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The Laguerre Transform and its Applications

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Contents

Acknowledgements i

Abstract iv

Résumé iv

Notations v

General Introduction 1

1 Basic notions 3
- 1.1 Falling Factorial . . . 3
- 1.2 Binomial Coefficients . . . 3
- 1.3 Generalization of Binomial Coefficients . . . 5
- 1.4 Generating Function . . . 5
- 1.5 The Kronecker Symbol . . . 6
- 1.6 Fibonacci Number Sequence . . . 6
- 1.7 Lucas Number Sequence . . . 7
- 1.8 Classical Fibonacci Polynomials . . . 8
- 1.9 Classical Lucas Polynomials . . . 9

2 Generalized Laguerre Transforms of Sequences 11
- 2.1 Classical Laguerre Polynomials . . . 11
- 2.2 Associated Laguerre Polynomials . . . 15
- 2.3 Associated Polynomials . . . 16
- 2.4 Generalized Laguerre Transform . . . 17
- 2.5 Fibonacci Associated Polynomials . . . 21
- 2.6 Lucas Associated Polynomials . . . 23
  - 2.6.1 Limit of the Ratio of Consecutive Associated Lucas Polynomials . . . 24
- 2.7 The Associated Matrix and their Properties . . . 32

3 The Application of Laguerre Transform 37
- 3.1 Orthogonal Polynomials . . . 37
- 3.2 Application . . . 38

3.2.1 Gauss-Laguerre Quadrature 38
3.2.2 Radial Wave Function in the Hydrogen Atom 38
3.2.3 Orthogonality of Laguerre Polynomials 39
3.2.4 Orthogonal Polynomials, Applications to Electrostatics 39
3.2.5 Application of V-Belt: continuously variable transmission system utilizing recurrent hybrid Laguerre orthogonal polynomials in neural network control systems and modified particle swarm optimization 40

Conclusion **42**

List of Tables

1.1 Pascal's Triangle Numbers 4

Abstract

Laguerre polynomials are an important family of orthogonal polynomials introduced by Edmond Laguerre in the study of differential equations. Because of their many properties, they play an important role in mathematical analysis, numerical methods, and mathematical physics.

From these properties, researchers developed the Laguerre transform and its generalization through the associated Laguerre polynomials [8]. As an application of these associated polynomials, we study the associated Fibonacci and Lucas polynomials.

This work is divided into three chapters. The first chapter presents some basic notions of enumerative combinatorics. The second chapter is devoted to the generalization of the Laguerre transform and the study of associated polynomials [8, 9]. Finally, the third chapter presents some applications of the Laguerre transform in different fields.

Keywords: Generating function, explicit formula, recurrence relation, Fibonacci numbers and polynomials, Lucas numbers and polynomials, Laguerre polynomials, associated Laguerre polynomials.

Résumé

Les polynômes de Laguerre sont une famille importante de polynômes orthogonaux introduite par Edmond Laguerre dans l'étude des équations différentielles. Grâce à leurs nombreuses propriétés, ils jouent un rôle important en analyse mathématique, en méthodes numériques et en physique mathématique.

À partir de ces propriétés, les chercheurs ont développé la transformation de Laguerre et sa généralisation à travers les polynômes associés de Laguerre [8]. Comme application de ces polynômes associés, nous étudions les polynômes associés de Fibonacci et de Lucas.

Ce travail est divisé en trois chapitres. Le premier présente quelques notions de base de la combinatoire énumérative. Le deuxième est consacré à la généralisation de la transformation de Laguerre et aux polynômes associés [8, 9]. Enfin, le troisième chapitre présente quelques applications de la transformation de Laguerre dans différents domaines.

Mots-clés : Fonction génératrice, formule explicite, relation de récurrence, nombres et polynômes de Fibonacci, nombres et polynômes de Lucas, les polynômes de Laguerre, polynômes associés de Laguerre.

الملخص

تُعدُّ متعددات حدود لاغير عائلةً مهمة من متعددات الحدود المتعامدة، وقد قدّمها إدموند لاغير في إطار دراسة المعادلات التفاضلية. وبفضل خصائصها العديدة، فإنها تؤدي دورًا مهمًا في التحليل الرياضي، والطرائق العددية، والفيزياء الرياضية.

انطلاقًا من هذه الخصائص، طوّر الباحثون تحويل لاغير وتعميمه من خلال متعددات حدود لاغير المرافقة. وكأحد تطبيقات هذه المتعددات المرافقة، ندرس متعددات حدود فيبوناتشي ولوكاس المرافقة.

يتكون هذا العمل من ثلاثة فصول. يقدّم الفصل الأول بعض المفاهيم الأساسية في التوافقيات التعدادية. أمّا الفصل الثاني فيُخصَّص لتعميم تحويل لاغير ودراسة متعددات الحدود المرافقة. وفي الفصل الثالث، نعرض بعض تطبيقات تحويل لاغير في مجالات مختلفة.

الكلمات المفتاحية: الدالة المولّدة، الصيغة الصريحة، العلاقة التراجعية، أعداد ومتعددات حدود فيبوناتشي، أعداد ومتعددات حدود لوكاس، متعددات حدود لاغير، متعددات حدود لاغير المرافقة.

Notations

$\mathbb{N}$ The set of natural numbers.

$[n]$ The set $\{1, 2, \ldots, n\}$, for $n \in \mathbb{N}$.

$\mathbb{Z}$ The set of integers.

$\mathbb{R}$ The field of real numbers.

$\mathbb{C}$ The field of complex numbers.

$x^{\underline{n}}$ The falling factorial.

$\binom{n}{k}$ The binomial coefficient.

$\delta_{n,k}$ The Kronecker symbol.

$:=$ Equality by definition (assignment).

F_n The Fibonacci sequence.

$\mathfrak{L}_n$ The Lucas sequence.

$f_n(x)$ The Fibonacci polynomials.

$l_n(x)$ The Lucas polynomials.

$L_n(x)$ The classical Laguerre polynomials.

$L_n^m(x)$ The associated Laguerre polynomials.

$a_{n,m}(x)$ The associated polynomials.

$\mathbb{F}_n(x)$ The associated Fibonacci polynomials.

$\mathbb{F}_n^m(x)$ The m-associated Fibonacci polynomials.

$\mathbb{L}_n(x)$ The associated Lucas polynomials.

$\mathbb{L}_n^m(x)$ The m-associated Lucas polynomials.

Introduction

General Introduction

Edmond Nicolas Laguerre was a French mathematician born in Bar-le-Duc on April 9, 1834, and died in Paris on August 14, 1886.

He is mainly known for his work in analysis, geometry, and differential equations. He published about 140 articles and worked with famous mathematicians such as Hermite and Poincaré [4].



Figure 0.1: Edmond Nicolas Laguerre

Laguerre introduced a family of orthogonal polynomials during his research on differential equations.

Laguerre polynomials are defined as the solutions of the following linear differential equation

$$xy'' + (1 - x)y' + ny = 0,$$

which is a second-order linear differential equation. This equation has non-singular solutions only when n is a positive integer.

These polynomials, usually denoted by $L_n(x)$, form a sequence of polynomials that can be defined by Rodrigues' formula:

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right).$$

They are orthogonal with respect to the inner product defined by

$$\int_0^\infty L_n(x)L_m(x)e^{-x}\,dx.$$

From these analytical properties, several researchers developed what is called the Laguerre transform.

This transform is a mathematical transformation related to Laguerre polynomials and applied to numerical sequences or functions.

According to the article [8], the Laguerre transform is defined by

$$b_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!} a_k.$$

The authors then proposed a generalization of this transform through the associated Laguerre polynomials defined by

$$L_n^m(x) = (-1)^m \frac{d^m}{dx^m} L_{n+m}(x).$$

This work is divided into three chapters.

In the first chapter, entitled "Basic Notions", we present the basic notions of enumerative combinatorics and recall some important elements that will be useful later, such as the binomial coefficient, generating functions, Fibonacci and Lucas numbers and polynomials, and their properties.

In the second chapter [8, 9], entitled "The Generalized Laguerre transforms of sequences", we present Laguerre polynomials and associated polynomials. As an application of associated polynomials, we study the associated Fibonacci polynomials and the associated Lucas polynomials.

In the third chapter, entitled "Applications of the Laguerre Transform", we present the applications of the Laguerre transform in mathematics and in other fields.

1 Basic notions

In this chapter, we present some basic elements which will be useful for the rest, including a falling factorial, binomial coefficients, Fibonacci numbers, Lucas polynomials and the Laguerre polynomials. For more precise and detailed concepts the reader can refer to [1, 3, 5, 12, 14, 15, 16, 21]

1.1 Falling Factorial

Let $x \in \mathbb{C}$ and $n \in \mathbb{N}$, the falling factorial of order n denoted $x^{\underline{n}}$, is defined by:

$$x^{\underline{n}} = \begin{cases} 1 & \text{if } n = 0 \\ x(x-1)\cdots(x-n+1) & \text{if } n \geq 1. \end{cases}$$

Example 1.1

$$\begin{aligned} x^{\underline{1}} &= x \\ x^{\underline{2}} &= x(x-1) = x^2 - x \\ x^{\underline{3}} &= x(x-1)(x-2) = x^3 - 3x^2 + 2x \\ x^{\underline{4}} &= x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x. \end{aligned}$$

1.2 Binomial Coefficients

Let $n \in \mathbb{N}$, the binomial coefficient, denoted $\binom{n}{k}$, is defined by the following relation [18]:

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!} \quad (0 \leq k \leq n),$$

with $\binom{n}{k} = 0$ for $k > n$, and $\binom{n}{k} = 1$ for $k = 0$ or $k = n$.

Note that if n is a positive integer, then $\binom{n}{k}$ is the number of ways to choose k elements from a set of n elements.

Example 1.2 *In the set* $[5] = \{a, b, c, d, e\}$*, there are ten ways to choose three elements:*

$$\{a,b,c\}; \{a,b,d\}; \{a,b,e\}; \{a,c,d\}; \{a,c,e\}; \{a,d,e\}; \{b,c,d\}; \{b,c,e\}; \{b,d,e\}; \{c,d,e\}.$$

Therefore

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{1\times 2\times 3\times 4\times 5}{(1\times 2\times 3)(1\times 2)} = 10.$$

It is also worth mentioning that these coefficients play a central role in the expansion of $(a+b)^n$, where a and b are real or complex numbers [5]:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

and satisfy the following recurrence relation:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad \text{for every} \quad n \geq k \geq 1. \tag{1.1}$$

Pascal's Triangle

From the recurrence relation (1.1), we can construct Pascal's triangle

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Table 1.1: Pascal's Triangle Numbers

Property 1.2.1 *The binomial coefficients satisfy nearly an infinite number of identities, including [7]:*

1. *The absorption identity:*

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

2. *The symmetry:*

$$\binom{n}{k} = \binom{n}{n-k}, \text{ with } n \geqslant k \geqslant 0.$$

3. *The top summation:*

$$\binom{n+1}{k+1} = \sum_{j=0}^{n} \binom{k}{j}, \text{ with } n, k \geqslant 0.$$

4. *The parallel summation:*

$$\binom{m+n+1}{n} = \sum_{n \geqslant j} \binom{m+j}{j}.$$

5. *The trinomial transformation :*

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}.$$

6. *The Vandermonde convolution:*

$$\binom{n+m}{k} = \sum_{j=0}^{n} \binom{n}{j}\binom{m}{k-j}.$$

1.3 Generalization of Binomial Coefficients

A generalization of the binomial coefficient $\binom{n}{k}$, consists in replacing n by a complex number x, and to define $\binom{x}{k}$ by posing, $x \in \mathbb{C}$ and $k \in \mathbb{N}$ by [7]:

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!} = \frac{x^{\underline{k}}}{k!}$$

and having almost the same properties as $\binom{n}{k}$.

1.4 Generating Function

Generating functions are an algebraic tool that allows us to reformulate combinatorial problems in order to transform them into problems of manipulating algebraic expressions. In particular, in combinatorics, it is often a matter of determining the number of objects of certain types that are of size n, which gives rise to a sequence $(a_n)_{n\geq 0}$ whose n-th term we seek to determine. Throughout the following, we denote by $\mathbb{K}$ a commutative ring [7].

