R}^n} u(-\Delta)^s \phi\, dx, \quad \forall \phi \in C_c^\infty(\Omega).$$

Saying that $(-\Delta)^s u = f$ *in* $\mathscr{D}'(\Omega)$ *is equivalent to the very weak formulation*

$$\int_{\mathbb{R}^n} u(-\Delta)^s \phi\, dx = \int_\Omega f\phi\, dx, \quad \forall \phi \in C_c^\infty(\Omega).$$

Definition 1.12 *We define* $D^{s,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^s}}$*, the completion of* $C_c^\infty(\Omega)$*, which is a Hilbert*

space with respect to the norm

$$\|u\|_{D^{s,2}(\Omega)} := \left(\iint_{\mathbb{R}^{2n}} \frac{(\phi(x)-\phi(y))^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{\frac{1}{2}}.$$

If $u \in D^{s,2}(\Omega) \subset L^1_s$ *satisfies*

$$(-\Delta)^s u = f \quad \text{in } \mathscr{D}'(\Omega),$$

we have the weak formulation

$$\langle u, \phi \rangle_{D^{s,2}(\Omega)} = \int_\Omega f\phi\, dx, \quad \phi \in \mathscr{D}(\Omega),$$

where

$$\langle u, \phi \rangle_{D^{s,2}(\Omega)} := C(n,s) \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\, dx\, dy = \int_{\mathbb{R}^n} |\xi|^{2s} \hat{u}\hat{\phi}\, d\xi.$$

Relation between the "Classical" Laplacian and the Fractional Laplacian

The following proposition describes the relation between the "classical" Laplacian and the fractional Laplacian .

Proposition 1.3 *Let* $u \in \mathscr{S}(\mathbb{R}^N)$. *Then*

(i) $\lim_{s\to 0^+}(-\Delta)^s u = u$,

(ii) $\lim_{s\to 1^-}(-\Delta)^s u = -\Delta u$.

Proof. Let $u \in \mathscr{S}(\mathbb{R}^N)$, where $\mathscr{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decreasing functions. We demonstrate the following limits for the fractional Laplacian $(-\Delta)^s u$:

(i) $\lim_{s\to 0^+}(-\Delta)^s u = u$

The fractional Laplacian $(-\Delta)^s u$ is defined via the Fourier transform:

$$(-\Delta)^s u = \mathscr{F}^{-1}\left(|\xi|^{2s}\mathscr{F}(u)(\xi)\right),$$

where $\mathscr{F}$ denotes the Fourier transform. As $s \to 0^+$, the symbol $|\xi|^{2s}$ converges to 1 for all $\xi \neq 0$. Thus:

$$\lim_{s\to 0^+}(-\Delta)^s u = \mathscr{F}^{-1}\big(\mathscr{F}(u)(\xi)\big) = u.$$

This shows that $(-\Delta)^s u$ converges to u as $s \to 0^+$.

(ii) $\lim_{s\to 1^-}(-\Delta)^s u = -\Delta u$

As $s \to 1^-$, the symbol $|\xi|^{2s}$ converges to $|\xi|^2$. Therefore:

$$\lim_{s\to 1^-}(-\Delta)^s u = \mathscr{F}^{-1}\big(|\xi|^2 \mathscr{F}(u)(\xi)\big) = -\Delta u,$$

since the Fourier transform of the Laplacian Δu is $-|\xi|^2\mathscr{F}(u)(\xi)$. This shows that $(-\Delta)^s u$ converges to $-\Delta u$ as $s \to 1^-$. ■

Properties of the Fractional Laplacian

Let $\psi, \phi \in C_c^\infty(\mathbb{R}^n)$. We have:

1. **Translation.** If we set $\phi_a(x) = \phi(x+a)$, $a \in \mathbb{R}$,

$$((-\Delta)^s \phi_a)(x) = ((-\Delta)^s \phi)(x+a), \quad x \in \mathbb{R}^n.$$

2. **Scaling.** If we set $\phi_\lambda(x) = \phi(\lambda x)$, $\lambda \in \mathbb{R}$,

$$((-\Delta)^s \phi_\lambda)(x) = \lambda^{2s}((-\Delta)^s \phi)(\lambda x). \tag{1.3}$$

3. **Product.**

$$(-\Delta)^s(\phi\psi)(x) = \phi(x)(-\Delta)^s\psi(x) + \psi(x)(-\Delta)^s\phi(x) - C(n,s)\int_{\mathbb{R}^n} \frac{(\phi(x)-\phi(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dy.$$

4. **Convolution.** If we take ρ_n a regularization sequence,

$$\rho_n * ((-\Delta)^s \phi)(x) = ((-\Delta)^s(\rho_n * \phi))(x), \quad \forall x \in \mathbb{R}^n. \tag{1.4}$$

Note that the same holds for the inverse of $(-\Delta)^s$.

Proof. Let $\psi, \phi \in C_c^\infty(\mathbb{R}^n)$.

1. For **1)**, we set the change of variable $z = y + a$,

$$((-\Delta)^s \phi_a)(x) = C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{\phi_a(x) - \phi_a(y)}{|x-y|^{n+2s}}\, dy$$

$$= C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{\phi(x+a) - \phi(y+a)}{|(x+a)-(y+a)|^{n+2s}}\, dy$$

$$= C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{\phi(x+a) - \phi(z)}{|(x+a)-z|^{n+2s}}\, dz$$

$$= ((-\Delta)^s \phi)(x+a).$$

2. For **2)**, with $\phi_\lambda(x) = \phi(\lambda x)$, $\lambda \in \mathbb{R}$, by a simple change of variable $z = \lambda y$,

$$((-\Delta)^s \phi_\lambda)(x) = C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{\phi_\lambda(x) - \phi_\lambda(y)}{|x-y|^{n+2s}}\, dy$$

$$= C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{\phi(\lambda x) - \phi(\lambda y)}{|x-y|^{n+2s}}\, dy$$

$$= \lambda^{-n} C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{\phi(\lambda x) - \phi(z)}{|x - z/\lambda|^{n+2s}}\, dz$$

$$= \lambda^{2s} C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{\phi(\lambda x) - \phi(z)}{|\lambda x - z|^{n+2s}}\, dz$$

$$= \lambda^{2s} ((-\Delta)^s \phi)(\lambda x).$$

3. For **3)**, with $x \in \mathbb{R}^n$, the matter is to do nothing, i.e., to add and remove the same value,

$$((-\Delta)^s \phi\psi)(x) = C(n,s)\,\text{P.V.} \int_{\mathbb{R}^n} \frac{\phi(x)\psi(x) - \phi(x)\psi(y) + \phi(x)\psi(y) - \phi(y)\psi(y)}{|x-y|^{n+2s}}\, dy$$

$$= \phi(x)(-\Delta)^s \psi(x) + \int_{\mathbb{R}^n} \frac{\phi(x)\psi(x) - \phi(x)\psi(x) + \phi(y)\psi(x) - \phi(y)\psi(x) + \phi(x)\psi(y) - \phi(y)\psi(y)}{|x-y|^{n+2s}}\, dy$$

$$= \phi(x)(-\Delta)^s \psi(x) + \psi(x)(-\Delta)^s \phi(x) - C(n,s) \int_{\mathbb{R}^n} \frac{(\phi(x) - \phi(y))(\psi(x) - \psi(y))}{|x-y|^{n+2s}}\, dy.$$

4. For **4)**,we have

$$\mathscr{F}(\rho_n * ((-\Delta)^s \phi))(\xi) = \widehat{\rho_n}(\xi)\mathscr{F}((-\Delta)^s \phi)(\xi) = \widehat{\rho_n}(\xi)(|\xi|^{2s}\widehat{\phi}(\xi)) = |\xi|^{2s}\widehat{\rho_n}(\xi)\widehat{\phi}(\xi)$$

$$= |\xi|^{2s}\widehat{\rho_n * \phi}(\xi) = \mathscr{F}((-\Delta)^s(\rho_n * \phi))(\xi).$$

For the inverse of the fractional Laplacian, the proof is similarly the same.

■

1.4 Critical Point Theory

1.4.1 Variational Principles

Definition 1.13 ***(Variational Framework)****: Let $(X, \|\cdot\|)$ be a Banach space and $J : X \to \mathbb{R}$ a C^1-functional.*

(i) A point $u \in X$ is a critical point if $J'(u) = 0$

(ii) The value $c = J(u)$ is a critical value

(iii) The set $K_c = \{u \in X : J(u) = c,\ J'(u) = 0\}$ is the critical set at level c

Theorem 1.7 ***First (Variation Principle):*** *For $J \in C^1(X, \mathbb{R})$, if u_0 is a local minimizer of J, then $J'(u_0) = 0$.*

Proof. For any $v \in X$ and $t > 0$, consider the difference quotient:

$$\frac{J(u_0 + t\,v) - J(u_0)}{t} \geq 0$$

Taking $t \to 0^+$ gives $\langle J'(u_0), v\rangle \geq 0$. Replacing v with $-v$ completes the proof. ■

1.4.2 Main Critical Point Theorems

Theorem 1.8 ***(Ekeland's Variational Principle)*** *Let (X, d) be a complete metric space and $J : X \to \mathbb{R} \cup \{+\infty\}$ proper, l.s.c., and bounded below. For any $\epsilon > 0$ and $u \in X$ satisfying:*

$$J(u) < \inf_X J + \epsilon$$

there exists $v_\epsilon \in X$ *such that:*

1. $J(v_\epsilon) \leq J(u)$
2. $d(u, v_\epsilon) \leq 1$
3. $J(w) > J(v_\epsilon) - \epsilon d(v_\epsilon, w) \quad \forall w \neq v_\epsilon$

Definition 1.14 (Palais-Smale Condition) *A functional* $J \in C^1(H, \mathbb{R})$ *satisfies the Palais-Smale condition (PS) if every sequence* $\{u_n\} \subset H$ *such that:*

1. $|J(u_n)|$ *is bounded*
2. $J'(u_n) \to 0$ *in* H^*

admits a convergent subsequence.

Theorem 1.9 (Mountain Pass Theorem) *Let* H *be a Banach space and* $J \in C^1(H, \mathbb{R})$ *satisfy the (PS) condition. Suppose:*

1. $J(0) = 0$
2. $\exists\, \alpha, \rho > 0$ *such that* $J(u) \geq \alpha$ *for all* $\|u\| = \rho$
3. $\exists\, v \in H$ *with* $\|v\| > \rho$ *such that* $J(v) \leq 0$

Then J *has a critical value* $c \geq \alpha$ *characterized by:*

$$c = \inf_{\rho \in \rho} \max_{t \in [0,1]} J(\rho(t))$$

where $\Gamma = \{\rho \in C([0,1], H) : \rho(0) = 0,\ \rho(1) = v\}$.

