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By

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The dam problem with leaky boundary condition

In front of the jury composed of :

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Dedication

I dedicate this work as a sign of respect, appreciation and thanks to my mather ***ZIANE FATMA*** *and my father* ***ALI****, for their great support and who love always believed in me and provided me with foundation to pursue my dreams. I want to express my deep gratitude and appreciation to my dear mother, who stood by my side with unwavering strength and patience despite her illness I am profoundly grateful for the unwavering loyalty and support of my father. His steadfast presence in my life has been a source of strength and inspiration. Through every challenge and triumph, he has stood by me with unwavering dedication, guiding me with wisdom and love .*

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Abstract

In this work, we will study the phenomenon of flow in porous media, specifically the problem of dams. After presenting the mathematical model of the problem, we will, under certain conditions, prove the existence and uniqueness of the solution to the problem using the Schauder theory. Finally, we will conclude by providing some properties of the solution and special examples .

Key Words : Porous media, Existence, Uniqueness, Schauder theory .

Résumé

Dans ce travail, nous étudierons le phénomène d'écoulement dans les milieux poreux, en particulier le problème des barrages. Après avoir présenté le modèle mathématique de phénomène, nous prouverons, sous certaines conditions, l'existence et l'unicité de la solution au problème en utilisant la théorie de Schauder. Enfin, nous conclurons en fournissant certaines propriétés de la solution et des exemples particuliers .

Mots clés : Milieu poreux, Existence, Unicité, Théorème de Schauder .

Notation

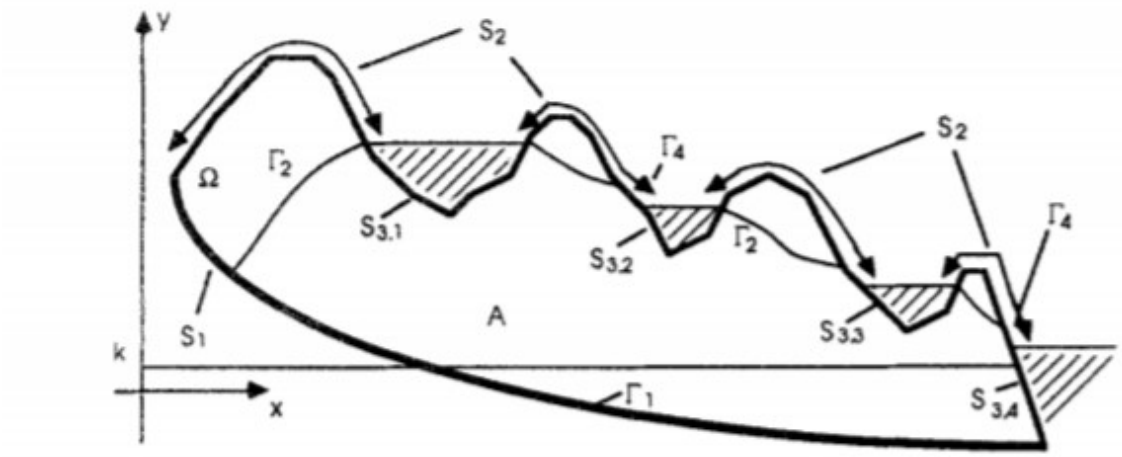
Ω	Open bounded set of $\mathbb{R}^N$.
$x=(x_1,x_2)$	A typical point in $\mathbb{R}^2$ and $\|x\|=\sqrt{x_1^2+x_2^2}$.
$x(S)$	the characteristic function of S .
$\|S\|$	The Lebesgue measure of S in $\mathbb{R}^2$.
$C^{0,\alpha}_{loc}(\Omega)$	The set of all functions $u:\Omega\to\mathbb{R}^2$ that are locally Hölder continuous in Ω .
$C^{1,\alpha}_{loc}(\Omega\cup S)$	The set of all functions $u:\Omega\to\mathbb{R}^2$ whose first partial derivatives are in $C^{0,\alpha}_{loc}(\Omega\cup S)$, where $S\subset\partial\Omega$.
$C^{0,\alpha}_{loc}(\Omega\cup S)$	The set of all functions $u:\Omega\to\mathbb{R}^2$ that possess an extension to $C^{0,\alpha}_{loc}(\Omega')$ for some open set Ω' which contains $\Omega\cup S$, where $S\subset\partial\Omega$.
$C^1(\Omega)$	The set all continuously differentiable functions $u:\Omega\to\mathbb{R}$.
$C^1(\bar{\Omega})$	The set of all functions $u:\bar{\Omega}\to\mathbb{R}$ that possess an extension to $C^1(\mathbb{R}^2)$.
$\mathscr{D}(\Omega)$	The set of all indefinitely continuously differentiable functions $u:\Omega\longrightarrow\mathbb{R}$ with compact support .
$L^p(\Omega)$	The set of all Lebesgue measurable functions $u:\Omega\longrightarrow\mathbb{R}$ such that $\|u\|_{p,\Omega}<\infty$, where $\|u\|_{p,\Omega}=(\int_\Omega \|u(x)\|^p\,dx)^{\frac{1}{p}}$ if $1\le p<\infty$ and $\|u\|_{\infty,\Omega}=\text{ess sup}_\Omega\|u\|$.
$L^{p'}(\Omega)$	The dual space of $L^p(\Omega)$,where $\frac{1}{p}+\frac{1}{p'}=1$.
$W^{1,p}(\Omega)$	(Sobolev) space of functions with distributional gradient in $(L^p(\Omega))^N$.
$W^{1,p}_0(\Omega)$	Closure of $C^1(\Omega)$ functions with compact support in Ω with respect to the norm of $W^{1,p}(\Omega)$.
$H^1(\Omega)$	$W^{1.2}(\Omega)$.
ξ	Test function .
γ_0	The trace on S_3 .
K	$=\{v\in L^2(S_3)\|\|v-\varphi\|_2\le R\}$.
$[f<g]$	$=\{(x,y)\in\Omega\|f(x,y)<g(x,y)\}$.
$[f\le g]$	$=\{(x,y)\in\Omega\|f(x,y)\le g(x,y)\}$.
$[p>0]$	$=\{(x,y)\in\Omega\|p(x,y)>0\}$.