Ordinary Generating Function

Let n be a natural integer and $(a_n)_{n\geq 0}$ a sequence of complex numbers. We call the *ordinary generating function* of the sequence $(a_n)_{n\geq 0}$, the function:

$$A(x) = \sum_{n\geq 0} a_n x^n.$$

The generating series in one indeterminate (denoted x) with coefficients $(a_n)_{n\geq 0}$ in a ring $\mathbb{K}$. The addition and multiplication of two ordinary generating series are defined by:

$$\sum_{n\geq 0} a_n x^n + \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0} (a_n + b_n) x^n$$

and

$$\sum_{n\geq 0} a_n x^n \cdot \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0} c_n x^n,$$

with

$$c_n = \sum_{k=0}^{n} a_k \, b_{n-k}.$$

Equipped with these two operations, the set of series forms a commutative ring, denoted $\mathbb{K}[[x]]$.

Exponential Generating Function

Let n be a natural integer and $(a_n)_{n\geq 0}$ a sequence of complex numbers. We call the *exponential generating function* of the sequence $(a_n)_{n\geq 0}$, the function:

$$B(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}.$$

The addition and multiplication of two exponential generating series are defined by:

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} + \sum_{n\geq 0} b_n \frac{x^n}{n!} = \sum_{n\geq 0} (a_n + b_n) \frac{x^n}{n!}$$

and

$$\sum_{n\geq 0} a_n \frac{x^n}{n!} \cdot \sum_{n\geq 0} b_n \frac{x^n}{n!} = \sum_{n\geq 0} c_n \frac{x^n}{n!},$$

with

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k \, b_{n-k}.$$

1.5 The Kronecker Symbol

The Kronecker Symbol is given by:

$$\delta_{n,k} = \begin{cases} 1 & \text{if} \quad n = k \\ 0 & \text{otherwise.} \end{cases}$$

1.6 Fibonacci Number Sequence

The Fibonacci sequence is a sequence of integers in which each term is the sum of the two preceding terms. It usually begins with the terms 0 and 1, and its first terms are:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21, \ldots$$

It is named after Leonardo Fibonacci who, in a recreational problem presented in his book *Liber Abaci*, published in 1202, described the growth of a rabbit population:

> "A man places a pair of rabbits in an enclosure surrounded on all sides by a wall. How many pairs will there be after one year if each pair produces a new pair every month starting from the third month of its life?"

Properties:

1. **Generating Function:** The generating function of the Fibonacci number sequence is [6, 11]:
$$\sum_{n=0}^{\infty} F_n z^n = \frac{z}{1-z-z^2}. \tag{1.2}$$

2. **Recurrence Formula:**The Fibonacci number sequence satisfies the following recurrence relation [22]:
$$F_{n+2} = F_{n+1} + F_n \qquad n \geq 0,$$
with $F_0 = 0$ and $F_1 = 1$.

3. **Binet's Formula:** The Binet formula for the Fibonacci sequence (F_n) is:
$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \tag{1.3}$$
where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ are the roots of the characteristic equation $z^2 - z - 1 = 0$.

Remark 1.6.1 *The positive root α is known as the "golden ratio".*

1.7 Lucas Number Sequence

The Lucas sequence $(\mathfrak{L}_n)_{n\geq 0}$, is an integer sequence closely related to the Fibonacci sequence, and satisfies the following properties:

1. **Generating Function:** The ordinary generating function of the Lucas number sequences is:
$$\sum_{n=0}^{\infty} \mathfrak{L}_n z^n = \frac{2-z}{1-z-z^2}. \tag{1.4}$$

2. **Recurrence Formula :**The Lucas number sequence satisfies the following recurrence relation [12]:
$$\mathfrak{L}_n = \mathfrak{L}_{n-1} + \mathfrak{L}_{n-2} \quad n \geq 2,$$
with $\mathfrak{L}_0 = 2$ and $\mathfrak{L}_1 = 1$.

   The first terms of the Lucas sequence are:
$$2, 1, 3, 4, 7, 11, 18, 29, \ldots$$

3. **Binet's Formula:** The Binet formula for the Lucas sequence is:
$$\mathfrak{L}_n = \alpha^n + \beta^n.$$

The following theorem presents some fundamental identities relating the Lucas and Fibonacci sequences.

Theorem 1.1

1. *For all $n \geq 1$, we have:*
$$\mathfrak{L}_n = F_{n-1} + F_{n+1}.$$
2. *For all $n \geq 2$, we have:*
$$F_{n+2} - F_{n-2} = \mathfrak{L}_n.$$
3. *For all $n \geq 0$, we have:*
$$F_{2n} = F_n \mathfrak{L}_n.$$

The following theorem establishes an explicit connection between the Lucas numbers, the Fibonacci numbers, and the golden ratio.

Theorem 1.2
$$\alpha^n = \frac{\mathfrak{L}_n + \sqrt{5}\, F_n}{2} \quad \textit{and} \quad \beta^n = \frac{\mathfrak{L}_n - \sqrt{5}\, F_n}{2}.$$

1.8 Classical Fibonacci Polynomials

Fibonacci polynomials are a natural generalization of the Fibonacci numbers and enjoy a wide range of interesting properties. They are defined as follows:

1. **Generating Function:** The ordinary generating function of $F_n(x)$ is given by:
$$\sum_{n=0}^{\infty} F_n(x) z^n = \frac{z}{1 - xz - z^2}.$$
2. **Recurrence Formula:** The classical Fibonacci polynomials $F_n(x)$, are defined by the recurrence relation [14, 16]:
$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x) \quad n \geq 0,$$
with initial values:
$$F_0(x) = 0 \quad \textit{and} \quad F_1(x) = 1.$$
The first terms of these polynomials are:
$$\begin{aligned}
F_0(x) &= 0 \\
F_1(x) &= 1 \\
F_2(x) &= x \\
F_3(x) &= x^2 + 1 \\
F_4(x) &= x^3 + 2x \\
F_5(x) &= x^4 + 3x^2 + 1 \\
F_6(x) &= x^5 + 4x^3 + 3x \\
F_7(x) &= x^6 + 5x^4 + 6x^2 + 1 \\
F_8(x) &= x^7 + 6x^5 + 10x^3 + 4x \\
F_9(x) &= x^8 + 7x^6 + 15x^4 + 10x^2 + 1 \\
F_{10}(x) &= x^9 + 8x^7 + 21x^5 + 20x^3 + 5x.
\end{aligned}$$
For $x = 1$, these polynomials reproduce the classical Fibonacci sequence $(F_n)_{n \geq 0}$.

3. **Binet's Formula:** The Binet formula for the classical Fibonacci polynomials is:

$$F_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}$$

where

$$\alpha(x) = \frac{x + \sqrt{x^2+4}}{2}, \quad \beta(x) = \frac{x - \sqrt{x^2+4}}{2}$$

are the roots of the characteristic equation $z^2 - xz - 1 = 0$.

1.9 Classical Lucas Polynomials

Lucas polynomials are a natural generalization of the Lucas numbers and are closely related to the Fibonacci polynomials. They are defined as follows:

1. **Generating Function:** The ordinary generating function of $\mathfrak{L}_n(x)$ is given by:

$$\sum_{n=0}^{\infty} \mathfrak{L}_n(x)\, z^n = \frac{2 - xz}{1 - xz - z^2}.$$

2. **Recurrence Formula:** The classical Lucas polynomials $\mathfrak{L}_n(x)$, are defined by the recurrence relation [1]:

$$\mathfrak{L}_n(x) = x\, \mathfrak{L}_{n-1}(x) + \mathfrak{L}_{n-2}(x) \quad n \geq 2,$$

with initial values:

$$\mathfrak{L}_0(x) = 2 \quad \textit{and} \quad \mathfrak{L}_1(x) = x.$$

The first terms of the sequence $(\mathfrak{L}_n(x))_{n\geq 0}$ are:

$$\begin{aligned}
\mathfrak{L}_0(x) &= 2 \\
\mathfrak{L}_1(x) &= x \\
\mathfrak{L}_2(x) &= x^2 + 2 \\
\mathfrak{L}_3(x) &= x^3 + 3x \\
\mathfrak{L}_4(x) &= x^4 + 4x^2 + 2 \\
\mathfrak{L}_5(x) &= x^5 + 5x^3 + 5x \\
\mathfrak{L}_6(x) &= x^6 + 6x^4 + 9x^2 + 2 \\
\mathfrak{L}_7(x) &= x^7 + 7x^5 + 9x^3 + 7x \\
\mathfrak{L}_8(x) &= x^8 + 8x^6 + 20x^4 + 4x^2 + 2 \\
\mathfrak{L}_9(x) &= x^9 + 9x^7 + 27x^5 + 30x^3 + 9x \\
\mathfrak{L}_{10}(x) &= x^{10} + 10x^8 + 35x^6 + 50x^4 + 25x^2 + 2.
\end{aligned}$$

For $x = 1$, these polynomials yield the classical Lucas numbers $\mathfrak{L}_n$.

3. **Binet's Formula:** The Binet formula for the classical Lucas polynomials is:

$$\mathfrak{L}_n(x) = \alpha^n(x) + \beta^n(x),$$

where

$$\alpha(x) = \frac{x + \sqrt{x^2+4}}{2}, \quad \beta(x) = \frac{x - \sqrt{x^2+4}}{2}.$$

are the roots of the characteristic equation:

$$z^2 - xz - 1 = 0.$$

The following theorem establishes some fundamental identities connecting the Fibonacci and Lucas polynomials.

Theorem 1.3 *For all $n \geq 1$, we have:*

$$\mathfrak{L}_n(x) = F_{n-1}(x) + F_{n+1}(x)$$

and

$$F_{2n}(x) = F_n(x)\,\mathfrak{L}_n(x).$$

where $F_n(x)$ and $\mathfrak{L}_n(x)$ denote the classical Fibonacci and Lucas polynomials, respectively.

2

Generalized Laguerre Transforms of Sequences

Laguerre polynomials are special mathematical functions used in mathematics and physics. They are solutions of a differential equation and are very important in quantum mechanics.

In this chapter, we study two articles related to associated polynomials. In the first article [8], which is about the generalized Laguerre transform of sequences, we study the associated polynomials, establish a three-term recurrence formula, and introduce the associated Fibonacci polynomials with some of their properties.

Then, in the second article [9], we study the Lucas-type associated polynomials in one variable as an application of the generalized Laguerre transform. We also derive an explicit formula using Leibniz's rule and Rodrigues' formula, present their generating function, and study some relations between the associated Lucas polynomials and the associated Fibonacci polynomials.

2.1 Classical Laguerre Polynomials

In this section, we introduce the classical Laguerre polynomials $L_n(x)$ and study some of their main properties, such as the generating function, recurrence relations, and important identities.

These properties help us understand the associated Laguerre polynomials, their properties, and their matrix representation, which will be presented in the following sections.

We call the following equation the Laguerre differential equation [21]:

$$xy'' + (1 - x)\, y' + ny = 0, \tag{2.1}$$

where $n = 0, 1, 2, 3, \ldots$

The solutions of this equation are the Laguerre polynomials denoted $L_n(x)$, written by the following Rodrigues formula:

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} \left(x^n e^{-x} \right). \tag{2.2}$$

We next present some fundamental properties of these polynomials.

Properties

1. **Generating Function:** The classical Laguerre polynomials $L_n(x)$, are usually defined by the following generating function:

$$\frac{1}{1-z}\exp\left(-\frac{xz}{1-z}\right)=\sum_{n=0}^{\infty}L_n(x)z^n. \tag{2.3}$$

2. **Recurrence Formula:** The classical Laguerre polynomials $L_n(x)$, are defined by the recurrence relation:

$$(n+1)L_{n+1}(x)=(2n+1-x)L_n(x)-nL_{n-1}(x),\quad n\geq 1.$$

The first seven Laguerre polynomials are:

$$\begin{aligned}
L_0(x)&=1\\
L_1(x)&=1-x\\
L_2(x)&=\frac{1}{2}\left(x^2-4x+2\right)\\
L_3(x)&=\frac{1}{6}\left(-x^3+9x^2-18x+6\right)\\
L_4(x)&=\frac{1}{24}\left(x^4-16x^3+72x^2-96x+24\right)\\
L_5(x)&=\frac{1}{120}\left(-x^5+25x^4-200x^3+600x^2-600x+120\right)\\
L_6(x)&=\frac{1}{720}\left(x^6-36x^5+450x^4-2400x^3+5400x^2-4320x+720\right).
\end{aligned}$$

The Laguerre polynomials $L_n(x)$ can be expressed explicitly as follows.