CHAPTER 2

Variational method for fractional elliptic equation $(-\Delta)^s u = f$

In this chapter we use the variational method to shall do the weak formulation and prove existence and uniqueness of weak solution of the problem

$$\begin{cases} (-\Delta)^s u = f, & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.1}$$

we show also the regularity estimates for the last problem, While we could use Lax-Milgram to do the same as we did with the classical Poisson's equation. First of all, we will define some basic notions and some notations we'll use through this chapter. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $s \in (0,1)$. Consider the bilinear form

$$\langle \cdot, \cdot \rangle_{D^{s,2}(\Omega)} : C_c^\infty(\Omega) \times C_c^\infty(\Omega) \to \mathbb{R}$$

defined by

$$\langle u, v \rangle_{D^{s,2}(\Omega)} = \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \, dx \, dy,$$

which is a scalar product on $C_c^\infty(\Omega)$. The Hilbert space $D^{s,2}(\Omega)$ is defined as the completion of $C_c^\infty(\Omega)$ with respect to this scalar product.

Many problems are equivalent to:

$$Au = 0, \tag{2.2}$$

where $A\colon X \to Y$ is an operator between two Banach spaces. When the problem is variational, there exists a differentiable functional
$\varphi : X \to \mathbb{R}$ such that $A = \varphi'$, i.e.,

$$\langle Au, v \rangle = \lim_{t \to 0} \frac{\varphi(u + tv) - \varphi(u)}{t}. \tag{2.3}$$

The space Y then corresponds to the dual space X' of X, and equation (2.2) is equivalent

to $\varphi'(u)=0$, i.e.,

$$\langle \varphi'(u), v\rangle = 0 \quad \forall v \in X, \tag{2.4}$$

which implies that every critical point of φ is a weak solution of (2.2).

Lemma 2.1 *If Ω is a bounded Lipschitz open set, then*

$$D^{s,2}(\Omega) = \{u \in H^s(\mathbb{R}^n) \text{ such that } u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

Proof. Since Ω is Lipschitz, $C_c^\infty(\Omega)$ is dense in $H_0^s(\Omega)$ with respect to the $D^{s,2}(\Omega)$-norm. Thus, $H_0^s(\Omega) \subseteq D^{s,2}(\Omega)$ Let $u \in D^{s,2}(\Omega)$. Since $D^{s,2}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{s,2}}$, there exists a sequence $(u_n) \subset C_c^\infty(\Omega)$ such that $u_n \to u$ a.e. in $\mathbb{R}^n$. For a.e. $x \in \mathbb{R}^n \setminus \Omega$ and $\varepsilon > 0$, there exists $n_\varepsilon > 0$ such that for all $n > n_\varepsilon$,

$$|u(x)| = |u_n(x) - u(x)| < \varepsilon \quad \forall \varepsilon > 0,$$

hence $u = 0$ in $\mathbb{R}^n \setminus \Omega$. As Ω is bounded, $u \in H^s(\mathbb{R}^n)$. Therefore, $u \in H_0^s(\Omega)$, implying $D^{s,2}(\Omega) \subseteq H_0^s(\Omega)$.

For $s = 1$, $D^{1,2}(\Omega) = H_0^1(\Omega)$. The norm in $D^{s,2}(\Omega)$ is

$$\|u\|_{D^{s,2}(\Omega)} = \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{1/2},$$

which is equivalent to the H^s-norm by the following proposition. ■

Proposition 2.1 *Let $s \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u : \Omega \to \mathbb{R}$ be a measurable, compactly supported function. Then there exists a constant $C = C(n,s,\Omega) > 0$ such that*

$$\|u\|_{L^2(\Omega)} \le C \|u\|_{D^{s,2}(\Omega)}.$$

Proof. Consider the following estimate:

$$\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \ge \int_\Omega u^2(x) \left(\int_{\mathbb{R}^n \setminus \Omega} |x-y|^{-n-2s}\, dy \right) dx. \tag{2.5}$$

Since

$$\int_{\mathbb{R}^n \setminus \Omega} |x-y|^{-n-2s}\, dy \ge C(n,s) |\Omega|^{-2s/n}, \tag{2.6}$$

This follows from the proof in [9, Lemma 6.1]. Let

$$\rho := \left(\frac{|\Omega|}{\omega_n}\right)^{1/n},$$

Notice that Ω and the ball $B_\rho(x)$ of radius ρ have the same measure. Thus it follows that

$$|(\mathscr{C}\Omega \cap B_\rho(x))| = |B_\rho(x)| - |\Omega \cap B_\rho(x)| = |\Omega| - |\Omega \cap B_\rho(x)| = |\Omega \cap \mathscr{C}B_\rho(x)|.$$

Therefore,

$$\begin{aligned}
\int_{\mathscr{C}} \frac{dy}{|x-y|^{n+2s}} &= \int_{(\mathscr{C}\Omega)\cap B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} + \int_{(\mathscr{C}\Omega\cap\mathscr{C}B_\rho(x))} \frac{dy}{|x-y|^{n+2s}} \\
&\geq \int_{(\mathscr{C}\Omega)\cap B_\rho(x))} \frac{dy}{\rho^{n+2s}} + \int_{(\mathscr{C}\Omega)\cap\mathscr{C}B_\rho(x))} \frac{dy}{|x-y|^{n+2s}} \\
&= \frac{|\mathscr{C}\Omega \cap B_\rho(x)|}{\rho^{n+2s}} + \int_{(\mathscr{C}\Omega)\cup\mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} \\
&= \frac{|\Omega \cup \mathscr{C}B_\rho(x)|}{\rho^{n+2s}} + \int_{(\mathscr{C}\Omega)\cup\mathscr{C}B_\rho(x))} \frac{dy}{|x-y|^{n+2s}} \\
&\geq \int_{\Omega\cup\mathscr{C}B_\rho(x)} \frac{dy}{|x-y|^{n+2s}} + \int_{(\mathscr{C}\Omega)\cap\mathscr{C}B_\rho(x))} \frac{dy}{|x-y|^{n+2s}} \\
&= \int_{\mathscr{C}\cap B_\rho(x)} \frac{dy}{|x-y|^{n+2s}}.
\end{aligned}$$

Using polar coordinates centered at x, i.e., $y = x + rz$ with $r = |x-y|$,

$$\begin{aligned}
\int_{\mathscr{C}\Omega} \frac{dy}{|x-y|^{n+2s}} &= |S_1| \int_\rho^\infty r^{-2s-1}\, dr \\
&= |S_2| \frac{\rho^{-2s}}{2s} \\
&= C(n,s)|\Omega|^{\frac{-2s}{n}}.
\end{aligned}$$

Combining (2.5) and (2.6), we obtain:

$$\|u\|_{L^2(\Omega)} \leq C(n,s,\Omega)\|u\|_{D^{s,2}(\Omega)}. \tag{2.7}$$

■

2.1 Existence and Uniqueness of a Weak Solution

In this section, we are interested in the existence and the uniqueness of a weak solution of the above Dirichlet problem, that is the well-posedness of the equation. The problem (2.1) is weakly formulated as follows:

$$\int_{\mathbb{R}^{2n}} \frac{(v(x)-v(y))(u(x)-u(y))}{|x-y|^{n+2s}}\,dy\,dx - \int_{\Omega} f(x)v(x)\,dx = 0, \quad v \in D^{s,2}(\Omega). \tag{2.8}$$

After we establish the weak formulation of the problem, we want to check if the latter is well-posed, i.e., there exists an unique solution depending continuously on f.

Proposition 2.2 *Consider the following variational problem:*

$$\text{Find } u \in D^{s,2}(\Omega) \text{ such that, } J(u,u) = \min_{v \in D^{s,2}(\Omega)} J(v,v), \tag{2.9}$$

where

$$J(u,v) = \langle u, v \rangle_{D^{s,2}(\Omega)} - \int_{\Omega} f\,v\,dx.$$

Then, (2.9) admits an unique solution.

We need to prove some results before the proof of the Proposition.

Lemma 2.2 *Let J be such that*

$$J(u,v) = \langle u, v \rangle_{D^{s,2}(\Omega)} - \int_{\Omega} f\,v\,dx.$$

Then J is continuous.

Proof. Let $J(u,v) = a(u,v) - l(v)$ where

$$a(u,v) = \int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dy\,dx,$$
$$l(v) = \int_{\Omega} f\,v\,dx.$$

The bilinear form $a(u,v)$ is continuous on $D^{s,2}(\Omega) \times D^{s,2}(\Omega)$ as it defines the inner product

on this Hilbert space. For the linear form $l(v)$, we have

$$|l(v)| = \left| \int_\Omega f\, v \, dx \right| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}.$$

By the continuous embedding of $D^{s,2}(\Omega)$ into $L^2(\Omega)$, there exists a constant $C > 0$ such that $\|v\|_{L^2(\Omega)} \leq C\|v\|_{D^{s,2}(\Omega)}$. Thus,

$$|l(v)| \leq C\|f\|_{L^2(\Omega)} \|v\|_{D^{s,2}(\Omega)},$$

implying the continuity of l. Therefore, $J(u,v)$ is continuous. ■

Lemma 2.3 *Let $(u_n)_{n\geq 1} \subset D^{s,2}(\Omega)$ be a minimizing sequence, i.e.,*

$$\lim_{k\to+\infty} J(u_k, u_k) = \inf_{v\in D^{s,2}(\Omega)} J(v,v) = I(\Omega). \tag{2.10}$$

Then, $(u_k)_{k\geq 1}$ is bounded in $D^{s,2}(\Omega)$.