$W^{-1,s}(\Omega)$	The dual space of $W_0^{1,s}(\Omega)$.
$W_0^{1,s}(\Omega)$	Closure of $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,s}(\Omega)$.
$(u)^+$	Denotes the positive part of the functions u.
$f_n \rightharpoonup f$	f_n converges weakly to f.

General Introduction

The study of flow in porous media is a fascinating area of mathematics that has significant applications in various fields,

including hydrology, petroleum engineering, and environmental science. This research focuses on understanding how fluids move through materials with a network of interconnected pores, such as soil, rock, or biological tissues. The behavior of fluid flow in these complex structures is governed by a set of partial differential equations, notably the Darcy's Law and the Navier- Stokes equations for more detailed models. Mathematical modeling of porous media flow involves the use of techniques from calculus, linear algebra, and numerical methods to solve these equations and predict fluid behavior under different conditions. This knowledge is crucial for designing efficient extraction processes in oil and gas industries, managing water resources, and mitigating environmental impacts such as groundwater contamination. Advances in this field also contribute to innovations in medical diagnostics and treatments, where understanding fluid flow in biological tissues is essential. The interdisciplinary nature of this study makes it a dynamic and continually evolving area of research, integrating insights from mathematics, physics, engineering, and computational science.



Our work is mainly based on the article by the esteemed authors **J. Carrillo and M. Chipot** in their work - **The Dam Problem with Leaky Boundary Conditions** [24] -. We

simplify the mathematical concepts and transitions to make them easily understandable for students and those interested in mathematics in general. This work is divided into four Chapter. In the first Chapter, we present the main theories and spaces that we will need in our work. Then, in the second Chapter, we present the physical phenomenon with its mathematical model and the appropriate initial conditions to obtain the weak formulation of our problem. The third Chapter was to proving the existence of a solution to the weak formulation using Schauder's theorem, by adding conditions to the boundary functions. We conclude this section by verifying the uniqueness of the solution to problem. Finally, in the fourth Chapter, we study some properties that the solution of the problem satisfies, and we present different examples under special conditions to conclude our work.

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Chapter 1

Preliminary

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The aim of this chapter is to recapitulate the fundamental concepts and findings employed throughout this study.

1.1 Reminders on some functional spaces and theoremes

1.1.1 Lebesgue Spaces

Definition 1.1

We define the Lebesgue space $L^p(\Omega)$ by:

1. let $1 \leq p < +\infty$; we set

$$L^p(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \quad \text{mesurable}, \quad \int_\Omega |u(x)|^p \, dx < +\infty \right\},$$

with

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

We shall check later on that $\| \ \|_p$ is a norm.

2. let $p = +\infty$

$$L^\infty(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \quad \text{mesurable}, \quad \sup_{x \in \Omega} ess|u(x)| < +\infty \right\},$$

with

$$\|u\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} ess|u(x)|.$$

Proposition 1.1: (Reflexivity. Separability. Dual of L^p)

We shall consider separately the following three cases:

1. For $1 < p < \infty$. L^p is reflexive, separable, and the dual of L^p is $L^{p'}$.
2. $L^1(\Omega)$ is separable, never reflexive and the dual space of $L^1(\Omega)$ is $L^\infty(\Omega)$.
3. $L^\infty(\Omega)$ is not sparable, not reflexive and the dual of $L^\infty(\Omega)$ is Strictly bigger than $L^1(\Omega)$.

Theorem 1.1. **(Hölder's inequality)**
Assume that $f \in L^P$ and $g \in L^q$with $1 \leq p \leq \infty$. Then $fg \in L^1$ whe have

$$\int_\Omega |f(x)g(x)dx| \leq \|f(x)\|_p \|g(x)\|_q.$$

For $p = q = 2$, it is called the **Cauchy-Schwarz Inequality**:

$$\int_\Omega |fg| \leq |f|_2 |g|_2.$$

1.1.2 Hilbert space

A Hilbert space is a vector space H equipped with a scalar product such that H is complete for the norm $|\ |$.

In all follows, H will always denote a Hilbert space

Exemple 1.1: Basic example
$L^2(\Omega)$ equipped with the scalar product

$$(u, v) = \int_\Omega u(x)v(x)d\mu.$$

is a Hilbert space. In particular , L^2 is a Hilbert space

Definition 1.2: (Carathéodory Function)

A function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is called a Carathéodory function if it satisfies the following conditions:

1. Measurability in the First Argument: For every fixed $y \in \mathbb{R}$, the function $x \mapsto f(x, y)$ is measurable with respect to x.

2. Continuity in the Second Argument : For almost every fixed $x \in \mathbb{R}^n$, the function

$y \mapsto f(x,y)$ is continuous with respect to y.

1.1.3 Distribution

Distributions make it possible to differentiate functions whose derivatives do not exist in the classical sense. In particular, any locally integrable function has a distributional derivative.

Definition 1.3: L'espace $\mathscr{D}(\Omega)$

Let Ω be an open set in $\mathbb{R}^n$ and f be a function defined on Ω with real values. The support of a function $f(x)$, denoted supp(f), is the smallest closed set of values of $x \in \Omega$ outside of which the function $f(x)$ is identically zero. It is therefore the closure of the set of points $x \in \Omega$ such that $f(x) \neq 0$, that is

$$\text{supp}(f) = \overline{\{x \in \Omega \quad \text{such that} \quad f(x) \neq 0\}}.$$

Definition 1.4

Let Ω be an open set in $\mathbb{R}^n$. All functions defined on Ω that we consider are real-valued. We denote by $D(\Omega)$ the vector space of infinitely differentiable functions on Ω with compact support in Ω.