Theorem 2.1

$$L_n(x)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{x^k}{k!}. \tag{2.4}$$

Proof. Using (2.2) and applying the Leibniz formula, we get:

$$\begin{aligned}
L_n(x)&=\frac{e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right)\\
&=\frac{e^x}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}\left(e^{-x}\right)\frac{d^{n-k}}{dx^{n-k}}\left(x^n\right)\\
&=\frac{e^x}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k e^{-x}\,(n)(n-1)(n-2)\cdots(n-(n-k)+1))\,x^k\\
&=\frac{e^x}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^k e^{-x}\cdot\frac{n!}{k!}x^k\\
&=\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{x^k}{k!}.
\end{aligned}$$

This completes the proof of Theorem 2.1.

Theorem 2.2 *We have:*

$$L_n(0) = 1. \tag{2.5}$$

$$L'_n(0) = -n. \tag{2.6}$$

$$L''_n(0) = \frac{1}{2}n(n-1). \tag{2.7}$$

$$\int_0^\infty e^{-tx} L_n(x)\, dx = \frac{1}{t}\left(1 - \frac{1}{t}\right)^n. \tag{2.8}$$

Proof. Using Definition (2.3), we prove (2.5).

For $x = 0$,

$$\begin{aligned} \frac{e^0}{1-z} &= \frac{1}{1-z} \\ &= \sum_{n\geq 0} L_n(0) z^n \\ &= \sum_{n\geq 0} z^n. \end{aligned}$$

Then,

$$L_n(0) = 1.$$

We now prove Formula (2.6).

The expression of $L_n(x)$ satisfies the Laguerre equation (2.1). Therefore, we have:

$$x\frac{d^2}{dx^2}L_n(x) + (1-x)\frac{d}{dx}L_n(x) + nL_n(x) = 0.$$

By setting $x = 0$ in this equation and using (2.4), we obtain:

$$\begin{aligned} L'_n(0) + nL_n(0) &= L'_n(0) + n \\ &= 0. \end{aligned}$$

Thus,

$$L'_n(0) = -n.$$

We now prove (2.6).

In the expression of $L_n(x)$ defined by Relation (2.4), only the terms containing x contribute to $L''_n(0)$. Therefore, we have

$$L_n''(0) = \sum_{k=2}^{n} (-1)^k \binom{n}{k} k(k-1) \frac{x^{k-2}}{k!}.$$

For $k = 2, \ldots, n$, we obtain

$$L_n''(x) = (-1)^2 \binom{n}{2} \frac{2x^0}{2!} + (-1)^3 \binom{n}{3} \frac{6x^1}{3!} + \cdots + (-1)^n \binom{n}{n} \frac{n(n-1)x^{n-2}}{n!}.$$

Then

$$\begin{aligned} L_n''(0) &= (-1)^2 \binom{n}{2} 2(1) \frac{1}{2!} + 0 \\ &= \frac{n!}{4(n-2)!} 2 \\ &= \frac{n(n-1)}{2}. \end{aligned}$$

Let us show that (2.8).

Replacing the expression of $L_n(x)$ defined by (2.4) into the integral, we get

$$\begin{aligned} \int_0^\infty e^{-tx} L_n(x)\, dx &= \int_0^\infty e^{-tx} \left(\sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2 (n-k)!} x^k \right) dx \\ &= \sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2 (n-k)!} \int_0^\infty x^k e^{-tx}\, dx. \end{aligned}$$

Set $z = tx$, so that $dz = t\, dx$. Then

$$\begin{aligned} \int_0^\infty e^{-tx} L_n(x)\, dx &= \sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2 (n-k)!} \int_0^\infty \left(\frac{z}{t} \right)^k e^{-z} \frac{dz}{t} \\ &= \sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2 (n-k)!} \frac{1}{t^{k+1}} \int_0^\infty z^k e^{-z}\, dz \\ &= \sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2 (n-k)!} \frac{\Gamma(k+1)}{t^{k+1}} \\ &= \frac{1}{t} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \left(\frac{1}{t} \right)^k \\ &\qquad (\Gamma(k+1) = k!) \\ &= \frac{1}{t} \sum_{k=0}^{n} \binom{n}{k} \left(-\frac{1}{t} \right)^k \\ &= \frac{1}{t} \left(1 - \frac{1}{t} \right)^n. \end{aligned}$$

■

2.2 Associated Laguerre Polynomials

In this section, we introduce the associated Laguerre polynomials $L_n^m(x)$ and study some of their main properties [8], such as the generating function, recurrence relations, important identities, and their matrix representation. These properties show that the associated Laguerre polynomials are an extension of the classical Laguerre polynomials and explain their importance in mathematics and physics.

For an arbitrary m, we define the following generalized differential equation:

$$xy'' + (m + 1 - x)y' + ny = 0.$$

The polynomial solutions of this equation are called *associated Laguerre polynomials* or *generalized Laguerre polynomials*. The associated Laguerre polynomials, denoted by $L_n^m(x)$, are defined by the following formula [3]:

$$L_n^m(x) = \frac{x^{-m}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+m}\right) \quad \text{for } n \geq 0,$$

with $L_n^0(x) = L_n(x)$.

We next present some fundamental properties of these polynomials.

Properties

1. **Generating Function:** The generating function of $L_n^m(x)$, is given by the following relation [15]:
$$\sum_{n=0}^{\infty} L_n^m(x)\, z^n = \frac{1}{(1-z)^{m+1}} \exp\left(\frac{-xz}{1-z}\right) \quad |z| < 1.$$

2. **Recurrence Formula:** The associated Laguerre polynomials $L_n^m(x)$, are defined by the recurrence relation [8]:
$$(n+1)L_{n+1}^{m-1}(x) = (m+n)L_n^{m-1}(x) - xL_n^m(x). \tag{2.9}$$
The first associated Laguerre polynomials as follows:
$$\begin{aligned} L_0^m(x) &= 1 \\ L_1^m(x) &= 1 + m - x \\ L_2^m(x) &= \frac{1}{2}\left((1+m)(2+m) - 2(2+m)x + x^2\right). \end{aligned}$$

3. **Explicit Formula:** The associated Laguerre polynomials [2] are defined by the explicit formula:
$$L_n^m(x) = (-1)^m \frac{d^m}{dx^m} L_{n+m}(x).$$

These polynomials can be put in an infinite matrix $\mathcal{L} = (L_n^m(x))_{n,m\geq 0}$ in which the first column $L_n^0(x) = L_n(x)$ is the classical Laguerre polynomials [8].

$$\mathcal{L} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1-x & 2-x & 3-x & 4-x & \cdots \\ \frac{1}{2}x^2-2x+1 & \frac{1}{2}x^2-3x+3 & \frac{1}{2}x^2-4x+6 & \vdots & \\ -\frac{1}{6}x^3+\frac{3}{2}x^2-3x+1 & -\frac{1}{6}x^3+2x^2-6x+4 & \vdots & \vdots & \\ \frac{1}{24}x^4-\frac{2}{3}x^3+3x^2-4x+1 & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \\ L_n^0(x) := L_n & L_n^1(x) & L_n^2(x) & & \end{pmatrix} \tag{2.10}$$

The following theorem gives some useful recurrence relations for $L_n^m(x)$.

Theorem 2.3 *The associated Laguerre polynomials satisfy the following properties:*

1. $L_{n-1}^m(x) + L_n^{m-1}(x) = L_n^m(x)$.
2. $(n+1)L_{n+1}^m(x) = (2n+m+1-x)L_n^m(x) - (n+1)L_{n-m}^m(x)$.
3. $x\dfrac{d}{dx}L_n^m(x) = nL_n^m(x) - (n+m)L_{n-1}^m(x)$.
4. $\dfrac{d}{dx}L_n^m(x) = -\displaystyle\sum_{k=0}^{n-1} L_k^m(x)$.
5. $\dfrac{d}{dx}L_n^m(x) = -L_{n-1}^{m-1}(x)$.
6. $L_n^{m+1}(x) = \displaystyle\sum_{k=0}^{n} L_k^m(x)$.

2.3 Associated Polynomials

In this section, we study some properties of the associated polynomials introduced in [8], denoted by $a_{n,m}(x)$, which satisfy the following conditions:

$$a_{n,m}(x) = (-1)^m \frac{d^m}{dx^m} a_{n+m}(x) \tag{2.11}$$

and

$$a_{n+1,m}(x) = \left(1 + \frac{m}{n+1}\right) a_{n,m}(x) - \frac{x}{n+1}\, a_{n,m+1}(x). \tag{2.12}$$

Where $(a_n(x))_{n\geq 0}$ is any sequence of polynomials of the form

$$a_n(x) = \sum_{k=0}^{n} \binom{n}{k} \lambda_k x^k$$

and $(\lambda_k)_{k\geq 0}$ is a sequence that depends on the index k.

The aim of this paper [8], is to construct an infinite matrix $\mathcal{M} := (a_{n,m}(x))_{n,m\geq 0}$ whose first row $\alpha_m := a_{0,m}$ corresponds to the initial sequence and whose first column $\beta_n := a_{n,0}(x)$ corresponds to the final sequence. For x a nonzero complex number, our goal is to examine the three-term recurrence relation (2.12), and more broadly, to extend the following Laguerre transformation. Recall that the Laguerre transform [19] of a sequence $(\alpha_n)_{n\geq 0}$ is the sequence $(\beta_n(x))_{n\geq 0}$ given by:

$$\beta_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!} \alpha_k \tag{2.13}$$

and the inverse transform is:

$$\alpha_n = n! \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{-n} \beta_k(x).$$

Remark 2.3.1 *We note that if $a_{n,m}(x)$ satisfies (2.11) then the initial sequence α_m does not depend on x.*

2.4 Generalized Laguerre Transform

Theorem 2.4 *Given an initial sequence $\alpha_m := a_{0,m}$, define the matrix $\mathcal{M}$ associated with the initial sequence by (2.12), then*

i) The entries of the matrix M are given by:

$$a_{n,m}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+m}{n-k} \frac{x^k}{k!} \alpha_{m+k}. \tag{2.14}$$

ii) Suppose that the initial sequence α_{m+s} has the following exponential generating function

$$A_s(z) = \sum_{k\geq 0} \alpha_{k+s} \frac{z^k}{k!}.$$

Then, the sequence $(a_{n,s}(x))_{n\geq 0}$ of the sth columns of the matrix $\mathcal{M}$ has an ordinary generating function $B_s(z;\, x) = \sum_{n\geq 0} a_{n,s}(x)\, z^n$, given by:

$$B_s(z;x) = \frac{1}{(1-z)^{s+1}} A_s\left(-\frac{xz}{1-z}\right). \tag{2.15}$$

Proof.

i) According to: $a_{n,m}(x) = (-1)^m \dfrac{d^m}{dx^m} a_{n+m}(x)$

and using the Laguerre transform (2.13), we find:

$$
\begin{aligned}
a_{n,m}(x) &= (-1)^m \frac{d^m}{dx^m} \sum_{k=0}^{n+m} \binom{n+m}{k} \frac{(-x)^k}{k!} a_k \\
&= (-1)^m \sum_{k=0}^{n+m} \binom{n+m}{k} \frac{(-1)^k a_k}{k!} \frac{d^m}{dx^m} x^k \\
&= (-1)^m \sum_{k=0}^{n+m} \binom{n+m}{k} \frac{(-1)^k a_k}{k!} \cdot \frac{x^{k-m} \cdot k!}{(k-m)!} \\
&= (-1)^m \sum_{k=0}^{n+m} \binom{n+m}{k} (-1)^k a_k \frac{x^{k-m}}{(k-m)!}.
\end{aligned}
$$

Setting $k = j + m$, we find:

$$
\begin{aligned}
a_{n,m}(x) &= (-1)^m \sum_{j=0}^{n} \binom{n+m}{j+m} \frac{(-1)^{j+m}}{j!} a_{j+m}\, x^j \\
&= \sum_{j=0}^{n} (-1)^j \binom{n+m}{n-j} \frac{x^j}{j!} a_{j+m},
\end{aligned}
$$

si $j = k$:

$$
a_{n,m}(x) = (-1)^k \sum_{k=0}^{n} \binom{n+m}{n-k} \frac{x^k}{k!} a_{m+k},
$$

which gives the result (2.11).

ii) Now, using (2.11), we obtain:

$$
\begin{aligned}
B_s(z,x) &= \sum_{n\geq 0} \left(\sum_{k=0}^{n} (-1)^k \binom{n+s}{n-k} \frac{x^k}{k!} a_{s+k} \right) z^n \\
&= \sum_{k\geq 0} \frac{(-1)^k x^k}{k!} a_{s+k} \sum_{n\geq 0} \binom{n+m}{k-m} z^n.
\end{aligned}
$$

We know that:

$$
\sum_{n\geq 0} \binom{n+m}{k-m} z^n = \frac{z^k}{(1-z)^{s+k+1}}.
$$

Then:

$$
\begin{aligned}
B_s(z,x) &= \sum_{k\geq 0} \frac{(-1)^k x^k}{k!} a_{s+k} \cdot \frac{z^k}{(1-z)^{s+k+1}} \\
&= \frac{1}{(1-z)^{s+1}} \sum_{k\geq 0} \frac{a_{s+k}}{k!} \left(\frac{-xz}{1-z} \right)^k \\
&= \frac{1}{(1-z)^{s+1}} A_s\left(\frac{-xz}{1-z} \right),
\end{aligned}
$$

which gives the formula (2.15).