Proof. By the minimizing property (2.10), for all $k \geq 1$,

$$I(\Omega) \leq J(u_k, u_k) < I(\Omega) + \frac{1}{k}. \tag{2.11}$$

For any $\varphi \in C_c^\infty(\Omega)$, applying the Cauchy-Schwarz inequality:

$$I(\Omega) \leq J(\varphi,\varphi) = \|\varphi\|^2_{D^{s,2}(\Omega)} - \int_\Omega f\varphi \, dx \leq \|\varphi\|^2_{D^{s,2}(\Omega)} + \|f\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}. \tag{2.12}$$

For the sequence (u_k), we have:

$$J(u_k, u_k) = \|u_k\|^2_{D^{s,2}(\Omega)} - \int_\Omega f\, u_k \, dx. \tag{2.13}$$

Using Young's inequality on the integral term:

$$\left| \int_\Omega f\, u_k \, dx \right| \leq \frac{1}{2}\|f\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_k\|^2_{L^2(\Omega)}. \tag{2.14}$$

Substituting (2.15) into (2.13) :

$$\|u_k\|^2_{D^{s,2}(\Omega)} \leq J(u_k, u_k) + \frac{1}{2}\|f\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_k\|^2_{L^2(\Omega)}.$$

By the continuous embedding $D^{s,2}(\Omega) \hookrightarrow L^2(\Omega)$, there exists $C > 0$ such that $\|u_k\|_{L^2(\Omega)} \leq C\|u_k\|_{D^{s,2}(\Omega)}$. Thus,

$$\|u_k\|^2_{D^{s,2}(\Omega)} \leq J(u_k, u_k) + \frac{1}{2}\|f\|^2_{L^2(\Omega)} + \frac{C^2}{2}\|u_k\|^2_{D^{s,2}(\Omega)}.$$

Rearranging terms (assuming $1 - \frac{C^2}{2} > 0$):

$$\|u_k\|^2_{D^{s,2}(\Omega)}\left(1 - \frac{C^2}{2}\right) \leq J(u_k, u_k) + \frac{1}{2}\|f\|^2_{L^2(\Omega)}.$$

Since $J(u_k, u_k)$ is bounded and $I(\Omega)$ is finite, (u_k) is bounded in $D^{s,2}(\Omega)$. Combining inequalities (2.10) and (2.12) with Proposition 2.2, we obtain:

$$\|u_k\|^2_{D^{s,2}(\Omega)} < I(\Omega) + \frac{1}{k} + \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} + \|u_k\|^2_{L^2(\Omega)}\right).$$

Using the continuous embedding $D^{s,2}(\Omega) \hookrightarrow L^2(\Omega)$, there exists a constant $C(n, s, \Omega) > 0$ such that:

$$\|u_k\|_{L^2(\Omega)} \leq C(n, s, \Omega)\|u_k\|_{D^{s,2}(\Omega)}.$$

Substituting this into the previous inequality gives:

$$\|u_k\|^2_{D^{s,2}(\Omega)} < I(\Omega) + \frac{1}{k} + \frac{1}{2}\left(\|f\|^2_{L^2(\Omega)} + C(n, s, \Omega)^2\|u_k\|^2_{D^{s,2}(\Omega)}\right).$$

Rearranging terms, we find:

$$C(n, s, \Omega)\|u_k\|^2_{D^{s,2}(\Omega)} < I(\Omega) + \frac{1}{k} + \frac{1}{2}\|f\|^2_{L^2(\Omega)} \to R \in \mathbb{R} \quad \text{as } k \to +\infty,$$

where $R = I(\Omega) + \frac{1}{2}\|f\|^2_{L^2(\Omega)}$ (after adjusting the constant $C(n, s, \Omega)$ if necessary). This establishes that $(u_k)_{k\geq 1}$ is bounded in $D^{s,2}(\Omega)$. ■

Proof of Proposition 2.2 Under the regularity assumptions on Ω, $D^{s,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$. Since $(u_k)_{k\geq 1}$ is bounded in $D^{s,2}(\Omega)$, there exists a subsequence $(u_{k_l})_{l\geq 1}$ of

$(u_k)_{k\geq 1}$ such that:

$$u_{k_l} \rightharpoonup u \text{ in } D^{s,2}(\Omega) \quad \text{(weak convergence)}$$
$$u_{k_l} \to u \text{ in } L^2(\Omega) \quad \text{(strong convergence)}$$
$$u_{k_l} \to u \text{ a.e } \Omega$$

Note that while we have different types of convergence, the uniqueness of limits ensures they all share the same limit u.

From the minimizing property, we have:

$$J(u_{k_l}, u_{k_l}) = \int_{\mathbb{R}^{2n}} \frac{|u_{k_l}(x) - u_{k_l}(y)|^2}{|x-y|^{n+2s}}\, dy\, dx - \int_\Omega f\, u_{k_l}\, dx \leq I(\Omega) + \frac{1}{l}, \quad \forall l > 0.$$

This implies:

$$\int_{\mathbb{R}^{2n}} \frac{|u_{k_l}(x) - u_{k_l}(y)|^2}{|x-y|^{n+2s}}\, dy\, dx < \int_\Omega f\, u_{k_l}\, dx + I(\Omega) + \frac{1}{l}, \quad \forall l > 0.$$

Applying Fatou's lemma, we obtain:

$$\int_{\mathbb{R}^{2n}} \liminf_{l\to+\infty} \frac{|u_{k_l}(x) - u_{k_l}(y)|^2}{|x-y|^{n+2s}}\, dy\, dx \leq \liminf_{l\to+\infty} \int_\Omega f\, u_{k_l}\, dx + I(\Omega).$$

Since $u_{k_l} \rightharpoonup u$ in $L^2(\Omega)$, we have:

$$\liminf_{l\to+\infty} \int_\Omega f\, u_{k_l}\, dx = \lim_{l\to+\infty} \int_\Omega f\, u_{k_l}\, dx = \int_\Omega f\, u\, dx.$$

Therefore:

$$\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dy\, dx \leq \int_\Omega f\, u\, dx + I(\Omega).$$

since $(u_k)_{\leq 1}$ converges weakly to u in $L^2(\Omega)$, we have

$$\lim inf_{l\to+\infty} \int_\Omega f\, u_{k_l}\, dx = \lim_{l\to+\infty} \int_\Omega f\, u_{k_l}\, dx = \int_\Omega f\, u\, dx.$$

This completes the proof, establishing the required convergence properties and the mini-

mality of u.

$$\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dy\, dx \le \int_\Omega f\, u\, dx + I(\Omega).$$

Therefore,

$$J(u,u) = \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dy\, dx - \int_\Omega f\, u\, dx \le I(\Omega).$$

Thus,

$$J(u,u) \le I(\Omega) \Longrightarrow J(u,u) = I(\Omega).$$

This establishes that J has a minimizer $u \in D^{s,2}(\Omega)$.

To prove uniqueness, let $u_1, u_2 \in D^{s,2}(\Omega)$ be solutions of (2.1), i.e.,

$$\begin{cases} (-\Delta)^s u_1 = f & \text{in } \Omega \\ u_1 = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta)^s u_2 = f & \text{in } \Omega \\ u_2 = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Set $w = u_1 - u_2$. Then we have:

$$\begin{cases} (-\Delta)^s w = 0 & \text{in } \Omega \\ w = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.15}$$

The weak formulation of (2.15), using w as the test function, gives:

$$\int_{\mathbb{R}^{2n}} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\, dy\, dx = 0, \quad w \in D^{s,2}(\Omega).$$

Combining this with inequality (2.7), we obtain:

$$\|w\|^2_{L^2(\Omega)} \le C(n,s,\Omega)\|w\|^2_{D^{s,2}(\Omega)} = 0 \Longrightarrow \|w\|^2_{L^2(\Omega)} = 0.$$

Hence $w = 0$ almost everywhere in Ω, which implies $u_1 = u_2$ almost everywhere in Ω. This completes the proof of uniqueness. ■

Proposition 2.3 *Equation (2.10) has an unique solution that depends continuously on the data $f \in L^2(\Omega)$.*

Proof. By Proposition 2.2, equation (2.1) admits an unique solution.

To prove continuous dependence on the data, let $u_1, u_2 \in D^{s,2}(\Omega)$ be solutions correspond-

ing to data $f_1, f_2 \in L^2(\Omega)$ respectively. From the weak formulation:

$$\langle u_i, v\rangle_{D^{s,2}(\Omega)} = \int_\Omega f_i v \, dx = \ell_i(v), \quad \forall v \in D^{s,2}(\Omega),\ i = 1,2.$$

Considering the difference $w = u_1 - u_2$, we have:

$$\langle w, v\rangle_{D^{s,2}(\Omega)} = \int_\Omega (f_1 - f_2) v \, dx, \quad \forall v \in D^{s,2}(\Omega).$$

Taking $v = w$ and applying the Cauchy-Schwarz inequality:

$$\|w\|^2_{D^{s,2}(\Omega)} = \int_\Omega (f_1 - f_2) w \, dx \leq \|f_1 - f_2\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}.$$

Using the coercivity inequality (2.7):

$$\|w\|_{L^2(\Omega)} \leq C(n,s,\Omega)\|w\|_{D^{s,2}(\Omega)}.$$

Combining these results:

$$\|w\|_{D^{s,2}(\Omega)} \leq C(n,s,\Omega)\|f_1 - f_2\|_{L^2(\Omega)}.$$

This inequality establishes the continuous dependence:

$$\|u_1 - u_2\|_{D^{s,2}(\Omega)} \leq C\|f_1 - f_2\|_{L^2(\Omega)},$$

proving the solution map $f \mapsto u$ is continuous in the specified spaces. Given by the weak formulation. Using inequality (2.7), we get

$$\begin{aligned}
\|u_1 - u_2\|_{D^{s,2}(\Omega)} &= \frac{\langle u_1 - u_2, u_1 - u_2\rangle_{D^{s,2}(\Omega)}}{\|u_1 - u_2\|_{D^{s,2}(\Omega)}} \\
&= \frac{\ell(u_1 - u_2)}{\|u_1 - u_2\|_{D^{s,2}(\Omega)}} \\
&\leq \frac{\|u_1 - u_2\|_{L^2(\Omega)}\|f_1 - f_2\|_{L^2(\Omega)}}{\|u_1 - u_2\|_{D^{s,2}(\Omega)}} \\
&\leq C(n,s,\Omega)\frac{\|u_1 - u_2\|_{D^{s,2}(\Omega)}\|f_1 - f_2\|_{L^2(\Omega)}}{\|u_1 - u_2\|_{D^{s,2}(\Omega)}} \\
&\leq C(n,s,\Omega)\|f_1 - f_2\|_{L^2(\Omega)};
\end{aligned}$$

we are done. ■

2.2 Regularity

In this section, we study the regularity of the equation (2.1) by using the Riesz potential and the Green function to estimate the regularity up to the boundary for a distributional solution of the equation

$$(-\Delta)^s u = f \tag{2.16}$$

Before we prove some regularity results, we need to recall the definition of some spaces. Recall

$$L^1_s := \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ such that } \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+2s}}\, dx < +\infty \right\},$$

And $\Omega \subset \mathbb{R}^n$ be an open set. We say $u : \Omega \to \mathbb{R}$ is in $C^{0,\alpha}(\Omega)$, i.e., Hölder continuous with exponent $\alpha \in (0,1)$, if

$$\|u\|_{C^{0,\alpha}(\Omega)} := \sup_{\substack{x \neq y \\ x,y \in \Omega}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < +\infty.$$

Lemma 2.4 *Let $f \in L^\infty(B_1)$, and let $u \in L^1_s \cap L^\infty_{\text{loc}}(B_1)$ such that $(-\Delta)^s u = f$ in B_1. Then for $r \in (0,1)$ and every $\alpha \in (0, \min(1,2s))$, there exists a constant $C(n,s,r,\alpha) > 0$ such that*

$$\|u\|_{C^{0,\alpha}(B_r)} \leq C(n,s,r,\alpha)\left(\|u\|_{L^\infty(B_r)} + \|f\|_{L^\infty(B_1)}\right).$$

Proof. Let $\eta \in C^\infty_c(\mathbb{R}^n)$ be a non-negative function satisfying:

- $\eta \equiv 1$ on B_r
- $\eta \equiv 0$ on $\mathbb{R}^n \setminus B_1$
- $\|D\eta\|_{L^\infty} \leq 1$

Consider the Riesz kernel:

$$I_{2s}(x,y) = \begin{cases} C(n,s)|x-y|^{2s-n} & \text{n} > 2\text{s} \\ \frac{1}{\pi}\log\frac{1}{|x|} & \text{n} = 2\text{s} = 1 \\ -C(1,s)|x-y|^{2s-n} & \text{n} = 1 < 2\text{s} \end{cases}$$

where $C(n,s)$ is a positive normalization constant.