1.1.4 Sobolev Space

Definition 1.5: $H^1(\Omega)$ Space

Let $\Omega \in \mathbb{R}$ be an open set. The Sobolev space $H^1(\Omega)$ is the space of functions in $L^2(\Omega)$ whose partial derivatives are also in $L^2(\Omega)$

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) \quad \text{such that}: \forall i \in \{1,...,n\}, \frac{\partial u}{\partial x_i} \in L^2(\Omega) \right\}.$$

In other words, a function $u \in L^2(\Omega)$ is in $H^1(\Omega)$ if there exist $v_1, \ldots, v_n \in L^2(\Omega)$ such that :

$$\forall v \in L^2(\Omega) : \int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega v_i \varphi \, dx, \quad \forall \varphi \in \mathscr{D}(\Omega), i \in \{1,2,\ldots,N\}.$$

And according to the theory of distributions, the v_i are denoted by $\frac{\partial u}{\partial x_i}$.

Definition 1.6: The Space $H_0^1(\Omega)$

Let Ω be an open subset of $\mathbb{R}^N$. The Sobolev space $H_0^1(\Omega)$ is defined as the closure of $\mathscr{D}(\Omega)$ in $H^1(\Omega)$. That is,

$$H_0^1(\Omega) = \overline{\mathscr{D}(\Omega)}^{H^1(\Omega)}$$

We denote

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\}$$

That is, the functions in $H_0^1(\Omega)$ are the limits in $H^1(\Omega)$ of sequences of functions from $\mathscr{D}(\Omega)$.

Definition 1.7

Let X be a Banach space and $T : X \to X$ an operator. A fixed point of T is any point $x \in X$ such that $T(x) = x$.

Theorem 1.2. **(of trace)** So let there be a bounded open set regularly of class C^1, or else $\Omega = \mathbb{R}^N_+$. We define the trace application γ_0.

$$\begin{aligned} H^1(\Omega) \cap C(\overline{\Omega}) &\longrightarrow L^2(\partial\Omega) \cap C(\overline{\partial\Omega}) \\ &\longrightarrow \gamma_0(u) = u|_{\partial\Omega} \end{aligned}$$

This application γ_0 is extended by continuity to a continuous linear application from $H^1(\Omega)$ to $L^2(\Omega)$, still denoted by γ_0. In particular, there exists a constant $C > 0$ such that, for any function u in $H^1(\Omega)$, we have

$$\|u\|_{L^2(\partial\Omega)} \leq C\|u\|_{H^1(\Omega)}.$$

1.1.5 Green's formula

Theorem 1.3. **(Green's formula)** If Ω is a bounded open set of class C^1, and if u and v are functions in $H^1(\Omega)$, they satisfy

$$\int_\Omega \frac{\partial u}{\partial x_i(x)} v(x) dx = -\int_\Omega u(x) \frac{\partial v}{\partial x_i}(x) dx + \int_{\partial\Omega} u(x) v(x) n_i d\sigma.$$

where $n = (n_i)_{1 \leq i \leq n}$ is the unit outward normal to $\partial\Omega$, and $d\sigma$ is the surface measure of the boundary.

Corollary 1.1. Let Ω be a bounded open set regularly of class C^1. The space $H_0^1(\Omega)$ coincides with the subspace of $H^1(\Omega)$ consisting of functions that vanish on the boundary $\partial\Omega$.

1.1.6 Mesurable function

Definition 1.8: Mesurable function

Let (X, A) be a measurable space. We say that $f : X \to \mathbb{R}$ is measurable if, for every open set Ω in $\mathbb{R}$, $f^{-1}(\Omega) \in A$.

Theorem 1.4. **Dominated Convergence Theorem of Lebesgue**
Let (f_n) be a sequence of functions in $L^p(\Omega)$, where $1 \leq p \leq +\infty$. We assume that

$$f_n \xrightarrow{a.e.} f \quad \text{on } \Omega.$$

There exists a non-negative function $g \in L^p(\Omega)$ such that for all n, $|f_n(x)| \leq g(x)$ almost everywhere on Ω.
Then $f \in L^p(\Omega)$ and

$$\|f_n - f\|_{L^p(\Omega)} \longrightarrow 0.$$

Theorem 1.5. **(Density Theorem)**
If the open set Ω is Lipschitz, the space $D(\overline{\Omega})$ is dense in $H^1(\Omega)$, i.e., for any element u in $H^1(\Omega)$, there exists a sequence (u_n) in $D(\overline{\Omega})$ such that $\|u_n - u\|_{H^1(\Omega)}$ converges to 0.

Proposition 1.2

The space $W_0^{1,p}$ is a separable Banach space. Moreover, it is reflexive for $1 < p < \infty$.
The space H_0^1 is a separable Hilbert space.

1.1.7 Monotone operators

Definition 1.9

Let V be a reflexive Banach space, and $A : V \to V'$ be an operator.
It is said that

1) A is monotone if and only if

$$\langle Au - Av, u - v \rangle \geq 0, \qquad \forall u, v \in V$$

.
2) A is strictly monotone if and only if

$$\langle Au - Av, u - v \rangle > 0, \qquad \forall u, v \in V,\ u \neq v.$$

1.1.8 Technical lemmas

Lemma 1.1. **(Young's inequality)** Let p and q be two strictly positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. For any positive real numbers a and b, we have

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

1.1.9 Convergence in $L^p(\Omega)$

Definition 1.10: (Strong convergence)

Let $1 \leq p < \infty$, $f \in L^p(\Omega)$, and $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^p(\Omega)$. We say that f_n converges strongly to f in $L^p(\Omega)$ if:

$$\lim_{n\to\infty} \|f_n - f\|_p = 0,$$

and we write $f_n \to f$ in $L^p(\Omega)$.

Definition 1.11: (Weak convergence)

Let $1 \leq p < \infty$, $f \in L^p(\Omega)$, and $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(\Omega)$. We say that f_n converges weakly to f in $L^p(\Omega)$ if:

$$\forall g \in L^{p'}(\Omega), \langle g, f_n\rangle \to \langle g, f\rangle,$$

and we write $f_n \rightharpoonup f$ in $L^p(\Omega)$.

1.1.10 Maximum principle

Theorem 1.6.

- Let f be a holomorphic function in a domain U. If $|f|$ has a local maximum at a point a in U, then f is constant in U.