Theorem 2.5 *Given final sequence $\beta_n := a_{n,0}(x)$, define the matrix $\mathcal{M}$ associated with the final sequence by:*

$$a_{n,m+1}(x) = \frac{n+m+1}{x}\, a_{n,m}(x) - \frac{n+1}{x}\, a_{n+1,m}(x), \tag{2.16}$$

then

i) The entries of the matrix $\mathcal{M}$ are given by:

$$a_{n,m}(x) = x^{-m} \sum_{k=0}^{m} (-1)^k \frac{(n+m)!}{n!} \binom{m}{m-k} \beta_{n+k}(x). \tag{2.17}$$

ii) Suppose that the final sequence β_{n+s} has the following ordinary generating function $B_s(z) = \sum_{k\geq 0} \beta_{k+s}\, z^k$. Then, the sequence $(a_{s,m}(x))_{m\geq 0}$ of the sth row of the matrix $\mathcal{M}$ has an exponential generating function $A_s(z;\, x) = \sum_{m\geq 0} a_{s,m}(x)\, \frac{z^m}{m!}$, given by:

$$A_s(z;x) = \frac{(-1)^s x}{s!(x-z)^{s+1}} \left(\left((x-z)^2 \frac{d}{dz} \right)^s \left(\frac{z}{z-x} \right)^s B_s\Big(\frac{z}{z-x} \Big) \right). \tag{2.18}$$

Proof.
The verification of (2.17) follows by induction on m. According to (2.17), we have:

$$\begin{aligned} A_s(z;x) &= \sum_{m\geq 0} \left(x^{-m} \sum_{k=0}^{m} (-1)^k \frac{(s+m)!}{s!} \binom{m}{m-k} \beta_{s+k} \right) \frac{z^m}{m!} \\ &= \sum_{k\geq 0} (-1)^k \beta_{s+k} \sum_{m\geq 0} \frac{(s+m)!}{s!m!} \binom{m}{m-k} \left(\frac{z}{x} \right)^m \\ &= \sum_{k\geq 0} (-1)^k \beta_{s+k} \sum_{m\geq 0} \binom{s+m}{m+k} \left(\frac{z}{x} \right)^m. \end{aligned}$$

Using the fact that

$$\sum_{m\geq 0} \binom{s+m}{s+k} \left(\frac{z}{x} \right)^m = \frac{x^{s+1}}{(x-z)^{k+s+1}}\, z^k$$

and

$$\binom{k+s}{s} t^k = \frac{1}{s!} \frac{d^s}{dt^s} t^{k+s},$$

we obtain

$$\begin{aligned} A_s(z;x) &= \frac{x^{s+1}}{(x-z)^{s+1}} \sum_{k\geq 0} \beta_{s+k} \binom{k+s}{s} \left(\frac{-z}{x-z} \right)^k \\ &= \frac{x^{s+1}}{(x-z)^{s+1}} \sum_{k\geq 0} \beta_{s+k} \frac{1}{(-x)^s s!} \left((x-z)^2 \frac{d}{dz} \right)^s \left(\frac{z}{z-x} \right)^{k+s} \\ &= \frac{(-1)^s x}{s!(x-z)^{s+1}} \left((x-z)^2 \frac{d}{dz} \right)^s \left[\left(\frac{z}{z-x} \right)^s B_s\Big(\frac{z}{z-x} \Big) \right], \end{aligned}$$

which completes the proof.

According to the two previous theorems, we obtain the following identity, which gives the generalized Laguerre transformation presented in the following corollary.

Corollary 2.4.1 *For $n, m \geq 0$, we have:*

$$\sum_{k=0}^{n}(-1)^k \binom{n+m}{n-k} \frac{x^k}{k!}\, \alpha_{m+k} = x^{-m} \sum_{k=0}^{m}(-1)^k \frac{(n+m)!}{n!} \binom{m}{k} \beta_{n+k}. \tag{2.19}$$

Remark 2.4.1 *The identity (2.19) can be viewed as the generalized Laguerre transform which can be reduced, for $m = 0$, to the Laguerre transform of the sequence α_n, and for $n = 0$, to the inverse Laguerre transform of the sequence β_m.*

Example 2.1 *Setting the initial sequence $a_{0,m} = 1$ in (2.12), we get (2.10). Since $A_m(z) = \exp(z)$, it follows from (2.15) that the final sequence has an ordinary generating function given by:*

$$\sum_{n\geq 0} L_n^m(x)\, z^n = \frac{1}{(1-z)^{m+1}} \exp\left(\frac{-xz}{1-z}\right). \tag{2.20}$$

Now, from Corollary 2.4.1, we have:

$$\begin{aligned} L_n^m(x) &= \sum_{k=0}^{n}(-1)^k \binom{n+m}{n-k} \frac{x^k}{k!} \\ &= x^{-m} \sum_{k=0}^{m}(-1)^k \frac{(n+m)!}{n!} \binom{m}{k} L_{n+k}(x). \end{aligned}$$

Theorem 2.6 *We consider a matrix $\mathcal{L} = \left(L_n^m(x)\right)_{n,m\geq 0}$ of order $n+1$ arising from (2.12) with initial sequence $a_{0,m} = 1$, then*

$$\det \mathcal{L} = 1,$$

where det *denotes determinant.*

Proof. Denote by C_j the j-th column of $\mathcal{L}$. We apply repeatedly, for $1 \leq r \leq n$, the following column operation:

$$C_{n-j} \longleftarrow C_{n-j} - C_{n-j-1} \qquad (0 \leq j \leq n-r).$$

This operation consists of subtracting from each column the one that precedes it, which does not alter the value of the determinant.

After applying these operations successively, the matrix $\mathcal{L}$ transforms into a lower triangular matrix whose entries are:

$$a_{i,j} = \sum_{k=0}^{i}(-1)^k \binom{i}{k+j} \frac{x^k}{k!}, \qquad 0 \leq i, j \leq n.$$

On the main diagonal, that is for $i = j$, the binomial coefficient $\binom{i}{k+i}$ vanishes for all $k \geq 1$, so that only the term $k = 0$ contributes:

$$a_{i,i} = \binom{i}{i} \frac{x^0}{0!} = 1.$$

The resulting matrix is therefore lower triangular with ones on the diagonal, and since the column operations performed do not change the value of the determinant, we conclude:

$$\det \mathcal{L} = \prod_{i=0}^{n} a_{i,i} = 1.$$

■

2.5 Fibonacci Associated Polynomials

As an application of the associated polynomials studied above, we introduce the associated Fibonacci polynomials and present some of their basic properties. To this end, we consider the initial sequence $a_{0,m} := -m!\, F_m$ in (2.12), this formula will be represented by the matrix $\mathcal{F}$ as follows:

$$\mathcal{F} = \begin{pmatrix} 0 & -1 & -2 & -12 & -72 & \cdots \\ x & 2x-2 & 12x-6 & 72x-48 & \cdots & \\ -x^2+2x & -6x^2+6x-3 & -36x^2+48x-12 & \cdots & & \\ 2x^3-3x^2+3x & 12x^3-24x^2+12x-4 & \cdots & & & \\ -3x^4+8x^3-6x^2+4x & \cdots & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \mathbb{F}_n^0(x) := \mathbb{F}_n(x) & \mathbb{F}_n^1(x) & \mathbb{F}_n^2(x) & \mathbb{F}_n^3(x) & \mathbb{F}_n^4(x) & \cdots \end{pmatrix}$$

The first column of the matrix $\mathcal{F}$ gives the Fibonacci-type associated polynomials, denoted $\mathbb{F}_n(x)$. Moreover, by applying the generalized Laguerre transform to the sequence $(-m!\, F_m)_{m\geq 0}$, we obtain the sequence $\mathbb{F}_n^m(x)$, known as the m-associated Fibonacci polynomials, defined by:

1. **Generating Function:** The m-associated Fibonacci polynomials $\mathbb{F}_n^m(x)$ are defined by the following generating function:

$$\frac{m!}{\alpha-\beta}\left(\frac{\beta^m}{(1-z+\beta xz)^{m+1}} - \frac{\alpha^m}{(1-z+\alpha xz)^{m+1}}\right) = \sum_{n\geq 0} \mathbb{F}_n^m(x)\, z^n.$$

Remark 2.5.1 *For $x = 1$ and $m = 0$, we obtain the ordinary generating function for the classical Fibonacci numbers defined by (1.2).*

2. **Explicit Formula:** The explicit formula of the m-associated Fibonacci polynomials is given by:

$$\begin{aligned}\mathbb{F}_n^m(x) &= (-1)^m \frac{d^m}{dx^m}\mathbb{F}_{n+m}(x) \\ &= \sum_{k=0}^{n}(-1)^{k+1}\binom{n+m}{n-k}\frac{x^k}{k!}(m+k)!\,F_{m+k} \\ &= x^{-m}\sum_{k=0}^{m}(-1)^k\frac{(n+m)!}{n!}\binom{m}{k}\mathbb{F}_{n+k}(x).\end{aligned}$$

In what follows, we present the polynomials $\mathbb{F}_n^0(x) := \mathbb{F}_n(x)$, which correspond to the first column of the matrix $\mathcal{F}$, and we give some of their properties.

Properties

1. **Generating Function:** The associated Fibonacci polynomials $\mathbb{F}_n(x)$ are defined by the following generating function [7]:

$$\sum_{n=0}^{\infty}\mathbb{F}_n(x)z^n = \frac{xz}{1-(2-x)z-(x^2+x-1)z^2}.$$

2. **Recurrence Formula:** The associated Fibonacci polynomials $\mathbb{F}_n(x)$ satisfy the following recurrence relation:

$$\mathbb{F}_{n+2}(x) = (2-x)\mathbb{F}_{n+1}(x) + (x^2+x-1)\mathbb{F}_n(x), \tag{2.21}$$

with initial conditions

$$\mathbb{F}_0(x) = 0 \quad \text{and} \quad \mathbb{F}_1(x) = x.$$

3. **Binet's Formula:** The associated Fibonacci polynomials $\mathbb{F}_n(x)$ are characterized by the following Binet's formula:

$$\mathbb{F}_n(x) = \frac{1}{\sqrt{5}}\left(\beta^n(x) - \alpha^n(x)\right), \tag{2.22}$$

where $\alpha(x) = 1-\alpha x$ and $\beta(x) = 1-\beta x$ are the roots of the characteristic equation.