The inverse fractional Laplacian can be expressed as a Riesz potential for $n > 2s$:

$$(-\Delta)^{-s} u(x) = C(n,s) \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^{n-2s}} dy = I_{2s} * u(x), \tag{2.17}$$

where $C(n,s) = \pi^{2s+n/2} \frac{\Gamma(s+n/2)}{\Gamma(-s)}$.

Define the potential

$$v(x) := \int_{\mathbb{R}^n} I_{2s}(x,y)(\zeta f)(y) dy, \quad x \in \mathbb{R}^n.$$

We claim $(-\Delta)^s v = \eta f$ distributionally. Indeed, recognizing $I_{2s}(x,y) = I^0_{2s}(x-y)$, we calculate:

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^n} I^0_{2s}(x-y)(\eta f)(y) dy \\ &= I^0_{2s} * (\eta f)(x) \\ &= (-\Delta)^{-s}(\eta f)(x), \end{aligned}$$

hence by the inverse property, $(-\Delta)^s v = \eta f$.

Let $a, b > 0$, $\alpha \in (0,1)$, and $m \in \mathbb{R}$ with $m + \alpha > 0$. The remaining arguments follow through... For constants $C_{m,\alpha} > 0$ depending only on m and α:

$$|a^{-m} - b^{-m}| \le C_{m,\alpha} |a-b|^\alpha \max\left(a^{-(m+\alpha)}, b^{-(m+\alpha)}\right) \tag{2.18}$$

$$|\ln b - \ln a| \le \frac{1}{\alpha} |a-b|^\alpha \max\left(a^{-\alpha}, b^{-\alpha}\right) \tag{2.19}$$

Proof of (2.18)**:** Let $c > 1$. Using Hölder's inequality:

$$\frac{1}{m}(1 - c^{-m}) = \int_1^c t^{-m-1} dt \le \left(\int_1^c dt\right)^\alpha \left(\int_1^\infty t^{\frac{-m-1}{1-\alpha}} dt\right)^{1-\alpha}$$

Evaluating the integrals:

$$= (c-1)^\eta \left(\frac{1-\alpha}{m+\alpha}\right)^{1-\alpha} \le \left(\frac{1}{m+\alpha}\right)^{1-\eta} (c-1)^\alpha$$

Set $C_{m,\alpha} := \left(\frac{1}{m+\alpha}\right)^{1-\alpha}$. Then:

$$|1-c^{-m}| \le C_{m,\alpha}|m||c-1|^{\alpha}$$

Taking $c := \frac{b}{a}$ $(a>0)$:

$$\left|\frac{a^{-m}-b^{-m}}{a^{-m}}\right| \le C_{m,\alpha}|m|\left|\frac{b-a}{a}\right|^{\alpha}$$

which simplifies to:

$$|a^{-m}-b^{-m}| \le C_{m,\alpha}|b-a|^{\alpha}a^{-m-\alpha} \le C_{m,\alpha}|b-a|^{\alpha}\max\left(b^{-m-\alpha}, a^{-m-\alpha}\right).$$

■

Proof of (2.19)**:** Similarly for $c>1$:

$$\ln c = \int_1^c t^{-1}\,dt \le \left(\int_1^c dt\right)^{\eta}\left(\int_1^{\infty} t^{\frac{-1}{1-\eta}}\,dt\right)^{1-\eta}$$

Evaluating:

$$= (c-1)^{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{1-\alpha} \le \frac{1}{\alpha}(c-1)^{\alpha}$$

With $c = b/a$, $a>0$, we have:

$$|\ln b - \ln a| \le \frac{1}{\alpha}|b-a|^{\alpha}a^{-\alpha} \le \frac{1}{\alpha}|b-a|^{\alpha}\max(b^{-\alpha}, a^{-\alpha}).$$

Using the triangle inequality:

$$\big||x-z|-|y-z|\big| \le |x-y|,$$

and also:

$$\max(|x-z|,|y-z|) \le |x-z|+|y-z|.$$

Therefore, we get:

$$\big||x-z|^{-m}-|y-z|^{-m}\big| \le C_{m,\alpha}|x-y|^{\alpha}\left(|x-z|^{-m-\alpha}+|y-z|^{-m-\alpha}\right). \tag{2.20}$$

And also:

$$\big|\ln|x-z|-\ln|y-z|\big| \le \frac{1}{\alpha}|x-y|^{\alpha}\left(|x-z|^{-\alpha}+|y-z|^{-\alpha}\right). \tag{2.21}$$

■

If $n > 2s$, with $\alpha \in (0, \min(1, 2s))$ and $x, y \in B_r$,

$$v(x) - v(y) = C(n,s) \int_{\mathbb{R}^n} \left(|x-z|^{2s-n} - |y-z|^{2s-n} \right) (\eta f)(z) dz.$$

Using inequality (2.20) with $m = n - 2s$:

$$\begin{aligned} |v(x) - v(y)| &\leq C(n,s,\alpha) \int_{\mathbb{R}^n} |x-y|^{\alpha} \left(|x-z|^{2s-n-\alpha} + |y-z|^{2s-n-\alpha} \right) |(\eta f)(z)| dz \\ &\leq C(n,s,\alpha) |x-y|^{\alpha} \int_{\mathbb{R}^n} 2|x-z|^{2s-n-\alpha} |(\eta f)(z)| dz. \end{aligned}$$

Since $\frac{|y-z|}{|x-z|} > 1$, using the fact that the support $\operatorname{supp}(\eta) \subseteq B_1 \subset B(x,2)$ (the ball of radius 2 centered at $x \in B_r$), we obtain

$$\begin{aligned} |v(x) - v(y)| \leqslant \; & C(n,s,\alpha) |x-y|^{\alpha} \textstyle\int_{\mathbb{R}^n} |x-y|^{2s-n-\alpha} |(\eta f)(y)| dy. \\ & \leqslant C(n,s,\alpha) |x-y|^{\alpha} \textstyle\int_{B(x,2)} |x-y|^{2s-n-\alpha} dy \|f\|_{L^\infty(B_1)}. \\ & = C(n,s,\alpha) |x-y|^{\alpha} \textstyle\int_{B(0,2)} |y|^{2s-n-\alpha} dy \|f\|_{L^\infty(B_1)}. \\ & = 2^n C(n,s,\alpha) |x-y|^{\alpha} \textstyle\int_{B_1} |y|^{2s-n-\alpha} dy \|f\|_{L^\infty(B_1)}, \end{aligned}$$

up to relabeling the positive constant $C(n;s,\alpha)$ that depends on the dimension n, s and α. Using polar coordinates $y = rx$, $r > 0$, we get that

$$\int_{B_1} |y|^{2s-n-\alpha} dy = \int_{S_1} \int_0^1 r^{n-1} r^{2s-n-\beta} dr d\sigma(x) = \frac{|S_1|}{2s-\alpha}$$

Then,

$$|v(x) - v(y)| \leq C(n,s,\alpha) |x-y|^{\alpha} \|f\|_{L^\infty(B_1)}.$$

Consequently,

$$\|v\|_{C^{0,\alpha}(B_1)} \leq C(n,s,\alpha) \|f\|_{L^\infty(B_1)} \tag{2.22}$$

where $C(n,s,\alpha)$ is a positive constant depending on the dimension, s and α.

If $n = 1 = 2s$, $x, y \in B_r$, the inequality (2.21) lets us write that

$$\begin{aligned}|v(x)-v(y)| &\leq \frac{1}{\pi}\int_{\mathbb{R}} \big|\log|x|-\log|y|\big|\,|\eta f(z)|dz \\ &\leq \frac{1}{\alpha\pi}|x-y|^{\alpha}\int_{\mathbb{R}} \left(|x|^{-\alpha}+|y|^{-\alpha}\right)|(\eta f)(z)|dz \\ &\leq 2\frac{1}{\alpha\pi}|x-y|^{\alpha}\int_{\mathbb{R}} |x|^{-\alpha}|(\eta f)(z)|dz,\end{aligned}$$

since $\frac{|y|}{|x|} > 1$. Then

$$\begin{aligned}|v(x)-v(y)| &\leq 2\frac{1}{\alpha\pi}|x-y|^{\alpha}\int_{-2}^{2} |x|^{-\alpha}dz\|f\|_{L^{\infty}(B_1)} \\ &= C_{\alpha}|x-y|^{\alpha}\|f\|_{L^{\infty}(B_1)},\end{aligned}$$

where we obtain

$$\|v\|_{C^{0,\alpha}(B_1)} \leq C_{\alpha}\|f\|_{L^1(B_1)}, \tag{2.23}$$

with C_{α} a positive constant depending on α.

If $n = 1 > 2s$, $x, y \in B_r$, using again (2.20) with $m = 1-2s$ and $\alpha \in (0, \min(1, 2s))$,

$$\begin{aligned}|v(x)-v(y)| &\leq C_{1,s}C(s,\alpha)\int_{\mathbb{R}} |x-y|^{\alpha}\left(|x-z|^{2s-1-\alpha}+|y-z|^{2s-1-\alpha}\right)|(\eta f)(z)|dz \\ &\leq C_{s,\alpha}|x-y|^{\alpha}\int_{\mathbb{R}} 2|x-z|^{2s-1-\alpha}|(\eta f)(z)|dz \\ &\leq C_{s,\alpha}|x-y|^{\alpha}\int_{[x-2,x+2]} |x-z|^{2s-1-\alpha}dz\|f\|_{L^{\infty}(B_1)} \\ &= 2C_{s,\alpha}|x-y|^{\alpha}\int_{-1}^{1} |z|^{2s-1-\alpha}dz\|f\|_{L^{\infty}(B_1)} \\ &= C(s,\alpha)|x-y|^{\alpha}\|f\|_{L^{\infty}(B_1)}.\end{aligned}$$

Consequently,

$$\|v\|_{C^{0,\alpha}(B_1)} \leq C(n,s,\alpha)\|f\|_{L^{\infty}(B_1)} \tag{2.24}$$

where $C(s,\alpha)$ is a positive constant depending on s and α.