- Let U be a bounded open set in $\mathbb{C}$ and f be a holomorphic function in U and continuous on $\overline{U}$. Then, for every $z \in U$, we have

$$|f(z)| \leq \sup\{|f(w)| : w \in \partial U\}.$$

Theorem 1.7. **(Poincaré inequality)** Let Ω be a bounded open set in $\mathbb{R}^n$, and $1 \leq p < \infty$. Then there exists a positive constant $C_p(\Omega)$ such that for all u in $W_0^{1,p}(\Omega)$,

$$\|u\|_{L^p(\Omega)} \leq C_p(\Omega)\|\nabla u\|_{L^p(\Omega)^n}.$$

Theorem 1.8. **(Rellich Theorem)** If Ω is a bounded open set of class of regularity C^1, then from any bounded sequence in $H^1(\Omega)$, we can extract a convergent subsequence in $L^2(\Omega)$ (we say that the canonical injection from $H^1(\Omega)$ into $L^2(\Omega)$ is compact).

1.2 Fixed point theorem

Definition 1.12: (Linear form)

Let H be a Hilbert space and $f : H \to \mathbb{R}$. We say that f is a linear form on H if and only if:

$$\forall u, v \in H, \quad \forall \alpha, \beta \in \mathbb{R}: \quad f(\alpha u + \beta v) = \alpha f(u) + \beta f(v).$$

Definition 1.13: (Coercive form)

Let H be a Hilbert space and a a bilinear form on H. a is coercive on H if:

there exists $\alpha > 0$ such that $a(u,u) > \alpha \|u\|_H^2$ for all $u \in H$.

Definition 1.14: (Bilinear and continuous form)

Let $a : H \times H \to \mathbb{R}$. We say that a is a bilinear form on H if for every fixed $u \in H$, the following mappings:

$$a(u,.) : v \in H \to a(u,v) \in \mathbb{R},$$

$$a(.,u) : v \in H \to a(v,u) \in \mathbb{R},$$

are linear.
Recall that if a is a continuous bilinear form on H, then there exists a constant $C > 0$ such that:

$$|a(u,v)| \leq C \|u\|_H \|v\|_H \quad \forall u, v \in H.$$

1.2.1 Fixed point theorems

In nonlinear analysis, fixed point theorems are used to justify the existence of a solution through a technique of transitioning from the linear case to the nonlinear case, building upon well-understood linear cases. There are several fixed point theorems, and their applications vary depending on the model and assumptions.

Theorem 1.9. **(Schauder's fixed-point theorem)** Let E be a normed vector space and C a closed convex subset of E.
If T is a continuous compact mapping from C to itself, then it has at least one fixed point in C .

Definition 1.15: (Darcy's Law)

To understand the mathematical aspect behind liquid flow in the substance, Darcy's law can be described as:

Darcy's law describes the relationship among the instantaneous rate of discharge through porous medium and pressure drop at a distance.

Using the specific sign convention, Darcy's law is expressed as:

$$Q = -KA\frac{\Delta h}{\Delta l}.$$

Where:

- Q is the rate of water flow,
- K is the hydraulic conductivity,
- A is the column cross-section area, and
- $\frac{\Delta h}{\Delta l}$ indicates the hydraulic gradient.

1.3 Classification of PDE's

We focus on second-order equations in two variables, such as the wave equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = f(x,t), \quad \text{(Hyperbolic)}$$

Laplace or Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x,y), \quad \text{(Elliptic)}$$

or Fourier's heat equation

$$\frac{\partial^2 \phi}{\partial x^2} - \kappa\frac{\partial \phi}{\partial t} = f(x,t). \quad \text{(Parabolic)}$$

What do the names hyperbolic, elliptic and parabolic mean? In high school coordinate geometry, we learned that a real quadratic curve

$$a x^2 + 2b\,x y + c\,y^2 + f\,y + g\,y + h = 0$$

represents a hyperbola, an ellipse, or a parabola depending on whether the discriminant, $ac - b^2$, is less than zero, greater than zero, or equal to zero, these being the conditions for the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

to have signature $(+,-)$, $(+,+)$ or $(+,0)$.

By analogy, the equation

$$a(x,y)\frac{\partial^2 \phi}{\partial x^2} + 2b(x,y)\frac{\partial^2 \phi}{\partial x \partial y} + c(x,y)\frac{\partial^2 \phi}{\partial y^2} + (\text{lower orders}) = 0,$$

is said to be hyperbolic, elliptic, or parabolic at a point (x, y) if

$$\begin{vmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{vmatrix} = (ac - b^2)(x,y)$$

is less than, greater than, or equal to zero, respectively. This classification helps us understand what sort of initial or boundary data we need to specify the problem.

1.3.1 Some boundary conditions:

There are three broad classes of boundary conditions:

a) **Dirichlet boundary conditions**: The value of the dependent variable is specified on the boundary.

b) **Neumann boundary conditions**: The normal derivative of the dependent variable is specified on the boundary.

c) **Cauchy boundary conditions**: Both the value and the normal derivative of the dependent variable are specified on the boundary.

Less commonly met are Robin boundary conditions, where the value of a linear combination of the dependent variable and the normal derivative of the dependent variable is specified on the boundary.
Cauchy boundary conditions are analogous to the initial conditions for a second-order ordinary differential equation. These are given at one end of the interval only. The other two classes of boundary condition are higherdimensional analogues of the conditions we impose on an ODE at both ends of the interval.
Each class of PDE's requires a different class of boundary conditions in order to have a unique, stable solution.

1) **Elliptic** equations require either Dirichlet or Neumann boundary conditions on a closed boundary surrounding the region of interest. Other boundary conditions are either insufficient to determine a unique solution, overly restrictive, or lead to instabilities.

2)**Hyperbolic** equations require Cauchy boundary conditions on a open surface. Other boundary conditions are either too restrictive for a solution to exist, or insufficient to determine a unique solution .

3) **Parabolic** equations require Dirichlet or Neumann boundary conditions on a open surface. Other boundary conditions are too restrictive.

1.3.2 Lipschitz function

Definition 1.16: Lipschitz

Let E be an open subset of $\mathbb{R}^n$. A function $f : E \longrightarrow \mathbb{R}^n$ is Lipschitz if for all $x, y \in E$, there is a K such that

$$\|f(x)-f(y)\| \leq K\|x-y\|.$$

The smallest such constant K is called the *Lipschitz constant* of the function f.