4. **Explicit Formula:** The explicit formula of the associated Fibonacci polynomials $\mathbb{L}_n(x)$ is given by the following expression:

$$\mathbb{F}_n(x) = \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}F_k\cdot x^k. \tag{2.23}$$

2.6 Lucas Associated Polynomials

As another application of the associated polynomials studied above, we introduce the Lucas-type associated polynomials and present some of their basic properties. To this end, we consider the initial sequence $a_{0,m} := -m!\,\mathfrak{L}_m$ in (2.12), this formula will be represented by the matrix $\mathfrak{L}$ as follows:

$$\mathfrak{L} = \begin{pmatrix} 2 & 1 & 6 & 24 & \cdots \\ 2-x & 2-6x & 18-24x & \cdots & \\ 2-2x+3x^2 & 3-18x+12x^2 & \cdots & & \\ 2-3x+9x^2-4x^3 & \cdots & & & \\ \vdots & \vdots & \vdots & \vdots & \\ \mathbb{L}_n^0(x) := \mathbb{L}_n(x) & \mathbb{L}_n^1(x) & \mathbb{L}_n^2(x) & \mathbb{L}_n^3(x) & \cdots \end{pmatrix}$$

The first column of the matrix $\mathfrak{L}$ gives the Lucas-type associated polynomials, denoted $\mathbb{L}_n(x)$. Moreover, by applying the generalized Laguerre transform to the sequence $(-m!\,\mathfrak{L}_m)_{m\geq 0}$, we obtain the sequence $\mathbb{L}_n^m(x)$, known as the m-associated Lucas polynomials, defined by:

1. **Generating Function:** The generating function of the m-associated Lucas polynomials is given by:

$$\frac{m!}{\beta-\alpha}\left(\frac{(2\beta-1)\beta^m}{(1-\beta x\,z)^{m+1}} - \frac{(2\alpha-1)\alpha^m}{(1-\alpha x\,z)^{m+1}}\right) = \sum_{n\geq 0}\mathbb{L}_n^m(x)\,z^n.$$

Remark 2.6.1 *For $x=1$ and $m=0$, we obtain the ordinary generating function for the classical Lucas numbers defined by (1.4).*

2. **Explicit Formula:** The explicit formula of the m-associated Lucas polynomials is given by:

$$\begin{aligned}\mathbb{L}_n^m(x) &= (-1)^m \frac{d^m}{dx^m}\,\mathbb{L}_{n+m}(x) \\ &= \sum_{k=0}^{n}(-1)^k \binom{n+m}{n-k}\frac{x^k}{k!\,(m+k)!}\,\mathfrak{L}_{m+k} \\ &= x^{-m}\sum_{k=0}^{m}(-1)^k\frac{(n+m)!}{n!}\binom{m}{k}\,\mathbb{L}_{n+k}(x).\end{aligned}$$

In what follows, we present the polynomials $\mathbb{L}_n^0(x) := \mathbb{L}_n(x)$, which correspond to the first column of the matrix $\mathfrak{L}$, and we give some of their properties.

Properties

1. **Generating Function:** The associated Lucas polynomials $\mathbb{L}_n(x)$ are defined by the following generating function:

$$\frac{2-(2-x)z}{1-(2-x)z+(x^2+x-1)z^2} = \sum_{n=0}^{\infty} \mathbb{L}_n(x)\, z^n. \tag{2.24}$$

2. **Recurrence Formula:** The associated Lucas polynomials $\mathbb{L}_n(x)$ satisfy the following recurrence relation:

$$\mathbb{L}_n(x) = (2-x)\,\mathbb{L}_{n-1}(x) + (x^2+x-1)\,\mathbb{L}_{n-2}(x) \quad n \geq 2, \tag{2.25}$$

with

$$\mathbb{L}_0(x) = 2 \qquad \text{and} \qquad \mathbb{L}_1(x) = 2-x.$$

3. **Binet's Formula:** The associated Lucas polynomials $\mathbb{L}_n(x)$ are characterized by the following Binet's formula:

$$\mathbb{L}_n(x) = \beta^n(x) + \alpha^n(x), \tag{2.26}$$

where $\alpha(x) = -\alpha x$ and $\beta(x) = -\beta x$ denote the roots of the characteristic equation:

$$\lambda^2 - (2-x)\,\lambda - \left(x^2+x\right) = 0.$$

Moreover, these roots satisfy the following identities:

$$\begin{aligned} \alpha(x)\,\beta(x) &= -x^2 - x - 1. \\ \alpha(x) + \beta(x) &= 2 - x. \\ \alpha(x) - \beta(x) &= -\sqrt{5}\,x. \end{aligned}$$

4. **Explicit Formula:** The explicit formula of the associated Lucas polynomials $\mathbb{L}_n(x)$ in terms of the classical Lucas numbers is given by the following expression:

$$\mathbb{L}_n(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k \mathfrak{L}_k. \tag{2.27}$$

whose inverse is:

$$\mathfrak{L}_k = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\, x^{-n}\, \mathbb{L}_k(x). \tag{2.28}$$

2.6.1 Limit of the Ratio of Consecutive Associated Lucas Polynomials

We now establish that the sequence of ratios of consecutive associated Lucas-type polynomials converges to the Golden ratio α, as shown by the following expansion:

$$\lim_{n\to+\infty} \frac{\mathbb{L}_n(x)}{\mathbb{L}_{n+1}(x)} = \frac{1}{\beta(x)} = \frac{\alpha(x)}{(x+\alpha)(x+\beta)}.$$

More generally, we have:

$$\lim_{n\to+\infty}\frac{\mathbb{L}_{n-r}(x)}{\mathbb{L}_n(x)} = \lim_{n\to+\infty}\frac{\mathbb{L}_{n-1}(x)}{\mathbb{L}_n(x)}\cdot\frac{\mathbb{L}_{n-2}(x)}{\mathbb{L}_{n-1}(x)}\cdots\frac{\mathbb{L}_{n-r}(x)}{\mathbb{L}_{n-r+1}(x)}$$

$$= \frac{1}{\beta^r(x)}$$

$$= \frac{\alpha^r(x)}{(x+\alpha)^r(x+\beta)^r}.$$

By applying Leibniz's formula for derivatives, one can verify that the associated Lucas-type polynomials introduced above are precisely the degree $n+1$ polynomials defined by the Rodrigues-type formula for Lucas polynomials. To this end, we state the following result.

Lemma 2.6.1 *For $n \geq 0$, we have:*

$$\frac{d^n}{dx^n}\left(\frac{x-2}{x^2+x-1}\right) = \frac{(-1)^n\, n!}{\alpha-\beta}\left(\frac{2+\alpha}{(x+\alpha)^{n+1}} - \frac{2+\beta}{(x+\beta)^{n+1}}\right).$$

Proof.
Since $x^2+x-1=0$ is the characteristic equation, we know that its roots are:

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

By partial fraction decomposition, we write:

$$\frac{x-2}{x^2+x-1} = \frac{A}{x+\alpha} + \frac{B}{x+\beta},$$

which gives:

$$x-2 = A(x+\beta) + B(x+\alpha).$$

Setting $x=-\alpha$, we obtain:

$$A = \frac{\alpha+2}{\alpha-\beta}.$$

Setting $x=-\beta$, we obtain:

$$B = \frac{-(\beta+2)}{\alpha-\beta}.$$

Therefore:

$$\frac{x-2}{x^2+x-1} = \frac{1}{\alpha-\beta}\left[\frac{\alpha+2}{x+\alpha} - \frac{\beta+2}{x+\beta}\right].$$

Using the classical formula:

$$\frac{d^n}{dx^n}\left(\frac{1}{x+c}\right) = \frac{(-1)^n\, n!}{(x+c)^{n+1}},$$

we obtain:

$$\frac{d^n}{dx^n}\left(\frac{x-2}{x^2+x-1}\right) = \frac{(-1)^n\, n!}{\alpha-\beta}\left[\frac{\alpha+2}{(x+\alpha)^{n+1}} - \frac{\beta+2}{(x+\beta)^{n+1}}\right].$$

■

The following theorem provides a Rodrigues-type formula for the Lucas-type associated polynomials.

Theorem 2.7 *The Rodrigues-type formula for the Lucas-type associated polynomials is given by:*

$$\mathbb{L}_{n+1}(x) = \frac{(1-x-x^2)^{n+1}}{n!} \cdot \frac{d^n}{dx^n}\left(\frac{x-2}{x^2+x-1} \cdot x^n\right).$$

Proof. By applying the Leibniz formula for derivatives and using Lemma 2.6.1 we get:

$$\frac{d^k}{dx^k}x^n = \frac{n!}{(n-k)!}x^{n-k},$$

we can deduce that

$$\frac{d^n}{dx^n}\left(\frac{x-2}{x^2+x-1}x^n\right) = \frac{n!}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{2+\alpha}{(x+\alpha)^{n-k+1}} - \frac{2+\beta}{(x+\beta)^{n-k+1}}\right)(-x)^{n-k}.$$

Then

$$\begin{aligned}\frac{d^n}{dx^n}\left(\frac{x-2}{x^2+x-1}x^n\right) &= \frac{n!}{\alpha-\beta}\frac{(2+\alpha)(x+\beta)^{n+1}}{(x^2+x-1)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(x+\alpha)^k(-x)^{n-k} \\ &\quad - \frac{n!}{\alpha-\beta}\frac{(2+\beta)(x+\alpha)^{n+1}}{(x^2+x-1)^{n+1}}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(x+\beta)^k x^{n-k}.\end{aligned}$$

Now by Newton's formula and the fact

$$\begin{aligned}1-2\beta &= \sqrt{5}, \\ 1-2\alpha &= -\sqrt{5},\end{aligned}$$

we obtain

$$\begin{aligned}\frac{d^n}{dx^n}\left(\frac{x-2}{x^2+x-1}x^n\right) &= \frac{n!}{\alpha-\beta}\frac{(2+\alpha)(x+\beta)^{n+1}}{(x^2+x-1)^{n+1}}\alpha^n \\ &\quad - \frac{n!}{\alpha-\beta}\frac{(2+\beta)(x+\alpha)^{n+1}}{(x^2+x-1)^{n+1}}\beta^n \\ &= \frac{(-1)^{n+1}n!}{(x^2+x-1)^{n+1}}\left((1-\alpha x)^{n+1} + (1-\beta x)^{n+1}\right).\end{aligned}$$

Thus

$$\frac{d^n}{dx^n}\left(\frac{x-2}{x^2+x-1}x^n\right) = \frac{n!}{(1-x-x^2)^{n+1}}L_{n+1}(x),$$

which completes the proof.

Next, we define the associated Lucas polynomials of negative index by means of (2.25), through the following formula:

$$(x^2+x-1)\,\mathbb{L}_{-n}(x) = \mathbb{L}_{-n+2}(x) - (2-x)\,\mathbb{L}_{-n+1}(x). \tag{2.29}$$

The first few expressions are:

$$\mathbb{L}_0(x) = 2$$

$$\mathbb{L}_{-1}(x) = \frac{2-x}{1-x-x^2}$$

$$\mathbb{L}_{-2}(x) = \frac{-2+2x+3x^2}{(1-x-x^2)^2}.$$

As a consequence of (2.25), we can establish a formula for the associated Lucas polynomials with a negative index.

Theorem 2.8 *For $n \geq 0$, we have:*

$$\mathbb{L}_{-n}(x) = \frac{\mathbb{L}_n(x)}{(\alpha(x)\,\beta(x))^n},$$

where $\alpha(x) = 1-\alpha x$ and $\beta(x) = 1-\beta x$.

Proof. We know that $\mathbb{L}_n(x) = \alpha^n(x) + \beta^n(x)$, therefore:

$$\mathbb{L}_{-n}(x) = \alpha^{-n}(x) + \beta^{-n}(x),$$

and we know that $\alpha^{-n}(x) = \dfrac{1}{\alpha^n(x)}$ and $\beta^{-n}(x) = \dfrac{1}{\beta^n(x)}$. Hence, we deduce that:

$$\begin{aligned}\mathbb{L}_{-n}(x) &= \frac{1}{\alpha^n(x)} + \frac{1}{\beta^n(x)}\\ &= \frac{\beta^n(x)+\alpha^n(x)}{\alpha^n(x)\,\beta^n(x)}\\ &= \frac{\mathbb{L}_n(x)}{(\alpha(x)\,\beta(x))^n},\end{aligned}$$

which completes the proof.

The following theorem presents an integral link between the Lucas-type associated polynomials and the associated Fibonacci polynomials.