Therefore, combining (2.22), (2.23) and (2.24), we get for every $s \in (0,1)$ and $\alpha \in (0, \min(1, 2s))$

the following estimate:

$$\|v\|_{C^{0,\alpha}(B_1)} \le C(n,s,\alpha)\|f\|_{L^\infty(B_1)}, \tag{2.25}$$

up to relabeling the positive constant $C(n,s,\alpha)$.

Now, let $w := u - v$. Then w is s-harmonic in B_1, i.e., $(-\Delta)^s w = 0$ in B_1. for every $r' \in (0,r)$,

$$|\nabla w(x)| \le C(n,r',s)\big(\|w\|_{L^\infty(B_1)} - w(x)\big), \quad \forall x \in B_{r'}.$$

Thus we get that

$$\begin{aligned} |\nabla w(x)| &\le C(n,s,r')\big(\|u-v\|_{L^\infty(B_1)} - (u-v)(x)\big) \\ &\le C(n,s,r\infty)\big(\|u\|_{L^\infty(B_r)} + \|v\|_{L^\infty(B_r)} + \|u\|_{L^\infty(B_r)} + \|v\|_{L^\infty(B_r)}\big) \\ &= C(n,s,r')\big(\|u\|_{L^\infty(B_r)} + \|v\|_{L^\infty(B_r)}\big) < +\infty. \end{aligned} \tag{2.26}$$

Hence w is Lipschitz in B_r and therefore also Hölder continuous in B_r. Indeed, for $\alpha \in (0, \min(1,2s))$,

$$\frac{|w(x)-w(y)|}{|x-y|^\alpha} = \frac{|w(x)-w(y)|}{|x-y|}|x-y|^{1-\alpha} \le (2r)^{1-\alpha}\frac{|w(x)-w(y)|}{|x-y|} < +\infty.$$

Thus from (2.26), we obtain

$$\|w\|_{C^{0,\alpha}(B_r)} \le C(n,s,r')\big(\|u\|_{L^\infty(B_r)} + \|v\|_{L^\infty(B_r)}\big). \tag{2.27}$$

Finally, using the inequalities (2.25) and (2.27), we have

$$\begin{aligned} \|u\|_{C^{0,\alpha}(B_r)} &= \|w+v\|_{C^{0,\alpha}(B_r)} \\ &\le \|w\|_{C^{0,\alpha}(B_r)} + \|v\|_{C^{0,\alpha}(B_r)} \\ &\le C(n,s,r')\big(\|u\|_{L^\infty(B_r)} + \|v\|_{L^\infty(B_r)}\big) + C(n,s,\alpha)\|f\|_{L^\infty(B_1)} \\ &\le C(n,s,r')\big(\|u\|_{L^\infty(B_r)} + C(n,s,\alpha)\|f\|_{L^\infty(B_1)}\big) \\ &\le C(n,s,r,\alpha)\big(\|u\|_{L^\infty(B_r)} + \|f\|_{L^\infty(B_1)}\big), \end{aligned}$$

where $C(n,s,r,\alpha)$ is a positive constant depending only on n, s, r, and α. ■

The distributional solution u of the equation

$$(-\Delta)^s u = f,$$

is Holder continuous locally, that is the interior regularity. And this result will help to prove the regularity of the solution up to the boundary.

Boundary Regularity

The Green function provides an explicit representation for solutions of Poisson's equation involving the fractional Laplacian. Before proving boundary regularity in Proposition 3.3.3, we establish key properties of the Green function and utilize them in our proof.

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain. Consider the Green function G for the fractional Laplacian $(-\Delta)^s$ in Ω. For any function $f \in L^p(\Omega)$ satisfying $2sp > n$, we define the solution operator:

$$Gf(x) := \int_\Omega G(x,y) f(y) dy,$$

where G satisfies the boundary condition $G(x,y) = 0$ when $x \in \mathbb{R}^n \setminus \Omega$ or $y \in \mathbb{R}^n \setminus \Omega$.

Properties

Let $\varphi \in C_c^1(\mathbb{R}^n)$, then the following equalities hold:

$$(G(-\Delta)^s \varphi)(x) = \varphi(x), \qquad \forall x \in \mathbb{R}^n; \tag{2.28}$$

$$(-\Delta)^s (G\varphi)(x) = \varphi(x), \qquad \forall x \in \Omega. \tag{2.29}$$

For the Green function G, we have the following estimate:

$$G(x,y) \leq \begin{cases} C|x-y|^{2s-n} \min f\left(\frac{d^s(x) d^s(y)}{|x-y|^{2s}}, 1\right), & \text{n} > 2\text{s}, \\ C \log\left(1 + \frac{d^{\frac{1}{2}}(x) d^{\frac{1}{2}}(y)}{|x-y|}\right), & \text{n} = 1 = 2\text{s}, \\ C|x-y|^{2s-1} \min\left(\frac{(d(x)d(y)^{\frac{2s-1}{2}})}{|x-y|^{2s-1}}, \frac{d(x)^s d(y)^s}{|x-y|^{2s}}\right), & \text{n=1<2s} \end{cases} \tag{2.30}$$

where d is the distance function to $\mathbb{R}^n \setminus \Omega$, and C is a positive constant depending only on n, s, and Ω. Thus the following results hold.

Proposition 2.4 *Let $f \in L^\infty(\Omega)$, then*

- *If $n \geq 2s$:*

$$|Gf(x)| \leq C d^s(x) \|f\|_{L^\infty(\Omega)}, \tag{2.31}$$

- *If $n = 1 < 2s$, we have:*

$$|Gf(x)| \leq C d^{\frac{2s-1}{2}}(x) \|f\|_{L^\infty(\Omega)}, \tag{2.32}$$

where $C > 0$ depends only on n, s, and Ω. Furthermore, we get that

$$\|Gf\|_{C^{0,\beta}(\overline{\Omega})} \leq C \|f\|_{L^\infty(\Omega)}, \tag{2.33}$$

for $\beta \in (0, s]$ if $n \geq 2s$ and $\beta \in (0, \frac{2s-1}{2}]$ for $n = 1 < 2s$.

Proof. Assume $f \geq 0$ without loss of generality (for $f \leq 0$, take absolute value). For every $x \in \Omega$:

$$\int_\Omega G(x,y) f(y) dy = \int_{\Omega \cap B(x, d^s(x)/2)} G(x,y) f(y) dy + \int_{\Omega \cap (B(x, diam(\Omega)) \setminus B(x, d^s(x)/2))} G(x,y) f(y) dy.$$

We prove . For $n > 2s$, using the first estimate of (2.30) with polar coordinates:

$$\begin{aligned} \int_{\Omega \cap B(x, d^s(x)/2)} G(x,y) f(y) dy &\leq C \int_{B(x, d^s(x)/2)} |x-y|^{2s-n} f(y) dy \\ &\leq C \|f\|_{L^1(\Omega)} \int_{S_1} \int_0^{d^s(x)/2} r^{2s-1} dr d\sigma \\ &= \frac{C \|f\|_{L^\infty(\Omega)}}{s 2^{2s+1}} d^{2s}(x) \\ &= C(\Omega, s) d^s(x) \|f\|_{L^\infty(\Omega)}. \end{aligned} \tag{2.34}$$

By the first estimate of (2.26), with the fact that $d(y) \le |x-y| + d(x) \le 3|x-y|$ since $y \notin B(x, d(x)/2)$

$$\begin{aligned}
\int_{B(x,\mathrm{diam}(\Omega))\setminus B(x,d(x)/2)} G(x,y)f(y)dy &\le \int_{B(x,\mathrm{diam}(\Omega))\setminus B(x,d(x)/2)} |x-y|^{2s-n} \min\left(\frac{d^s(x)d^s(y)}{|x-y|^{2s}}, 1\right) f(y)dy, \\
&\le C\|f\|_{L^\infty(\Omega)} \int_{B(x,\mathrm{diam}(\Omega))\setminus B(x,d(x)/2)} |x-y|^{2s-n} \frac{d^s(x)d^s(y)}{|x-y|^{2s}} dy, \\
&\le 3^s C d^s(x)\|f\|_{L^\infty(\Omega)} \int_{B(x,\mathrm{diam}(\Omega))\setminus B(x,d(x)/2)} |x-y|^{2s-n} \frac{|x-y|^s}{|x-y|^{2s}} dy \\
&\le C d^s(x)\|f\|_{L^\infty(\Omega)} \int_{B(x,\mathrm{diam}(\Omega))\setminus B(x,d(x)/2)} |x-y|^{s-n} dy \\
&= C d^s(x)\|f\|_{L^1(\Omega)} \int_{S_1}\int_0^{\mathrm{diam}(\Omega)} r^{s-1} dr d\sigma, \\
&= C d^s(x)\|f\|_{L^\infty(\Omega)}
\end{aligned} \tag{2.35}$$

up to relabeling the positive constant C. Thus, (2.34) and (2.35) give the desired result for $n > 2s$.

For $n = 1 = 2s$, note that for any $a, b > 0$,

$$\int_0^{ba} \log\left(1 + \frac{ba}{t}\right) dt = ba\log 2 + ba\int_0^{ba} \frac{dt}{t+ba} = a(2b\log 2);$$

with the second estimate of(2.30), by setting $a = d^{1/2}(x)$ and $b = 2\,\mathrm{diam}(\Omega) > d(y)\ \forall y \in \Omega$, we have that

$$\begin{aligned}
\int_{B(x,d(x)/2)} G(x,y)f(y)dy &\le C\int_{B(x,d(x)/2)} \log\left(1 + \frac{d^{1/2}(x)d^{1/2}(y)}{|x-y|}\right) dy, \\
&\le C\|f\|_{L^\infty(\Omega)} \int_{B(x,d(x)/2)} \log\left(1 + \frac{d^{1/2}(x)\,\mathrm{diam}^{1/2}(\Omega)}{|x-y|}\right) dy \\
&= C\|f\|_{L^\infty(\Omega)} \int_0^{d^{1/2}(x)\mathrm{diam}(\Omega)} \log\left(1 + \frac{d^{1/2}(x)\,\mathrm{diam}^{1/2}(\Omega)}{t}\right) dt, \\
&= C d^{1/2}(x)\|f\|_{L^\infty(\Omega)}(2\,\mathrm{diam}(\Omega)\log 2) \\
&= C d^{1/2}(x)\|f\|_{L^\infty(\Omega)},
\end{aligned} \tag{2.36}$$

where the positive constant C has changed. Using the fact that $d(y) \le |x-y| + d(x) \le 3|x-y|$, we have that

$$\begin{aligned}\int_{B(x,\operatorname{diam}(\Omega))\setminus B(x,d(x)/2)} G(x,y)f(y)dy &\le C\int_{B(x,\operatorname{diam}(\Omega))}\log\left(1+\frac{d^{1/2}(x)d^{1/2}(y)}{|x-y|}\right)f(y)dy;\\ &\le C\|f\|_{L^\infty(\Omega)}\int_{B(x,\operatorname{diam}(\Omega))}\log\left(1+\frac{3^{1/2}d^{1/2}(x)|x-y|^{1/2}}{|x-y|}\right)dy\\ &= C\|f\|_{L^\infty(\Omega)}\int_{B(x,\operatorname{diam}(\Omega))}\log\left(1+3^{1/2}d^{1/2}(x)|x-y|^{-1/2}\right)dy;\\ &\le C\|f\|_{L^\infty(\Omega)}\int_0^{\operatorname{diam}(\Omega)} 3^{1/2}d^{1/2}(x)r^{-1/2}dy;\\ &= C\,d^{1/2}(x)\|f\|_{L^\infty(\Omega)}\end{aligned} \tag{2.37}$$

up to relabeling the positive constant C. Therefore, by (2.36) and (2.37), we obtain (2.31) for $n\ge 2s$.