Chapter 2

Formulation of the dam problem and Existence of a solution

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2.1 A unified formulation of the dam problem

2.1.1 Formulation of the problem

2.1.2 Strong Formulation

The boundary of A, denoted by ∂A, is partitioned into four distinct sections: Γ_1 represents an impervious segment, Γ_2 denotes a free boundary, Γ_3 signifies a region submerged beneath the fluid, and finally, Γ_4 designates a seepage front where fluid exits Ω but exerts negligible pressure modification (see Figure 1)."
$\Gamma_1 \subset S_1$ - the impervious part,
$\Gamma_2 \subset \Omega$ - the free boundary,
$\Gamma_3 \subset S_3$- the part covered by fluid,
$\Gamma_4 \subset S_2$ - the part where the fluid flows out of Ω

The velocity v of the fluid in A is given, according to Darcy's law, by

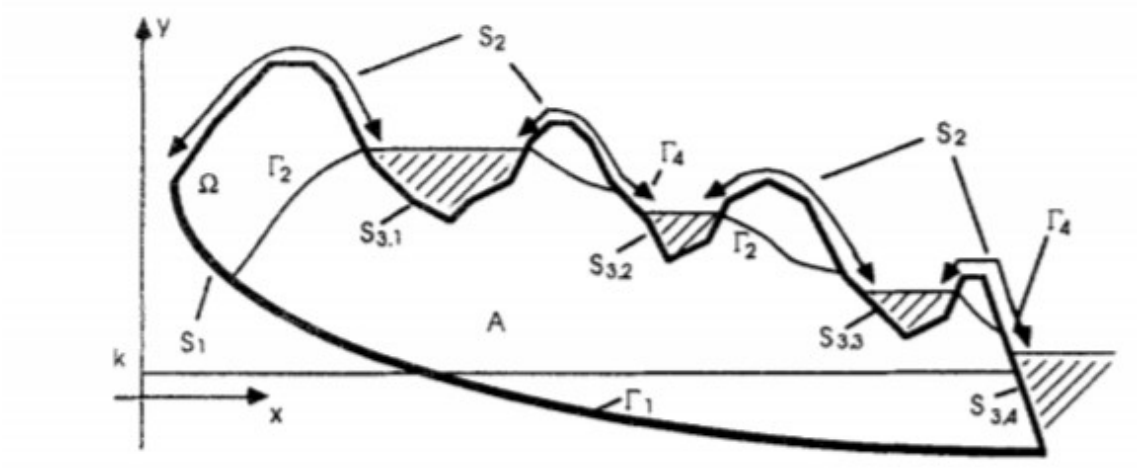


Figure 2.1

$$v = -k\nabla(p+y), \tag{2.1}$$

where p is the pressure and k is the coefficient of permeability of the medium.

We assume the medium is homogeneous, and for simplicity, we consider the fluid to be water with a specific weight equal to 1. Additionally, we've scaled all relevant parameters that $k = 1$.

Then, if the fluid is incompressible we have

$$v = 0 \quad \text{in } A,$$

or

$$\Delta p = 0 \quad \text{in} A. \tag{2.2}$$

Next, on $\Gamma_1 \cup \Gamma_2$ there is no flux of fluid through this part of the boundary. So, if ν denotes the outward unit normal to ∂A we have

$$v \cdot \nu = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2,$$

or, by (2.1)

$$\frac{\partial}{\partial \nu}(p+y) = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2. \tag{2.3}$$

On Γ_4 the fluid is free to exit the porous medium and thus we have

$$v\nu \geq \quad \text{on } \Gamma_4,$$

which reads, again by (2.1)

$$\frac{\partial}{\partial \nu}(p+y) \leq 0 \quad \text{on } \Gamma_4. \tag{2.4}$$

We denote by φ the outside pressure on $S_2 \cup S_3$. For instance, if we assume that we are in the case of Figure 1, and if the atmospheric pressure has been scaled to 0, then φ is given by

$$\varphi(x,y) = \begin{cases} 0 & \text{if } \ (x,y) \in S_2, \\ h_i - y & \text{if } \ (x,y) \in S_{3,i}, \ \ i = 1,...,N, \end{cases}$$

where h_i denotes the level of the reservoir covering $S_{3,i}$. (we assume that the fluid is water and the units are chosen in such a way that the hydrostatic pressure at (x,y) of a reservoir of level h is given by $h-y$.) Clearly, φ is a Lipschitz continuous function. In what follows we could consider the case of a general Lipschitz function φ. However, we restrict our analysis to φ given by the above expression, i.e., to the case where the external pressure is due to reservoirs hanging around.

Besides (2.3) there is a second condition on Γ_2, i.e., the pressure p coincides with the atmospheric pressure, in such a way that

$$p = 0 \quad \text{on } \Gamma_2. \tag{2.5}$$

In [13] or [12] we considered the case of the Dirichlet boundary condition, i.e., the case where p is prescribed equal to φ on S_3 . Here we would like to prescribe the flux on this part of the boundary. More precisely we would like to consider the case where the flux is governed by a function of the jump of pressure across S_3 . This kind

of conditions is called a "leaky boundary condition" and we refer the reader to [5] for physical justifications of such a model.

So, we assume that

$$\frac{\partial}{\partial \nu}(p+y)=\beta(x,\varphi-p) \quad \text{on} \quad S_3. \tag{2.6}$$

Here x denotes a point in $\mathbb{R}^2$. We use the same notation for the first entry of such a point but we do not think that this will create any confusion. On β we can assume, for instance,

$$\beta(x,0)\in L^2(S_3). \tag{2.7}$$

Where $L^2(S_3)$ denotes the usual space of functions of square integrable on S_3 for the superficial measure σ (see, for instance,[14]),

$$x\to\beta(x,u) \quad \text{is measurable for every} \quad u\in R, \tag{2.8}$$

and also $\exists C>0$ such that :

$$|\beta(x,u_1)-\beta(x,u_2)|\le C|u_1-u_2| \quad \sigma-a.e. \quad x\in S_3, \quad \forall u_1,u_2\in R, \tag{2.9}$$

$$u\to\beta(x,u) \ \text{ is nondecreasing for } \quad \sigma-a.e. \ x\in S_3, \tag{2.10}$$

$$\beta(x,u)\ge 0 \quad \sigma-a.e. \quad x\in S_3, \quad \forall u\ge 0. \tag{2.11}$$

A particular case is when

$$\beta(x,u)=\beta_i(u) \quad \text{for} \quad x\in S_3.$$

It may also be thought to extend our results to the case where β_i is a maximal monotone graph. For instance if β_i is, for every $i=1,...,N$, the rnultivalued graph

$$\beta_i=(0,R),$$

then the Difichlet conditions of [13] are recovered (see [16] for this theory and also [17] for results in this direction). However, since our concern is mainly to stress the differences between this model and the classical one we do not consider such a generalization here.