Theorem 2.9 *For $n \geq 1$, we have:*

$$\mathbb{L}_n(x) = 2 - \mathbb{F}_n(x) + 2n \int_0^x \mathbb{F}_{n-1}(t)\, dt,$$

where $\mathbb{F}_n(x)$ denotes the associated Fibonacci polynomials.

Proof. Since $F_{k+1} = F_k + F_{k-1}$, we have:

$$\begin{aligned}
\mathfrak{L}_k &= F_{k-1} + F_{k+1} \\
&= F_{k-1} + F_k + F_{k-1} \\
&= 2F_{k-1} + F_k.
\end{aligned}$$

Using the explicit formula (2.27) and the fact that $\mathfrak{L}_k = 2F_{k-1} + F_k$, we get:

$$\begin{aligned}
\mathbb{L}_n(x) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k (F_k + 2F_{k-1}) \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k F_k + 2 \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^k F_{k-1}.
\end{aligned}$$

Using the explicit formula (2.23), we obtain:

$$\begin{aligned}
\mathbb{F}_n(x) &= -\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_k x^k + 2 \sum_{k=0}^{n} (-1)^k x^k F_{k-1} \\
&= -\mathbb{F}_n(x) + 2 \sum_{k=0}^{n} (-1)^k x^k F_{k-1}.
\end{aligned}$$

Since for $k = 0$: $F_{-1} = 1$, we get:

$$\mathbb{F}_n(x) = -\mathbb{F}_n(x) + 2 + 2 \sum_{k=1}^{n} (-1)^k \binom{n}{k} x^k F_{k-1}.$$

Using $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$, we get:

$$\begin{aligned}
\mathbb{F}_n(x) &= -\mathbb{F}_n(x) + 2 + 2 \sum_{k=1}^{n} (-1)^k \frac{n}{k} \binom{n-1}{k-1} x^k F_{k-1} \\
&= -\mathbb{F}_n(x) + 2 + 2n \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \frac{x^k}{k} F_{k-1}.
\end{aligned}$$

Using $\dfrac{x^k}{k} = \displaystyle\int_0^x t^{k-1}\, dt$, we get:

$$\mathbb{F}_n(x) = -\mathbb{F}_n(x) + 2 + 2n \int_0^x \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} t^{k-1} F_{k-1}\, dt.$$

Setting $k' = k - 1$:

$$\begin{aligned}\mathbb{F}_n(x) &= -\mathbb{F}_n(x) + 2 + 2n \int_0^x \sum_{k'=0}^{n-1} (-1)^{k'+1} \binom{n-1}{k'} t^{k'} F_{k'}\, dt \\ &= -\mathbb{F}_n(x) + 2 + 2n \int_0^x \mathbb{F}_{n-1}(t)\, dt,\end{aligned}$$

which completes the proof.

Theorem 2.10 *For $n \geq 1$, we have:*

$$x\,\mathbb{L}_n(x) = (x^2 + x - 1)\,\mathbb{F}_{n-1}(x) + \mathbb{F}_{n+1}(x).$$

Proof. By induction on n. For $n = 1$, we have:

$$x\mathbb{L}_1(x) = (x^2 + x - 1)\mathbb{F}_0(x) + \mathbb{F}_2(x).$$

Assume that

$$x\mathbb{L}_{n-1}(x) = (x^2 + x - 1)\mathbb{F}_{n-2}(x) + \mathbb{F}_n(x)$$

and

$$x\mathbb{L}_{n-2}(x) = (x^2 + x - 1)\mathbb{F}_{n-3}(x) + \mathbb{F}_{n-1}(x).$$

From formula (2.25), we have:

$$\begin{aligned}x\mathbb{L}_n(x) &= x(2-x)\mathbb{L}_{n-1}(x) + x(x^2 + x - 1)\mathbb{L}_{n-2}(x) \\ &= (2-x)\,x\mathbb{L}_{n-1}(x) + (x^2 + x - 1)\,x\mathbb{L}_{n-2}(x) \\ &= (2-x)\Big((x^2 + x - 1)\mathbb{F}_{n-2}(x) + \mathbb{F}_n(x)\Big) \\ &\quad + (x^2 + x - 1)\Big((x^2 + x - 1)\mathbb{F}_{n-3}(x) + \mathbb{F}_{n-1}(x)\Big) \\ &= (x^2 + x - 1)\Big((2-x)\mathbb{F}_{n-2}(x) + (x^2 + x - 1)\mathbb{F}_{n-3}(x)\Big) \\ &\quad + (2-x)\mathbb{F}_n(x) + (x^2 + x - 1)\mathbb{F}_{n-1}(x).\end{aligned}$$

From (2.21), we get:

$$\mathbb{F}_{n-1}(x) = (2-x)\mathbb{F}_{n-2}(x) + (x^2 + x - 2)\mathbb{F}_{n-3}(x)$$

and

$$\mathbb{F}_{n+1}(x) = (2-x)\mathbb{F}_n(x) + (x^2 + x - 2)\mathbb{F}_{n-1}(x),$$

therefore

$$x\mathbb{L}_n(x) = (x^2 + x - 1)\mathbb{F}_{n-1}(x) + \mathbb{F}_{n+1}(x),$$

which completes the proof.

The next results follow from Theorem 2.10, and from the recurrence relation (2.21).

Corollary 2.6.1 *For $n \geq 1$, we have:*

$$x\,\mathbb{L}_n(x) = (2-x)\,\mathbb{F}_n(x) + 2(x^2+x-1)\,\mathbb{F}_{n-1}(x)$$

and for $n \geq 0$:

$$x\,\mathbb{L}_n(x) = 2\,\mathbb{F}_{n+1}(x) - (2-x)\,\mathbb{F}_n(x).$$

Corollary 2.6.2 *For $n \geq 0$, we have:*

$$\beta^n(x) = \frac{\mathbb{L}_n(x) + \sqrt{5}\,\mathbb{F}_n(x)}{2}$$

and

$$\alpha^n(x) = \frac{\mathbb{L}_n(x) - \sqrt{5}\,\mathbb{F}_n(x)}{2}.$$

Corollary 2.6.3 *For $n \geq 0$, we have:*

$$\mathbb{F}_{2n}(x) = \mathbb{F}_n(x)\,\mathbb{L}_n(x)$$

and

$$\mathbb{L}_n^2(x) - 5\,\mathbb{F}_n^2(x) = 4(-1)^n(x^2+x-1)^n.$$

The following theorem provides a Catalan-type identity for the Lucas-type associated polynomials.

Theorem 2.11 *For $n \geq r$, we have:*

$$\mathbb{L}_{n-r}(x)\,\mathbb{L}_{n+r}(x) - \mathbb{L}_n^2(x) = 5(-1)^{n-r}(x^2+x-1)^{n-r}\,\mathbb{F}_r^2(x). \tag{2.30}$$

Proof. From (2.26), we get:

$$\begin{aligned}
\mathbb{L}_{n-r}(x)\mathbb{L}_{n+r}(x) - \mathbb{L}_n^2(x) &= \left(\alpha^{n-r}(x) + \beta^{n-r}(x)\right)\left(\alpha^{n+r}(x) + \beta^{n+r}(x)\right) \\
&\quad - \left(\alpha^n(x) + \beta^n(x)\right)^2 \\
&= \alpha^{2n}(x) + \alpha^{n-r}(x)\beta^{n+r}(x) + \beta^{n-r}(x)\alpha^{n+r}(x) + \beta^{2n}(x) \\
&\quad - \alpha^{2n}(x) - \beta^{2n}(x) - 2(\alpha\beta)^n(x) \\
&= \alpha^{n-r}(x)\beta^{n+r}(x) + \beta^{n-r}(x)\alpha^{n+r}(x) + 2(\alpha\beta)^n(x).
\end{aligned}$$

We know that:

$$\alpha^{n-r}(x)\beta^{n+r}(x) = (\alpha\beta)^{n-r}(x)\cdot\beta^{2r}(x)$$

and

$$\beta^{n-r}(x)\alpha^{n+r}(x) = (\alpha\beta)^{n-r}(x)\cdot\alpha^{2r}(x)$$

and

$$2(\alpha\beta)^n(x) = (\alpha\beta)^{n-r}(x)\cdot 2(\alpha\beta)^r(x).$$

Therefore

$$\begin{aligned}
\mathbb{L}_{n-r}(x)\mathbb{L}_{n+r}(x) - \mathbb{L}_n^2(x) &= (\alpha\beta)^{n-r}(x)\left(\beta^{2r}(x) + \alpha^{2r}(x) - 2(\alpha\beta)^r(x)\right) \\
&= (\alpha\beta)^{n-r}(x)\left(\alpha^r(x) - \beta^r(x)\right)^2.
\end{aligned}$$

We know that:

$$(\alpha\beta)^{n-r}(x) = (-1)^{n-r}(x^2+x-1)^{n-r}$$

and

$$\begin{aligned}(\alpha^r(x)-\beta^r(x))^2 &= ((\alpha(x)-\beta(x))\,\mathbb{F}_r(x))^2\\ &= 5\,\mathbb{F}_r^2(x).\end{aligned}$$

therefore

$$\mathbb{L}_{n-r}(x)\mathbb{L}_{n+r}(x) - \mathbb{L}_n^2(x) = 5(-1)^{n-r}(x^2+x-1)^{n-r}\mathbb{F}_r^2(x),$$

which completes the proof.

Remark 2.6.2 *For $r=1$, we obtain Cassini's type identity:*

$$\mathbb{L}_{n-1}(x)\,\mathbb{L}_{n+1}(x) - \mathbb{L}_n^2(x) = 5(-1)^{n-1}(x^2+x-1)^{n-1}\,x^2.$$

In the following theorem, we present a D'Ocagne-type identity involving Lucas polynomials and Fibonacci polynomials.

Theorem 2.12 *If $m \geq n$, then:*

$$\mathbb{L}_m(x)\,\mathbb{L}_{n+1}(x) - \mathbb{L}_{m+1}(x)\,\mathbb{L}_n(x) = -5x(1-x-x^2)^n\,\mathbb{F}_{m-n}(x).$$

Proof. From Binet formula of Fibonacci associated polynomials (2.22) and Binet formula of Lucas associated polynomials (2.26):

$$\begin{aligned}&\mathbb{L}_m(x)\,\mathbb{L}_{n+1}(x) - \mathbb{L}_{m+1}(x)\,\mathbb{L}_n(x)\\ &= \left((\alpha^m(x)+\beta^m(x))\left(\alpha^{n+1}(x)+\beta^{n+1}(x)\right)\right)\\ &\quad - \left(\left(\alpha^{m+1}(x)+\beta^{m+1}(x)\right)(\alpha^n(x)+\beta^n(x))\right)\\ &= \alpha^{m+n+1}(x) + \alpha^m(x)\beta^{n+1}(x) + \beta^m(x)\alpha^{n+1}(x) + \beta^{m+n+1}(x)\\ &\quad - \left(\alpha^{m+n+1}(x) + \alpha^{m+1}(x)\beta^n(x) + \alpha^n(x)\beta^{m+1}(x) + \beta^{m+n+1}(x)\right)\\ &= \alpha^m(x)\beta^{n+1}(x) + \beta^m(x)\alpha^{n+1}(x) - \alpha^{m+1}(x)\beta^n(x) - \alpha^n(x)\beta^{m+1}(x)\\ &= \alpha^m(x)\beta^n(x)\beta(x) + \beta^m(x)\alpha^n(x)\alpha(x) - \alpha^m(x)\alpha(x)\beta^n(x) - \alpha^n(x)\beta^m(x)\beta(x)\\ &= \alpha^m(x)\beta^n(x)\,(\beta(x)-\alpha(x)) + \alpha^n(x)\beta^m(x)\,(\alpha(x)-\beta(x))\\ &= (\alpha(x)-\beta(x))\,(\alpha^n(x)\beta^m(x) - \alpha^m(x)\beta^n(x))\\ &= -\sqrt{5x}\,(\alpha^n(x)\beta^m(x) - \alpha^m(x)\beta^n(x))\,.\end{aligned}$$

We know that:

$$\alpha^n(x)\,\beta^m(x) = (\alpha\beta)^n(x)\,\beta^{m-n}(x)$$

$$\alpha^m(x)\,\beta^n(x) = (\alpha\beta)^n(x)\,\alpha^{m-n}(x)$$

.