Now, we prove (2.32), for $n=1<2s$ using the last estimate of (2.30),

$$\int_\Omega G(x,y)f(y)dy \le C\int_\Omega |x-y|^{2s-1}\frac{(d(x)d(y))^{\frac{2s-1}{2}}}{|x-y|^{2s-1}}dy \le C|\Omega|(\operatorname{diam}\Omega)^{\frac{2s-1}{2}}\|f\|_{L^\infty(\Omega)}.$$

Finally, let us prove (2.33). For $n\ge 2s$, using (2.31), we obtain

$$|Gf(x)|\le C\,d^s(x)\|f\|_{L^\infty(\Omega)} \quad\text{and}\quad |Gf(y)|\le C\,d^s(y)\|f\|_{L^\infty(\Omega)},\quad \forall x,y\in\mathbb{R}^n.$$

If we assume that $2|x-y|\ge\max(d(x),d(y))$, we have that

$$\begin{aligned}|Gf(x)-Gf(y)| &\le |Gf(x)|+|Gf(y)|\\ &\le C(d^s(x)+d^s(y))\|f\|_{L^\infty(\Omega)}\\ &\le 2^{s+1}C|x-y|^s\|f\|_{L^\infty(\Omega)}\\ &= C|x-y|^s\|f\|_{L^\infty(\Omega)},\quad \forall x,y\in\mathbb{R}^n\end{aligned} \tag{2.38}$$

where the value of the constant $C>0$ has been changed a little bit. For $n=1<2s$, using (2.32) and the same argument as above (case $n\ge 2s$), we obtain that

$$|Gf(x)-Gf(y)|\le 2^{(2s+1)/2}C|x-y|^{\frac{2s-1}{2}}\|f\|_{L^\infty(\Omega)} = C|x-y|^{\frac{2s-1}{2}}\|f\|_{L^\infty(\Omega)},\quad \forall x,y\in\mathbb{R}^n. \tag{2.39}$$

Now, assume that $|x-y|<\frac{d(y)}{2}$ and set $r:=\frac{d(y)}{2}$ such that $y+rz\in\Omega$, $\forall z\in B(0,2)$. Put

$F(z) := Gf(y+rz)$, $z \in B(0,2)$. Let us consider the case $n \geq 2s$, by (2.31)

$$|F(z)| = |Gf(y+rz)| \leq C d^s(y+rz)\|f\|_{L^\infty(\Omega)} \leq C d^s(y)\|f\|_{L^\infty(\Omega)} \leq C r^s \|f\|_{L^\infty(\Omega)}. \tag{2.40}$$

Remark that

$$(-\Delta)^s F(z) = r^{2s}(-\Delta)^s Gf(z) = r^{2s} f(z) \quad \text{in } B(0,2),$$

then by the interior regularity Lemma (2.4) and (2.40), we get

$$\begin{aligned}
\left| \frac{F\left(\frac{x-y}{r}\right) - F(0)}{\left(\frac{|x-y|}{r}\right)^\alpha} \right| &\leq \|F\|_{C^{0,\alpha}(B(0,1))}, \quad \alpha \in (0,1) \\
&\leq C(n,s)\left(\|F\|_{L^\infty(B(0,1))} + r^{2s}\|f\|_{L^\infty(\Omega)}\right) \\
&\leq C(n,s)\left(C r^s \|f\|_{L^\infty(\Omega)} + r^{2s}\|f\|_{L^\infty(\Omega)}\right) \\
&= C(n,s,\Omega) r^s \|f\|_{L^\infty(\Omega)}
\end{aligned}$$

where $C(n,s,\Omega)$ is a positive constant depending only on n, s and Ω. Therefore, we can conclude that

$$|Gf(x) - Gf(y)| \leq C(n,s,\Omega) r^{s-\alpha} |x-y|^\alpha \|f\|_{L^\infty(\Omega)},$$

for $\alpha \in (0,s]$. This, together with the estimate in (2.38), we obtain (2.33) for $n \geq 2s$. For $n = 1 < 2s$, the proof is similar to the case $n \geq 2s$. ■

Proposition 2.5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain. Let $f \in L^\infty(\Omega)$ and u solve the equation*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u \in D^{s,2}(\mathbb{R}^n) \end{cases}$$

Then $u(x) = Gf(x) = \int_\Omega G(x,y) f(y) dy$. In particular, $u \in C(\mathbb{R}^n)$ and there exists $C = C(\Omega) > 0$ such that

$$|u(x)| \leq C d^s(x) \|f\|_{L^\infty(\Omega)}.$$

This shows u is continuous up to the boundary.

Proof. Assume $f \geq 0$ (by decomposition $f = f^+ - f^-$), so $u \geq 0$. Define

$$v(x) = \int_\Omega G(x,y) f(y) dy,$$

and consider the subdomains

$$\Omega_n = \left\{ x \in \Omega : d(x) > \frac{1}{n} \right\}.$$

Let ρ_n be a symmetric mollifier (Chapter 1, (1.0.3)), and set $v_n = \rho_n * v$, $f_n = \rho_n * f$. Then

$$\begin{aligned} \rho_n * v(x) &= \int_{\mathbb{R}^n} \rho_n(x-y) v(y) dy \\ &= \int_{\mathbb{R}^n} \rho_n(x-y)(-\Delta)^{-s} f(y) dy \\ &= \rho_n * (-\Delta)^{-s} f(x) = (-\Delta)^{-s}(\rho_n * f)(x), \end{aligned}$$

so $(-\Delta)^s v_n(x) = f_n(x)$ for all $x \in \Omega_n$. Moreover, $v_n \in C_c^1(\mathbb{R}^n)$ and $v_n \geq 0$.

Let u_n solve

$$\begin{cases} (-\Delta)^s u_n = f_n & \text{in } \Omega_n \\ u_n \in D^{s,2}(\Omega_n) \end{cases}$$

we have $v_n \geq u_n$ in $\mathbb{R}^n$ since $u_n = 0$ in $\mathbb{R}^n \setminus \Omega_n$. Taking limits, $v \geq u$ in $\mathbb{R}^n$ because:

- $\rho_n * f \to f$ in $L^2(\Omega)$
- $u_n \to u$ in $D^{s,2}(\Omega)$ as $n \to \infty$
- $u_n \to 0$ in $\mathbb{R}^n \setminus \Omega$

Now consider the expanded domains

$$\Omega_n^- = \left\{ x \in \mathbb{R}^n : d(x) > -\frac{1}{n} \right\}$$

and let u_{-n} solve

$$\begin{cases} (-\Delta)^s u_n^- = f & \text{in } \Omega_n^- \\ u_n^- \in D^{s,2}(\Omega_n^-) \end{cases}$$

For $n_0 \geq 1$ fixed, the sequence $(u_n^-)_{n \geq n_0}$ is bounded in $D^{s,2}(\Omega_{n_0}^-)$ (by the Poincaré inequality). Then there exists a subsequence of $(u_{-n})_{n \geq n_0}$ (which we still denote by $(u_n^-)_{n \geq n_0}$) that converges strongly in $L^2(\Omega_{n_0}^-)$. Since $u_n^- = 0$ in $\mathbb{R}^n \setminus \Omega_n^-$, it follows that the limit $u_- = 0$ in $\mathbb{R}^n \setminus \Omega$. Therefore, $u_- = u \in D^{s,2}(\Omega)$.

Note that u_n^- is continuous in Ω_n^- because $Gf - u_n^-$ is s-harmonic in Ω_n^- (and thus smooth in Ω_n^-). Now observe that for every $n \geq n_0$, we have:

- $(-\Delta)^s(u_n^- - v) \geq 0$ in $D'(\Omega)$
- $u_n^- - v$ is continuous up to the boundary of Ω
- $u_n^- - v \geq 0$ in $\mathbb{R}^n \setminus \Omega$

Thus, $u_n^- - v \geq 0$ in $\mathbb{R}^n$. Taking the limit as $n \to \infty$, we obtain $v \leq u^- = u$ in $\mathbb{R}^n$.

We finally conclude that $u = v$ in $\mathbb{R}^n$. ■

CHAPTER 3

Existence of solution for non-local elliptic equation $(-\Delta)^s u = f(x,u)$

The purpose of this paper is to study the existence of solutions for equations driven by a non-local integrate-differential operator with homogeneous Dirichlet boundary conditions. we using the Mountain Pass Theorem To make the nonlinear methods work

$$\begin{cases} (-\Delta)^s u &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.1}$$

Theorem 3.1 *Let $s \in (0,1)$, $n > 2s$, and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ be a function satisfying conditions*

- $\gamma K \in L^1(\mathbb{R}^n)$, *where* $\gamma(x) = \min\{|x|^2, 1\}$;
- *there exists $\lambda > 0$ such that $K(x) \geq \lambda |x|^{-(n+2s)}$ for any $x \in \mathbb{R}^n \setminus \{0\}$;*
- $K(x) = K(-x)$ *for any $x \in \mathbb{R}^n \setminus \{0\}$.*

Let f be a Carathéodory function verifying

- $|f(x,t)| \leq a_1 + a_2 |t|^{q-1}$ *a.e. $x \in \Omega$, $t \in \mathbb{R}$;*
- $\lim_{|t|\to 0} \frac{f(x,t)}{|t|} = 0$ *uniformly in $x \in \Omega$;*
- *there exist $\mu > 2$ and $r > 0$ such that $0 < \mu F(x,t) \leq t f(x,t)$ a.e. $x \in \Omega$, $|t| \geq r$.*

Then,the follow problem admits a Mountain Pass type solution $u \in X_0$ which is not identically zero

$$\begin{cases} \int_{\mathbb{R}^{2n}} \big(u(x)-u(y)\big)\big(\phi(x)-\phi(y)\big) K(x-y)\,dx\,dy &= \int_\Omega f(x,u(x))\phi(x)\,dx \quad \forall \phi \in X_0, \\ u \in X_0. \end{cases} \tag{3.2}$$