So, assuming that everything is smooth the problem is to find a pair (p,A) such that (2.2)-(2.6) holds. This is what is referred to as the strong formulation.

2.1.3 Weak Formulation

First note that to find the pair (p,A) is equivalent to finding the pair (p,χ_A), where χ_A denotes the characteristic function of the set A. Then, following [11] (see also [1]), for any smooth function ξ we have

$$\begin{aligned}\int_A \nabla p\cdot\nabla\xi+\xi_y\,dx &= \int_A \nabla(p+y).\nabla\xi\,dx \\ &= \int_A -\Delta(p+y).\xi\,dx+\int_A \frac{\partial p+y}{\partial \nu}.\xi\,d\sigma(x).\end{aligned}$$

So, if we assume that p is a smooth function satisfying (2.2)-(2.6) we get, by (2.2),

$$\int_A \nabla p.\nabla \xi + \xi_y \, dx = \int_A \frac{\partial p + y}{\partial \nu}.\xi \, d\sigma(x).$$

Using (2.3)and(2.6)this implies

$$\int_A \nabla p.\nabla \xi + \xi_y \, dx = \int_A \beta(x, \varphi - p).\xi \, d\sigma + \int_A \frac{\partial p + y}{\partial \nu}.\xi \, d\sigma(x).$$

and if we assum that

$$\xi \geq 0 \text{ on } \Gamma_4. \tag{2.12}$$

We get by (2.4)

$$\int_A \nabla p.\nabla \xi + \xi_y \, dx - \int_{S_3} \beta(x, \varphi - p).\xi \, d\sigma \leq 0. \tag{2.13}$$

Now, see (2.5), assume that we extend p by 0 outside of A and that we still denote this extension by p. Then, clearly, if p is smooth up to Γ_2 we deduce from (2.15) that

$$\int_\Omega \nabla p.\nabla \xi + \chi_A \xi_y \, dx - \int_{S_3} \beta(x, \varphi - p).\xi \, d\sigma \leq 0, \quad \forall \xi \geq 0 \quad \text{on} \quad S_2. \tag{2.14}$$

The part Γ_4 is an unknown of the problem so we have assumed that $\xi \geq 0$ on S_2 and this implies, in particular, $\xi \geq 0$ on Γ_4 .

So, we are led to look for a pair $(p, \chi) = (P, \chi_A)$ satisfying (2.16). Recasting this within reasonable spaces the problem becomes :

$$\begin{cases} (P) \text{ Find} (p, \chi) \in H^1(\Omega) L^\infty(\Omega) \text{ such that :} \\ (i)\ p \geq 0, \quad 0 \leq \chi \leq 1 \text{ a.e. in } \Omega, \\ \chi = 1 \text{ on } [p > 0] = \{(x, y) \in \Omega | p(x, y) > 0\}, \\ (ii) \quad p = 0 \quad on \quad S_2, \\ (iii) \int_\Omega \nabla p \cdot \nabla \xi + \chi \xi_y \, dx - \int_{S_3} \beta(x, \varphi - p) \cdot \xi \, d\sigma(x) \leq 0, \\ \forall \xi \in H^1(\Omega), \quad \xi \geq 0 \text{ on } S_2. \end{cases}$$

We refer to (P) as the weak formulation of our initial problem. Clearly, if(2.2)-(2.6) has a solution (p, A) and if p also denotes the extension of p by 0 outside A, then we have shown above that (p, χ_A) is a solution to (P) and thus any strong solutions to our initial problem will be found among those of (P).

We now study the question of the existence of a solution to (P)

2.2 Existence of a solution

We argue as in [11] and we first introduce the following approximated problem

(p_ε) Find $p_\varepsilon \in H^1(\Omega)$ such that :

$$p_\varepsilon = 0 \text{ on } S_2,$$

and

$$\int_\Omega \nabla p_\varepsilon.\nabla\xi + H_\varepsilon(p_\varepsilon)\xi_y\,dx - \int_{S_3}\beta(x,\varphi-p_\varepsilon).\xi\,d\sigma = 0,$$
$$\forall \xi \in H^1(\Omega),\ \ \xi = 0 \ \ on\ S_2.$$

Here H_ε is the approximation of the heaviside graph defined by

$$H_\varepsilon(p) = 0 \vee \tfrac{1}{\varepsilon} \wedge 1,$$

Wher $\vee$ denotes the maximum of two functions and $\wedge$ the minimum, ε is positive.

Then we have

Theorem 2.1. *Assume that $\beta(x,u)$ is a function satisfying (2.7) – (2.11). Then, under the above assumptions, for any $\varepsilon > 0$ there exists a unique solution p_ε to (P_ε).*

Moreover, we have

$$0 \le P_\varepsilon \quad a.\text{e.in } \Omega. \tag{2.15}$$

Proof. set

$$V = \left\{ v \in H^1(\Omega) | v = 0 \ \text{ on } \ S_2 \right\}.$$

For $p \in V$ let us consider the map from V into R:

$$\xi \to \int_\Omega \nabla p \cdot \nabla\xi\,dx - \int_{S_3}\beta(x,\gamma_0(\varphi-p))\cdot\xi\,d\sigma(x), \tag{2.16}$$

Where γ_0 denotes the usual trace operator (see [14] or [22]). For simplicity we sometimes drop the notation γ_0 in the second integral of (2.16).