Then

$$\begin{aligned}
&\mathbb{L}_m(x)\,\mathbb{L}_{n+1}(x) - \mathbb{L}_{m+1}(x)\,\mathbb{L}_n(x)\\
&= -\sqrt{5x}\left((\alpha\beta)^n(x)\,\beta^{m-n}(x) - (\alpha\beta)^n(x)\,\alpha^{m-n}(x)\right)\\
&= -\sqrt{5x}\,(\alpha\beta)^n(x)\left(\beta^{m-n}(x) - \alpha^{m-n}(x)\right)\\
&= -\sqrt{5x}\left(-x^2-x+1\right)^n\left(\beta^{m-n}(x) - \alpha^{m-n}(x)\right)\\
&= \frac{\sqrt{5}}{\sqrt{5}}\left(-\sqrt{5x}\left(-x^2-x+1\right)^n\left(\beta^{m-n}(x) - \alpha^{m-n}(x)\right)\right)\\
&= -5x\left(-x^2-x+1\right)^n\frac{\left(\beta^{m-n}(x) - \alpha^{m-n}(x)\right)}{\sqrt{5}}\\
&= -5x\left(-x^2-x+1\right)^n\,\mathbb{F}_{m-n}(x).
\end{aligned}$$

Which completes the proof.

2.7 The Associated Matrix and their Properties

In the classical case, the Q-matrix associated with the Fibonacci numbers is defined by:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

This matrix is a very important tool for providing and constructing identities. Now, we introduce the matrix $G(x)$ that plays the role of the Q-matrix, which is defined as:

$$G(x) = \begin{pmatrix} 2-x & x^2+x-1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 2.13 *Let $n \geq 1$. Then:*

$$G^n(x) = \frac{1}{5x^2}\begin{pmatrix} (2-x)\,\mathbb{L}_{n+1}(x) + \varphi(x)\,\mathbb{L}_n(x) & \varphi(x)\,\mathbb{L}_{n+1}(x) + (x)\,\mathbb{L}_n(x) \\ (2-x)\,\mathbb{L}_n(x) + \varphi(x)\,\mathbb{L}_{n-1}(x) & \varphi(x)\,\mathbb{L}_n(x) + (x)\,\mathbb{L}_{n-1}(x) \end{pmatrix}. \tag{2.31}$$

where

$$\varphi(x) = 2x^2 + 2x - 2 \qquad \textit{and} \qquad (x) = x^3 - x^2 - 3x + 2.$$

Proof. We can prove the Theorem by induction on n.

For $n = 1$:
We have $G^1(x) = G(x)$:

$$G^1(x) = \begin{pmatrix} 2-x & x^2+x-1 \\ 1 & 0 \end{pmatrix}.$$

. We know that:

$$\mathbb{L}_0(x) = 2$$
$$\mathbb{L}_1(x) = 2 - x$$
$$\mathbb{L}_2(x) = 3x^2 - 2x + 2.$$

We compute each entry:

$$a_{11} = (2-x)\,\mathbb{L}_2(x) + \varphi(x)\,\mathbb{L}_1(x) = 5x^2(2-x)$$
$$a_{12} = \varphi(x)\,\mathbb{L}_2(x) + (x)\,\mathbb{L}_1(x) = 5x^2(x^2+x-1)$$
$$a_{21} = (2-x)\,\mathbb{L}_1(x) + \varphi(x)\,\mathbb{L}_0(x) = 5x^2$$
$$a_{22} = \varphi(x)\,\mathbb{L}_1(x) + (x)\,\mathbb{L}_0(x) = (3x^2+2x-2)(2-x) + (x^3 - x^2 - 3x + 2)\cdot 2 = 0.$$

Therefore:

$$\frac{1}{5x^2}\begin{pmatrix} 5x^2(2-x) & 5x^2(x^2+x-1) \\ 5x^2 & 0 \end{pmatrix} = \begin{pmatrix} 2-x & x^2+x-1 \\ 1 & 0 \end{pmatrix}$$
$$= G(x).$$

Hence the formula holds for $n = 1$.

Assume that the formula is true for $n \geq 1$

$$G^n(x) = \frac{1}{5x^2}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

where:

$$a_{11} = (2-x)\,\mathbb{L}_{n+1}(x) + \varphi(x)\,\mathbb{L}_n(x)$$
$$a_{12} = \varphi(x)\,\mathbb{L}_{n+1}(x) + (x)\,\mathbb{L}_n(x)$$
$$a_{21} = (2-x)\,\mathbb{L}_n(x) + \varphi(x)\,\mathbb{L}_{n-1}(x)$$
$$a_{22} = \varphi(x)\,\mathbb{L}_n(x) + (x)\,\mathbb{L}_{n-1}(x).$$

We compute $G^{n+1}(x) = G^n(x) \cdot G(x)$:

$$G^{n+1}(x) = \frac{1}{5x^2}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 2-x & x^2+x-1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{5x^2}\begin{pmatrix} a_{11}(2-x) + a_{12} & a_{11}(x^2+x-1) \\ a_{21}(2-x) + a_{22} & a_{21}(x^2+x-1) \end{pmatrix}.$$

Computing the entry a_{11}:

$$\begin{aligned} a_{11}(2-x)+a_{12} &= \mathbb{L}_{n+1}(x)\left[(2-x)^2+\varphi(x)\right]+\mathbb{L}_n(x)\left[(2-x)\varphi(x)+(x)\right] \\ &= (2-x)\,\mathbb{L}_{n+2}(x)+\varphi(x)\,\mathbb{L}_{n+1}(x). \end{aligned}$$

And similarly for the other entries, we conclude:

$$G^{n+1}(x)=\frac{1}{5x^2}\begin{pmatrix} (2-x)\,\mathbb{L}_{n+2}(x)+\varphi(x)\,\mathbb{L}_{n+1}(x) & \varphi(x)\,\mathbb{L}_{n+2}(x)+(x)\,\mathbb{L}_{n+1}(x) \\ (2-x)\,\mathbb{L}_{n+1}(x)+\varphi(x)\,\mathbb{L}_n(x) & \varphi(x)\,\mathbb{L}_{n+1}(x)+(x)\,\mathbb{L}_n(x) \end{pmatrix},$$

■

As a consequence of Theorem 2.13, the Lucas-type associated polynomials can be expressed in matrix form as:

$$\begin{pmatrix} \mathbb{L}_{n+1}(x) \\ \mathbb{L}_n(x) \end{pmatrix} = G^n(x)\begin{pmatrix} 2 \\ 2-x \end{pmatrix}$$

and:

$$\mathbb{L}_n(x)=\begin{pmatrix} 1 & 0 \end{pmatrix} G^{n-1}(x)\begin{pmatrix} 2 \\ 2-x \end{pmatrix} \quad n\geq 1.$$

Theorem 2.14 *The roots of the characteristic equation of $G^n(x)$ are $\alpha^n(x)$ and $\beta^n(x)$.*

Proof. The characteristic equation of $G^n(m)$ is:

$$\det(G^n(x)-\lambda I)=\det\begin{pmatrix} \frac{1}{x}F_{n+1}(x)-\lambda & \frac{1}{x}(x^2+x-1)F_n(x) \\ \frac{1}{x}F_n(x) & \frac{1}{x}(x^2+x-1)F_{n-1}(x)-\lambda \end{pmatrix}$$

then

$$\begin{aligned} \det(G^n(x)-\lambda I) &= \left(\frac{1}{x}F_{n+1}(x)-\lambda\right)\left(\frac{1}{x}(x^2+x-1)F_{n-1}(x)-\lambda\right) \\ &\quad -\left(\frac{1}{x}F_n(x)\right)\left(\frac{1}{x}(x^2+x-1)F_n(x)\right) \\ &= \frac{1}{x^2}(x^2+x-1)F_{n+1}(x)F_{n-2}(x) \\ &\quad -\frac{\lambda}{x}F_{n+1}(x)-\frac{\lambda}{x}(x^2+x-1)F_{n-1}(x)+\lambda^2 \\ &\quad -\frac{1}{x^2}F_n^2(x)(x^2+-1) \\ &= \lambda^2-\lambda\left(\frac{1}{x}F_{n+1}(x)+\frac{1}{x}(x^2+x-1)F_{n-1}(x)\right) \\ &\quad +\frac{1}{x^2}(x^2+x-1)\left(F_{n+1}(x)F_{n-1}(x)-F_n^2(x)\right). \end{aligned}$$

From Theorem 2.8 and Cassini's type identity of $F_n(x)$, we obtain:

$$\begin{aligned}\det(G^n(m) - \lambda I) &= \lambda^2 - \lambda\left(\frac{1}{x}(xL_n(x))\right) + \frac{1}{x^2}(x^2+x-1)(-1)^n(x^2+x-1)^{n-1}x^2 \\ &= \lambda^2 - \lambda L_n(x) + (-1)^n(x^2+x-1)^n.\end{aligned}$$

So Characteristic equation of $G^n(m)$ is:

$$\lambda^2 - \lambda L_n(x) + (-1)^n(x^2+x-1)^n = 0$$

and the roots of the characteristic equation are:

$$\lambda = L_n(x) \pm \frac{\sqrt{L_n^2(x) - 4(-1)^n(x^2+x-1)}}{2}.$$

From Corollary 2.6.3:

$$L_n^2(x) - 4(-1)^n(x^2+x-1)^n = 5F_n^2(x).$$

So:

$$\begin{aligned}\lambda &= \frac{L_n(x) \pm \sqrt{5F_n^2(x)}}{2} \\ &= \frac{L_n(x) \pm \sqrt{5}F_n(x)}{2}.\end{aligned}$$

From Corollary 2.6.2:

$$\alpha^n(x) = \frac{L_n(x) + \sqrt{5}F_n(x)}{2}$$

and

$$B^n(x) = \frac{L_n(x) - \sqrt{5}F_n(x)}{2}.$$

So:

$$\lambda = \alpha^n(m) \qquad and \qquad \lambda = B^n(m)$$

are the roots of characteristic equation,

wich complete the proof.

Theorem 2.15 *Let $R_n(x)$ denote the $n \times n$ tridiagonal matrix defined as:*

$$R_n(x) = \begin{pmatrix} 2 & x^2+x-1 & & & \\ 2-x & 2 & x^2+x-1 & & \\ & -1 & 2-x & \ddots & \\ & & \ddots & \ddots & x^2+x-1 \\ & & & -1 & 2-x \end{pmatrix}.$$

Then

$$\det R_n(x) = \mathbb{L}_{n-1}(x), \quad n \geq 1.$$

3

The Application of Laguerre Transform

Laguerre polynomials and the Laguerre transform have many important applications in mathematics, physics, and engineering. They are used in quantum mechanics, especially in the study of the hydrogen atom, and in numerical analysis for the approximation of improper integrals using Gauss–Laguerre quadrature. They are also used in several engineering fields such as signal processing, control theory, and differential equations.

In this chapter, we study the main applications of the Laguerre transform and present some of its important uses in different scientific and engineering fields.

3.1 Orthogonal Polynomials

Given two real polynomials in x, $P(x)$ and $Q(x)$, we define:

$$\langle P \mid Q \rangle = \int_a^b P(x)Q(x)\rho(x)\,dx,$$

where $\rho(x)$ is a defined and integrable function on (a, b), positive or zero, and piecewise continuous.

The integral $\langle P \mid Q \rangle$ is an inner product since

$$\langle P \mid Q \rangle = \langle Q \mid P \rangle$$

$$\langle P \mid P \rangle = \int_a^b P^2(x)\rho(x)\,dx \begin{cases} > 0, & \text{if } P(x) \neq 0 \\ = 0, & \text{if } P(x) = 0. \end{cases}$$

Polynomials P and Q are said to be orthogonal if:

$$\langle P \mid Q \rangle = 0.$$

The family of polynomials $\{P_n(x)\}$, where $P_n(x)$ is a polynomial of degree n, is called a family or sequence of orthogonal polynomials with respect to the weight $\rho(x)$ on the interval (a, b) if:

$$\langle P_n \mid P_m \rangle = 0, \qquad \text{for } n \neq m.$$

The sequence $\{P_n\}$ is said to be orthonormal if:

$$\langle P_n \mid P_m \rangle = \delta_{mn},$$

where δ_{mn} is the Kronecker delta symbol, defined by:

$$\delta_{mn} = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

3.2 Application

Laguerre polynomials appear in many areas of mathematics and physics. They are widely used in quantum mechanics, especially in solving Schrödinger's equation for the hydrogen atom. They also have important applications in electrostatics and in numerical analysis, particularly in the computation of improper integrals using the Gauss–Laguerre quadrature method [10, 17, 20].