First of all,we prove some preliminary results

Lemma 3.1 *Assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function , Then, for any $\varepsilon > 0$ there*

exists $\delta = \delta(\varepsilon)$ *such that a.e.* $x \in \Omega$ *and for any* $t \in \mathbb{R}$

$$|f(x,t)| \leq 2\varepsilon|t| + q\delta(\varepsilon)|t|^{q-1} \tag{3.3}$$

and so, as a consequence,

$$|F(x,t)| \leq \varepsilon|t|^2 + \delta(\varepsilon)|t|^q, \tag{3.4}$$

where F *is defined*

$$F(x,t) = \int_0^t f(x,\tau)d\tau \tag{3.5}$$

Proof. For any $\varepsilon > 0$ there exists $\sigma = \sigma(\varepsilon) > 0$ such that for any $t \in \mathbb{R}$ with $|t| < \sigma$ and a.e. $x \in \Omega$ we get

$$|f(x,t)| \leq 2\varepsilon|t|. \tag{3.6}$$

Moreover, there exists $\delta = \delta(\sigma) > 0$ such that a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$ with $|t| \geq \sigma$ we have

$$|f(x,t)| \leq q\delta(\sigma)|t|^{q-1}. \tag{3.7}$$

■

Lemma 3.2 *Let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Carathéodory function and let* F *be as in* (3.2)*. Then, there exist two positive measurable functions* $m = m(x)$ *and* $M = M(x)$ *such that a.e.* $x \in \Omega$ *and for any* $t \in \mathbb{R}$

$$F(x,t) \geq m(x)|t|^\mu - M(x). \tag{3.8}$$

then $m, M \in L^\infty(\Omega)$.

Proof. Let $r > 0$ then, a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$ with $|t| \geq r > 0$

$$\frac{tf(x,t)}{F(x,t)} \geq \mu.$$

Suppose $t > r$. Dividing by t and integrating both terms in $[r,t]$ we obtain

$$F(x,t) \geq \frac{F(x,r)}{r^\mu} t^\mu.$$

With the same arguments it is easy to prove that if $t<-r$ then it holds

$$F(x,t)\geq \frac{F(x,-r)}{r^{\mu}}|t|^{\mu},$$

so that for any $t\in\mathbb{R}$ with $|t|\geq r$ we get

$$F(x,t)\geq m(x)|t|^{\mu}, \tag{3.9}$$

where $m(x)=r^{-\mu}\min\{F(x,r),F(x,-r)\}$.

Since the function $t\mapsto F(\cdot,t)$ is continuous in $\mathbb{R}$, by the Weierstrass Theorem, it is bounded for any $t\in\mathbb{R}$ such that $|t|\leq r$, say

$$|F(x,t)|\leq \widetilde{M}(x)\text{in }\{|t|\leq r\},$$

where $\widetilde{M}(x)=\max\{|F(x,t)|:|t|\leq r\}$. taking $M(x)=\widetilde{M}(x)+m(x)r^{\mu}$.

Note that m and M are measurable functions, since $x\mapsto F(x,\cdot)$ is. Moreover, m and M are positive, being $F(x,t)>0$ a.e. $x\in\Omega$ and for any $t\in\mathbb{R}$ such that $|t|\geq r$. ■

Now, suppose that $\varepsilon=1$ we get

$$m(x)=r^{-\mu}\min\{F(x,r),F(x,-r)\}\leq r^{-\mu}\left(|r|^2+\delta(1)r^q\right)\in L^{\infty}(\Omega)$$

and

$$M(x)=\max\{|F(x,t)|:|t|\leq r\}\leq |r|^2+\delta(1)r^q\in L^{\infty}(\Omega),$$

so that the assertion follows.

Now, before going on, we need some preliminary results on X and X_0. In the sequel we denote $Q=\mathbb{R}^{2n}\setminus\mathcal{O}$, where

$$\mathcal{O}=(\mathcal{C}\Omega)\times(\mathcal{C}\Omega)\subset\mathbb{R}^{2n}, \tag{3.10}$$

and $\mathcal{C}\Omega=\mathbb{R}^n\setminus\Omega$.

The space X is endowed with the norm defined as

$$\|g\|_X=\|g\|_{L^2(\Omega)}+\left(\int_Q |g(x)-g(y)|^2K(x-y)\,dx\,dy\right)^{1/2}. \tag{3.11}$$

It is easy to check that $\|\cdot\|_X$ is a norm on X. We only show that if $\|g\|_X = 0$, then $g = 0$ a.e. in $\mathbb{R}^n$. Indeed, by $\|g\|_X = 0$ we get $\|g\|_{L^2(\Omega)} = 0$, which implies that

$$g = 0 \text{ a.e. in } \Omega, \tag{3.12}$$

and

$$\int_Q |g(x)-g(y)|^2 K(x-y)\,dx\,dy = 0. \tag{3.13}$$

we deduce that $g(x) = g(y)$ a.e. $(x,y) \in Q$, that is g is constant a.e. in $\mathbb{R}^n$, say $g = c \in \mathbb{R}$ a.e. in $\mathbb{R}^n$. it easily follows that $c = 0$, so that $g = 0$ a.e. in $\mathbb{R}^n$.

In the following we denote by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm (the so-called Gagliardo norm)

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \right)^{1/2}. \tag{3.14}$$

Lemma 3.3 *Let $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$. Then:*

(a) If $v \in X$, then $v \in H^s(\Omega)$. Moreover,

$$\|v\|_{H^s(\Omega)} \le c(\lambda)\|v\|_X.$$

(b) If $v \in X_0$, then $v \in H^s(\mathbb{R}^n)$. Moreover,

$$\|v\|_{H^s(\Omega)} \le \|v\|_{H^s(\mathbb{R}^n)} \le c(\lambda)\|v\|_X,$$

where $c(\lambda) = \max\{1, \lambda^{-1/2}\}$, and λ is the constant .

Proof. Let us prove part (a). we get

$$\begin{aligned}\iint_{\Omega\times\Omega} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &\le \frac{1}{\lambda}\iint_{\Omega\times\Omega} |v(x)-v(y)|^2 K(x-y)\,dx\,dy \\ &\le \frac{1}{\lambda}\iint_{Q} |v(x)-v(y)|^2 K(x-y)\,dx\,dy < +\infty,\end{aligned}$$

since $v \in X$. The first assertion is proved.

Let us prove part (b). Note that $v \in X$ and $v = 0$ a.e. in $\mathscr{C}\Omega$. As a consequence,

$$\|v\|_{L^2(\mathbb{R}^n)} = \|v\|_{L^2(\Omega)} < +\infty,$$

and

$$\iint_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \iint_{Q} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \leq \frac{1}{\lambda}\iint_{Q} |v(x)-v(y)|^2 K(x-y)\,dx\,dy < +\infty.$$

Hence $v \in H^s(\mathbb{R}^n)$. The estimate on the norm easily follows (we remark that, even if $v = 0$ on $\mathscr{C}\Omega$, it is not true that $\|v\|_{H^s(\Omega)} = \|v\|_{H^s(\mathbb{R}^n)}$, since in the latter norm the interaction between Ω and $\mathscr{C}\Omega$ gives, in general, a positive contribution). ■

Lemma 3.4 *Let $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$. Then*

(a) There exists a positive constant c, depending only on n and s, such that for any $v \in X_0$:

$$\|v\|^2_{L^{2^*}(\Omega)} = \|v\|^2_{L^{2^*}(\mathbb{R}^n)} \leq c \iint_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

(b) There exists a constant $C > 1$, depending only on n, s, λ and Ω, such that for any $v \in X_0$:

$$\iint_{Q} |v(x)-v(y)|^2 K(x-y)\,dx\,dy \leq \|v\|^2_X \leq C \iint_{Q} |v(x)-v(y)|^2 K(x-y)\,dx\,dy,$$

that is

$$\|v\|_{X_0} = \left(\iint_{Q} |v(x)-v(y)|^2 K(x-y)\,dx\,dy \right)^{1/2}.$$

Proof. Let v be in X_0, we know that $v \in H^s(\mathbb{R}^n)$ we get

$$\|v\|^2_{L^{2^*}(\mathbb{R}^n)} \leq c \iint_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

where c is a positive constant depending only on n and s. Since $v = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, we get

assertion (a).

For part (b), we note that

$$\|v\|_X^2 \geq \iint_Q |v(x)-v(y)|^2 K(x-y)\,dx\,dy.$$

Moreover, using the fact that $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ continuously (being Ω bounded and $2 < 2^* = \frac{2n}{n-2s}$), part (a) we get

$$\begin{aligned}
\|v\|_X^2 &= \left(\|v\|_{L^2(\Omega)} + \left(\iint_Q |v(x)-v(y)|^2 K(x-y)\,dx\,dy \right)^{1/2} \right)^2 \\
&\leq 2\|v\|_{L^2(\Omega)}^2 + 2\iint_Q |v(x)-v(y)|^2 K(x-y)\,dx\,dy \\
&\leq 2|\Omega|^{(2^*-2)/2^*}\|v\|_{L^{2^*}(\Omega)}^2 + 2\iint_Q |v(x)-v(y)|^2 K(x-y)\,dx\,dy.
\end{aligned}$$

■

Proof of Theorem 3.1 (Mountain Pass solutions)

For the proof of Theorem 1, we observe that problem (3.2) has a variational structure; indeed, it is the Euler-Lagrange equation of the functional $J : X_0 \to \mathbb{R}$ defined as follows:

$$J(u) = \frac{1}{2}\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \int_\Omega F(x,u(x))\,dx.$$

Note that the functional J is Fréchet differentiable at $u \in X_0$ and for any $\phi \in X_0$,

$$\langle J'(u), \phi\rangle = \int_Q (u(x)-u(y))(\phi(x)-\phi(y))K(x-y)\,dx\,dy - \int_\Omega f(x,u(x))\phi(x)\,dx.$$

Thus, critical points of J are solutions to problem (3.2). To find these critical points, we use the Mountain Pass Theorem. We need to verify that J has the required geometric structure and satisfies the Palais-Smale condition.

We start by proving the geometric structure of J.