Since β is Lipschitz continuous and satisfies (2.7), (2.9) we have :

$$\begin{aligned}\left|\int_{S_3}\beta(x,\gamma_0(\varphi-p))\cdot\xi\,d\sigma(x)\right| &\le C|\gamma_0(\varphi-p)|_{2,S_3} + |\beta(x,0)|_{2,S_3}|\xi|_{2,S_3} \\ &\le K|\xi|_{1,2},\end{aligned} \tag{2.17}$$

where $|\ |_{2,S_3}$ denotes the usual L^2-norm on $L^2(S_3)$ and $|\ \ |_{2,S_3}$ denotes the $H^1(\Omega)$- norm. We assum V is equipped with this last norm. We deduce from (2.17) that (2.16) defines a continuous linear form on V that we denote by $\mathbf{A}(p)$. Thus if $\langle\ \ \rangle$ denotes the pairing between V' and V, we have defined an operator $\mathbf{A}$ from V into V' through the formula

$$\left\langle A(p),\xi\right\rangle = \int_\Omega \nabla p\cdot\nabla\xi\,dx - \int_{S_3}\beta(x,\gamma(\varphi-p))\cdot\xi\,d\sigma(x). \tag{2.18}$$

We have :

$$\begin{aligned}\left\langle A(p)-A(p\prime),p-p\prime\right\rangle &= \int_\Omega \nabla(p-p')-\nabla(p-p') \\ &= \|p-p'\|^2_{H^1(\Omega)} \\ &\ge 0.\end{aligned}$$

So A is monotone on V. More precisely we have

$$\langle A(p)-A(p\prime), p-p\prime\rangle \geq \int_\Omega |\nabla(p-p\prime)|^2 d\,x. \tag{2.19}$$

And A is coercive on V (see [15]). Moreover, A restricted to finite-dimensional spaces is clearly continuous. So, (see [15]), A is a one-to-one operator from V into V'. Now, if $\upsilon \in L^2(\Omega)$, then

$$\xi \rightarrow -\int_\Omega H_\varepsilon(\upsilon)\xi_y\, d\,x,$$

defines a continuous linear form on V. Thus, for any $\upsilon \in L^2(\Omega)$, there exists a unique $u_\varepsilon = \tau_\varepsilon(\upsilon)$ such that :

$$u_\varepsilon \in V, \quad \langle A(u_\varepsilon), \xi\rangle = -\int_\Omega H_\varepsilon(\upsilon)\xi_y\, d\,x, \quad \forall \xi \in V. \tag{2.20}$$

So, to prove that there exists a solution to (P_ε) we now only need to show that the map τ_ε has a fixed point. For that, if φ denotes a Lipschitz continuous extension of φ to Ω, we note that $u_\varepsilon - \varphi \in V$, thus from (3.7) we obtain

$$\langle A(u_\varepsilon), u_\varepsilon-\varphi\rangle = -\int_\Omega H_\varepsilon(\upsilon)(u_\varepsilon-\varphi)_y\, d\,x,$$

hence, for some constant C independent of ε,

$$\begin{aligned}
\langle A(u_\varepsilon)-A(\varphi), u_\varepsilon-\varphi\rangle &= -\langle A(\varphi), u_\varepsilon-\varphi\rangle - \int_\Omega H_\varepsilon(v)\cdot(u_\varepsilon-\varphi)_y\, dx \\
&= -\int_\Omega \nabla\varphi\cdot\nabla(u_\varepsilon-\varphi)dx + \int_{S_3}\beta(x,0)(u_\varepsilon-\varphi)d\sigma(x) \\
&\quad -\int_\Omega H_\varepsilon(v)\cdot(u_\varepsilon-\varphi)_y\, dx
\end{aligned}$$

by **Youngs inequality** we have

$$\begin{aligned}
&\leq -\|\nabla\varphi\|_2\cdot\|\nabla(u_\varepsilon-\varphi)\|_2 \\
&\quad +|\beta(x,0)|_{2,S_{S_3}}\cdot\|u_\varepsilon-\varphi\|_2 - |\Omega|^{1/2}\cdot\|u_\varepsilon-\varphi\|_2
\end{aligned}$$

by **Poincaré inquality** we have

$$\begin{aligned}
&\leq -\|\nabla\varphi\|_2\cdot\|\nabla(u_\varepsilon-\varphi)\|_2 \\
&\quad +C|\beta(x,0)|_{2,S_{S_3}}\cdot\|\nabla u_\varepsilon-\varphi\|_2 - |\Omega|^{1/2}\cdot\|\nabla u_\varepsilon-\varphi\|_2
\end{aligned}$$

and finally we get

$$\leq \left(\|\nabla\varphi\|_2 + |\Omega|^{\frac{1}{2}} + C|\beta(x,0)|_{2,S_3}\right)\|\nabla(u_\varepsilon-\varphi)\|_2 \tag{2.21}$$

by the Cauchy-Schwarz inequality. ($|\ |$ denote the Euclidean norm in $\mathbf{R}^n$ or the Lebesgue measure, $|\ |_2$ is the L^2-norm on $L^2(\Omega)$ and we used the continuity of the trace operator, see [14]). Recalling (2.19) we obtain

$$\int_\Omega |\nabla(u_\varepsilon-\varphi)|^2 \leq (\|\nabla\varphi\|_2 + |\Omega|^{1/2} + C|\beta(x,0)|_{2,S_3})^2, \tag{2.22}$$

and thus

$$|u_\varepsilon|_{1,2}\leq C, \tag{2.23}$$

where C is some constant independent of ε. The existence of a fixed point for $\tau\varepsilon$

then follows easily from the Schauder fixed-point theorem. Hence the existence of p_ε.