3.2.1 Gauss-Laguerre Quadrature

For integrals of the form:

$$\int_0^\infty e^{-x} f(x)\, dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

where x_i are the roots of $L_n(x)$, and w_i are the corresponding weights.

3.2.2 Radial Wave Function in the Hydrogen Atom

The radial solution of Schrödinger's equation for the hydrogen atom is expressed as:

$$R_{n,l}(r) = N_{n,l}\, r^l\, e^{-\frac{r}{na_0}}\, L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right),$$

where

- $L_n^l(x)$ is the associated Laguerre polynomial.
- a_0 is the Bohr radius (the natural unit of length for the hydrogen atom).
- $N_{n,l}$ is the normalization constant ensuring the wavefunction is normalized.
- $R_{n,l}(r)$ is the radial wavefunction describing the probability amplitude of finding the electron at distance r.

3.2.3 Orthogonality of Laguerre Polynomials

Theorem 3.1

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\, dx = \delta_{nm}.$$

3.2.4 Orthogonal Polynomials, Applications to Electrostatics

Applications in electrostatics correspond to a relationship of the definitions of Stieltjes, Jacobi, Laguerre and Hermite, starting from the latter the definition of unbounded intervals.

Jan Stieltjes, at the end of the 19th century, found a definition of the zeros of classical orthogonal polynomials in terms of an electrostatic model. Thus, to briefly describe the problem, it will be necessary to resort to concepts related to electrostatics. For this, one can see the generalized relationship presented by Coulomb, which can be studied in inverse square functions and, consequently, in conservative fields. And, without losing the aforementioned generality, this electrostatic force can be studied in the complex plane, but the force must be given in modulus.

With these elementary principles from the domain of physics already established, the Stieltjes problem is presented. One of the most significant applications can be found in the problems of *electrostatic equilibrium with logarithmic potential*. Thus:

If the interaction between charges is defined by a logarithmic potential, the total energy of the system obeying electrostatic equilibrium is minimal at all points when they coincide with the zeros of the **Jacobi orthogonal polynomials** (charges of -1 and 1, respectively at the endpoints).

The following illustration of charges distributed in the interval $(-1, 1)$ describes the situation.

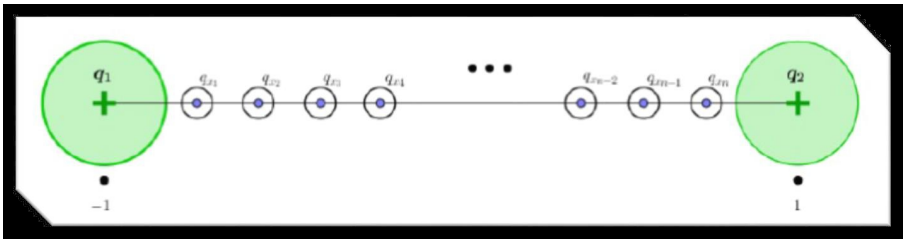


Figure 3.1: Electrostatic charge distribution

In accordance with the foregoing, Stieltjes considered interpretations quite similar to those of Laguerre, Jacobi and Hermite for the zeros of the polynomials described by them. Given that in this situation the free charges can move within an unbounded set, what prevents them from escaping to infinity? Stieltjes posed the following (for the case relevant to the present report, in relation exclusively to Laguerre.

$$\sum_{\substack{j=0 \\ j\neq i}} \frac{1}{x_i - x_j} < \frac{1}{2x_i}, \quad i = 1, 2, \ldots, n$$

Thus, the equilibrium position of certain charges (which is represented by the summation) is achieved only if certain solutions x coincide with the zeros of certain Laguerre orthogonal

polynomials. An example of the foregoing on $(0, +\infty)$ — which is where it applies to the problem — is:

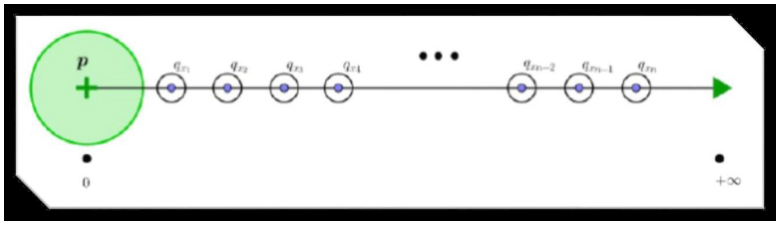


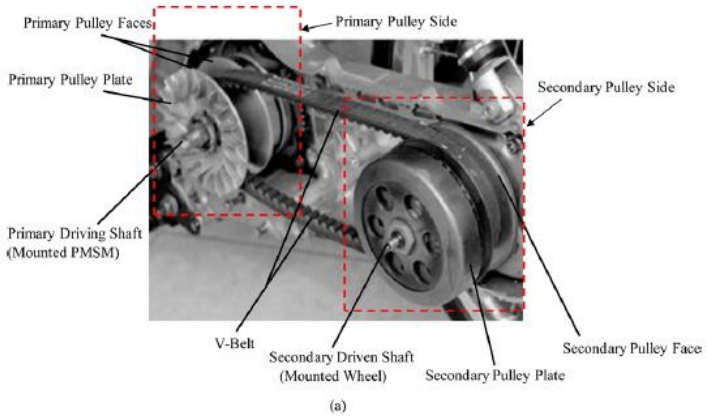
Figure 3.2: Electrostatic charge distribution

3.2.5 Application of V-Belt: continuously variable transmission system utilizing recurrent hybrid Laguerre orthogonal polynomials in neural network control systems and modified particle swarm optimization

A V-Belt CVT generally consists of a chain with a V-belt and two springs that actuate pulleys with variable radii. To launch a vehicle from rest, the radius of the input pulley will be smaller than the radius of the output pulley, resulting in a speed reduction and a multiplication of the torque transmitted to the transmission shaft. To increase the speed of the output shaft, the pulley radii are inversely manipulated simultaneously in order to maintain the constant length of the belt.

Given that the electric scooter system involves numerous uncertainties, unknown nonlinearities, and parameter variations such as load torque, rolling resistance, wind resistance, and braking force the V-Belt CVT and the clutch in the scooter model can be categorized as operating in one of two operating modes depending on the rotational speed of the output shaft of the V-belt CVT: *disconnected* or *fully coupled*. At the beginning of the PMSM management cycles, the scooter is idle. The clutch is initially disconnected, and the subsequent transition between modes is governed by the rotational speed of the clutch shaft. Except for mechanical losses, power is transmitted from the PMSM through the V-belt CVT and the clutch to the wheel of the electric scooter.

The following are some illustrations of the previously described system:



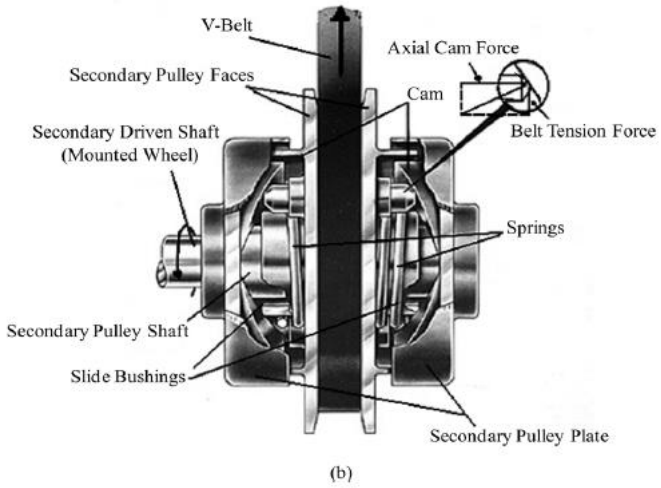


Fig. 1 Structure of the V-belt CVT system: (*a*) appearance of the primary pulley side and the secondary pulley side and (*b*) cross-sectional view of the secondary shaft

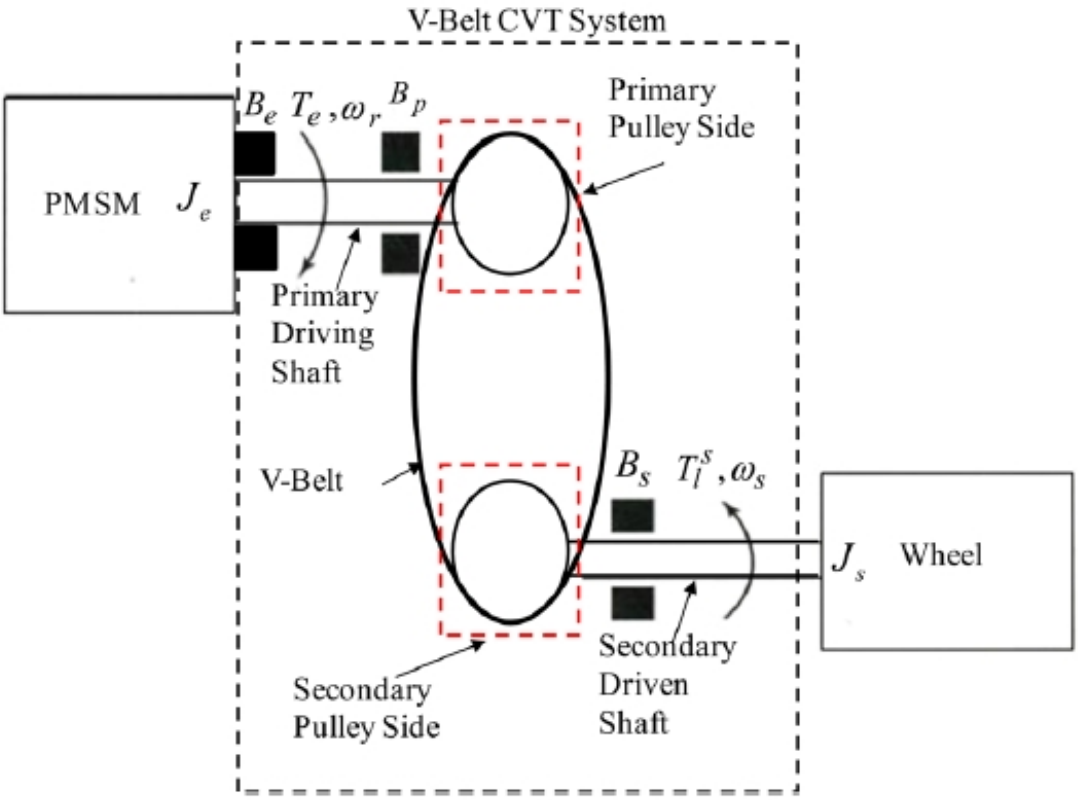


Fig. 2 Schematic diagram of the PMSM-wheel connection through the V-belt CVT system

Conclusion

Conclusion

In this work, we studied Laguerre polynomials and the generalization of the Laguerre transform through associated polynomials. First, we presented some basic notions of enumerative combinatorics, such as generating functions, recurrence relations, and Fibonacci and Lucas numbers and polynomials, which were important for this study.

In the second chapter, we introduced the associated polynomials defined from an initial sequence and established several important algebraic and analytical properties. We also obtained a generalization of the Laguerre transform, as well as different representations using generating functions and explicit formulas. Then, we applied this theory to the associated Fibonacci and Lucas polynomials. Several important properties were studied, such as recurrence relations, generating functions, Binet-type formulas, and some remarkable identities. We also studied more advanced results, including the Rodrigues formula and the asymptotic behavior of the associated Lucas polynomials.

Finally, in the last chapter, we presented some applications of the Laguerre transform in different mathematical and scientific fields. These applications show the importance of this transform and its usefulness in the study of special sequences, orthogonal polynomials, and analytical methods.

This work helped us better understand the structure of associated polynomials and the connection between the Laguerre transform, orthogonal polynomials, and the classical Fibonacci and Lucas sequences. This study may also open the way for future research on other families of special polynomials and new applications of the Laguerre transform.

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