Proposition 3.1 *Let f be a Carathéodory function. Then, there exist $\rho > 0$ and $\beta > 0$ such that for any $u \in X_0$ with $\|u\|_{X_0} = \rho$, it results that $J(u) \geq \beta$.*

Proof. Let u be a function in X_0, we get that for any $\varepsilon > 0$

$$\begin{aligned}
\mathcal{J}(u) &\geq \frac{1}{2}\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \varepsilon\int_\Omega |u(x)|^2\,dx - \delta(\varepsilon)\int_\Omega |u(x)|^q\,dx \\
&\geq \frac{1}{2}\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \varepsilon\|u\|^2_{L^2(\Omega)} - \delta(\varepsilon)\|u\|^q_{L^q(\Omega)} \\
&\geq \frac{1}{2}\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \varepsilon|\Omega|^{(2^*-2)/2^*}\|u\|^2_{L^{2^*}(\Omega)} \\
&\quad - |\Omega|^{(2^*-q)/2^*}\delta(\varepsilon)\|u\|^q_{L^{2^*}(\Omega)}.
\end{aligned}$$

Thanks to the fact that $L^{2^*}(\Omega) \hookrightarrow L^2(\Omega)$ and $L^{2^*}(\Omega) \hookrightarrow L^q(\Omega)$ continuously (being Ω bounded and $\max\{2,q\} = q < 2^*$), for any $\varepsilon > 0$:

$$\begin{aligned}
J(u) &= \frac{1}{2}\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy \\
&\quad - \varepsilon c|\Omega|^{(2^*-2)/2^*}\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad - \delta(\varepsilon)c^{q/2}|\Omega|^{(2^*-q)/2^*}\left(\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)^{q/2} \\
&= \left(\frac{1}{2} - \varepsilon c|\Omega|^{(2^*-2)/2^*\lambda}\right)\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy \\
&\quad - \delta(\varepsilon)c^{q/2}|\Omega|^{(2^*-q)/2^*\lambda}\left(\iint_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy\right)^{q/2}
\end{aligned}$$

Choosing $\varepsilon > 0$ such that $2\varepsilon c|\Omega|^{(2^*-2)/2^*} < \lambda$,

$$\begin{aligned}
J(u) &\geq \alpha\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy\left[1 - \kappa\left(\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy\right)^{(q/2)-1}\right] \\
&\geq \alpha\|u\|^2_{X_0}\left[1 - \kappa\|u\|^{q-2}_{X_0}\right]
\end{aligned}$$

for suitable positive constants α and κ.

Now, let $u \in X_0$ be such that $\|u\|_{X_0} = \rho > 0$. Since $q > 2$ by assumption, we can choose ρ

sufficiently small (i.e., ρ such that $1-\kappa\rho^{q-2}>0$), so that:

$$\inf_{\substack{u\in X_0\\ \|u\|_{X_0}=\rho}} J(u)\geq \alpha\rho^2\left[1-\kappa\rho^{q-2}\right]=:\beta>0.$$

■

Proposition 3.2 *Let f be a Carathéodory function Then there exists $e\in X_0$ such that $e\geq 0$ a.e. in $\mathbb{R}^n$, $\|e\|_{X_0}>\rho$, and $J(e)<\beta$, where ρ and β are given in Proposition 3.1.*

Proof. Fix $u\in X_0$ such that $\|u\|_{X_0}=1$ and $u\geq 0$ a.e. in $\mathbb{R}^n$. By Lemma 3.4

$$J(t\,u)=\frac{t^2}{2}\iint_Q |t\,u(x)-t\,u(y)|^2K(x-y)\,dx\,dy-\int_\Omega F(x,t\,u(x))\,dx,$$

we obtain:

$$J(t\,u)\leq \frac{t^2}{2}-t^\mu\int_\Omega m(x)|u(x)|^\mu\,dx+\int_\Omega M(x)\,dx.$$

Since $\mu>2$, as $t\to+\infty$, $J(t\,u)\to-\infty$. Thus, choosing $e=Tu$ with T sufficiently large completes the proof. ■

Proposition 3.3 *Let f be a Carathéodory function). Let $\{u_j\}$ be a sequence in X_0 such that $\{u_j\}$ is bounded in X_0 . Then there exists $u_\infty\in X_0$ such that, up to a subsequence,*

$$\|u_j-u_\infty\|_{X_0}\to 0 \quad as\ j\to+\infty.$$

Proof. Since $\{u_j\}$ is bounded in X_0 and X_0 is a reflexive space , up to a subsequence, still denoted by $\{u_j\}$, there exists $u_\infty\in X_0$ such that

$$\int_Q \big(u_j(x)-u_j(y)\big)\big(\phi(x)-\phi(y)\big)K(x-y)\,dx\,dy\to\int_Q \big(u_\infty(x)-u_\infty(y)\big)\big(\phi(x)-\phi(y)\big)K(x-y)\,dx\,dy$$

for any $\phi\in X_0$ as $j\to+\infty$. Up to a subsequence:

$$u_j\to u_\infty \quad \text{in } L^q(\mathbb{R}^n),$$
$$u_j\to u_\infty \quad \text{a.e. in } \mathbb{R}^n,$$

as $j \to +\infty$, and there exists $\varrho \in L^q(\mathbb{R}^n)$ such that

$$|u_j(x)| \le \varrho(x) \quad \text{a.e. in } \mathbb{R}^n \text{ for any } j \in \mathbb{N}.$$

The fact that the map $t \mapsto f(\cdot, t)$ is continuous in $t \in \mathbb{R}$, and the Dominated Convergence Theorem, we get

$$\int_\Omega f(x, u_j(x)) u_j(x)\, dx \to \int_\Omega f(x, u_\infty(x)) u_\infty(x)\, dx$$

and

$$\int_\Omega f(x, u_j(x)) u_\infty(x)\, dx \to \int_\Omega f(x, u_\infty(x)) u_\infty(x)\, dx$$

as $j \to +\infty$. We have that

$$0 \leftarrow \langle J'(u_j), u_j \rangle = \int_Q |u_j(x) - u_j(y)|^2 K(x-y)\, dx\, dy - \int_\Omega f(x, u_j(x)) u_j(x)\, dx,$$

we deduce that

$$\int_Q |u_j(x) - u_j(y)|^2 K(x-y)\, dx\, dy \to \int_\Omega f(x, u_\infty(x)) u_\infty(x)\, dx$$

as $j \to +\infty$. Furthermore,

$$\begin{aligned} 0 \leftarrow \langle J'(u_j), u_\infty \rangle = &\int_Q (u_j(x) - u_j(y))(u_\infty(x) - u_\infty(y)) K(x-y)\, dx\, dy \\ &- \int_\Omega f(x, u_j(x)) u_\infty(x)\, dx \end{aligned}$$

as $j \to +\infty$. we obtain

$$\int_Q |u_\infty(x) - u_\infty(y)|^2 K(x-y)\, dx\, dy = \int_\Omega f(x, u_\infty(x)) u_\infty(x)\, dx.$$

Thus,

$$\int_Q |u_j(x) - u_j(y)|^2 K(x-y)\, dx\, dy \to \int_Q |u_\infty(x) - u_\infty(y)|^2 K(x-y)\, dx\, dy,$$

so that

$$\|u_j\|_{X_0} \to \|u_\infty\|_{X_0},$$

as $j \to \infty$.

Finally we have that

$$\begin{aligned}\|u_j - u_\infty\|^2_{X_0} &= \|u_j\|^2_{X_0} + \|u_\infty\|^2_{X_0} - 2\int_Q (u_j(x)-u_j(y))(u_\infty(x)-u_\infty(y))K(x-y)\,dx\,dy \\ &\to 2\|u_\infty\|^2_{X_0} - 2\int_Q |u_\infty(x)-u_\infty(y)|^2 K(x-y)\,dx\,dy \\ &= 0.\end{aligned}$$

■

An equation driven by the fractional Laplacian

Theorem 3.2 *Let $s \in (0,1)$, $n > 2s$, and Ω be an open bounded set of $\mathbb{R}^n$ with Lipschitz boundary. Consider the equation*

$$\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega f(x,u(x))\phi(x)\,dx, \tag{3.15}$$

for any $\phi \in H^s(\mathbb{R}^n)$ with $\phi = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

If $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, then problem (3.15) admits a Mountain Pass type solution $u \in H^s(\mathbb{R}^n)$, which is not identically zero, and such that $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

Proof of Theorem 3.2 Theorem 3.2 follows from Theorem 3.1 by choosing

$$K(x) = |x|^{-(n+2s)},$$

and by recalling that $X_0 \subseteq H^s(\mathbb{R}^n)$.

CONCLUSION

In this dissertation, we studied a non-local elliptic problem involving the fractional Laplacian and a regular nonlinearity, employing two different variational techniques under distinct sets of assumptions. In Chapter 2, we used variational methods to investigate the existence and uniqueness of a weak solution to the corresponding Dirichlet problem, thereby establishing the well-posedness of the equation. We also derived global regularity estimates for the distributional solution of the equation $(-\Delta)^s = f$.

In Chapter 3, we applied the Mountain Pass Theorem to establish the existence of solutions for semilinear equations driven by the fractional Laplacian.

BIBLIOGRAPHY

[1] V. Ambrosio, *Nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$, Birkhäuser: Cham, Switzerland, 2021.

[2] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349-381.

[3] T. Bartsch, *Infinitely many solutions of a symmetric Dirichlet problem*, Nonlinear Anal. **20** (1993), 1205-1216.

[4] M.B. Benboubker, *On Some Quasi linear Elliptic Non-homogeneous Problems of Dirichlet or Neumann Type*, Thèse, 2014.

[5] H. Brézis, *Analyse fonctionnelle: Théorie et applications*, Masson, Paris, 1983.

[6] C. Bucur, E. Valdinoci, *Nonlocal diffusion and applications*, vol. 20. Springer, 2016.

[7] L.A. Caffarelli, *Nonlocal equations, drifts and games*, Nonlinear Partial Differential Equations, Abel Symposia **7**(2012), 37-52.

[8] L.A. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32**(2007), 1245-1260.

[9] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136**(2012), 521-573.

[10] B. Dyda, *Fractional calculus for power functions and eigenvalues of the fractional Laplacian*, Fractional Calculus and Applied Analysis, **15**(2012), 536-555.

[11] L.C. Evans. *Partial Differential Equations: Second Edition.* American Mathematical Society, Providence, R.I., 2nd edition, March 2010.

[12] G. Leoni, *A First Course in Fractional Sobolev Spaces*, American Mathematical Society, 2023.

[13] P. Pucci, V. Rădulescu, *The impact of the mountain pass theory in nonlinear analysis: A mathematical survey*, Boll. Unione Mat. Ital. (9) **3** (2010), 543–584.

[14] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. Math. **65**, Amer. Math. Soc., Providence, RI, 1986.

[15] R. Servadei, E. Valdinoci, *Lewy–Stampacchia type estimates for variational inequalities driven by (non)local operators*, preprint, available at: http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=11-103, 2011.

[16] M. Struwe, *Variational methods: Applications to nonlinear partial differential equations and Hamiltonian systems*, Ergeb. Math. Grenzgeb. (3) **34**, Springer-Verlag, Berlin–Heidelberg, 1990.