To prove uniqueness we argue like in [11] and see also [7] for more general results. More precisely, set, for $\delta > 0$,

$$f_\delta(x) = \begin{cases} \left(1-\frac{\delta}{x}\right)^+ & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x \leq 0, \end{cases} \tag{2.24}$$

f_δ is a Lipschitz continuous function and if p_ε, p'_ε are solutions to (P_ε), then

$$f_\delta(p_\varepsilon - p'_\varepsilon) \in V.$$

Thus it is deduced from (P_ε) written for p_ε and p'_ε that

$$\begin{aligned} &\int_\Omega \nabla(p_\varepsilon - p'_\varepsilon)\cdot\nabla f_\delta(p_\varepsilon - p'_\varepsilon)dx \\ &= -\int_\Omega (H_\varepsilon(p_\varepsilon) - H_\varepsilon(p'_\varepsilon)) f_\delta(p_\varepsilon - p'_\varepsilon)_y dx \\ &+ \int_{S_3} \beta(x, \varphi - p_\varepsilon) - \beta(x, \varphi - p'_\varepsilon)\cdot f_\delta(p_\varepsilon - p'_\varepsilon)d\sigma(x). \end{aligned} \tag{2.25}$$

Hence, by the monotonicity of β ,

$$\int_\Omega \nabla(p_\varepsilon - p'_\varepsilon)\cdot\nabla f_\delta(p_\varepsilon - p'_\varepsilon)dx \leq -\int_\Omega (H_\varepsilon(p_\varepsilon) - H_\varepsilon(p'_\varepsilon)) f_\delta(p_\varepsilon - p'_\varepsilon)_y dx.$$

So, if we set $q_\varepsilon = p_\varepsilon - p'_\varepsilon, [q_\varepsilon > \delta] = \{x \in \Omega : q_\varepsilon > \delta\}$ we easily obtain, using (2.24) and the Lipschitz continuity of H_ε,

$$\int_{[q_\varepsilon>\delta]} \frac{|\nabla q_\varepsilon|^2}{q_\varepsilon^2} dx \leq \frac{1}{\varepsilon}\int_{[q_\varepsilon>\delta]} \frac{|\nabla q_\varepsilon|}{q_\varepsilon} dx.$$

Hence , by the Cauchy-Schwarz inequality,

$$\int_\Omega \left|\nabla \ln\left(1 + \frac{(q_\varepsilon - \delta)^+}{\delta}\right)\right|^2 dx = \int_{[q_\varepsilon>\delta]} \frac{|\nabla q_\varepsilon|^2}{q_\varepsilon^2} dx \leq \frac{|\Omega|}{\varepsilon^2}.$$

By Poincaré's inequality we obtain

$$\int_\Omega \left|\ln\left(1 + \frac{(q_\varepsilon - \delta)^+}{\delta}\right)\right|^2 dx \leq C,$$

Where C is independent of δ. Letting $\delta \to 0$ we deduce

$$q_\varepsilon \leq 0 \quad \text{a.e.in } \Omega,$$

and the uniqueness of p_ε follows by exchanging the roles of p_ε and p'_ε .

To prouve (2.15) if we set $\xi = (p_\varepsilon^-)$ in (p_ε) we obtain

$$-\int_\Omega |\nabla p_\varepsilon^-|^2 dx - \int_{S_3} \beta(x, \varphi - p_\varepsilon)\cdot p_\varepsilon^- d\sigma(x) = 0,$$

hence, by (2.11) ,

$$\int_\Omega |\nabla p_\varepsilon^-|^2 dx \le 0,$$

and (2.15) follows. □

Remark 2.1. In fact,we have that if φ, φ' are such that

$$\varphi \le \varphi' \text{ on } S_2 \cup S_3,$$

then the corresponding solutions p_ε and p'_ε are such that

$$p_\varepsilon \le p'_\varepsilon \text{ a.e.in } \Omega.$$

Indeed $f_\delta(p_\varepsilon - p'_\varepsilon) \in V$ and the last integral in (2.25) written with φ and φ' is nonpositive. In fact, the existence of a nonnegative solution to (p_ε) when $u \to \beta(x,u)$ is not assumed to be nondecreasing can also be proved . Indeed we have

Theorem 2.2. *Assume that $\beta(x,u)$ is a Carathéodory function, i.e., measurable in x for every u and continuous in u for $\sigma - a.e$, x and such that, for some constant a, b, we have*

$$|\beta(x,u)| \le a|u| + b \quad \sigma - a.e. \quad x \in S_3. \quad \forall u \in \boldsymbol{R}. \tag{2.26}$$

Then if a is small enough there exists a solution to (p_ε) . Moreover,if (2.11) holds,then we have

$$p_\varepsilon \ge 0 \qquad a.\text{e.in } \Omega.$$

Proof. For $\upsilon \in L^2(S_3)$ there exists a unique $u_\varepsilon \in V$ such that

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla \xi \, dx = -\int_\Omega H_\varepsilon(u_\varepsilon)\xi_y \, dx + \int_{S_3} \beta(x, \varphi - \upsilon)\cdot \xi \, d\sigma(x). \tag{2.27}$$

By the shauder fixed-point theorem we have $u_\varepsilon \in V$ is a unique solution

$$L^2(S_3) \longrightarrow L^2(S_3)$$

$$\chi \longrightarrow S(\chi) = w$$

Such that W is a solution of probleme

$$\int_\Omega \nabla w \cdot \nabla \xi \, dx = -\int_\Omega H_\varepsilon(\chi)\xi_y \, dx + \int_{S_3} \beta(x, \varphi - \upsilon)\cdot \xi \, d\sigma(x).$$

We will demonstrate that S satisfies all the hypotheses of the Schauder fixed-point theorem
1-S fixes a certain closed ball $L^2(S_3)$

We choose $\xi = w$

$$\int_\Omega \nabla w \cdot \nabla w \, dx = -\int_\Omega H_\varepsilon(\chi) w_y \, dx + \int_{S_3} \beta(x, \varphi - \upsilon)\cdot w \, d\sigma(x).$$

by **Cauchy - Schwarz** inequality we have

$$\|w\|^2_{H^1(\Omega)} \leq -\|w\|_2 \cdot |\Omega|^{1/2}_{L^2} + |\beta(x, \varphi - v)|_2 \cdot \|w\|_2,$$

by **Poinca** inequality

$$\|w\|^2_{H^1(\Omega)} \leq C_\Omega(-|\Omega|^{1/2}_{L^2} + a|\varphi - v|_2 + b)\cdot \|w\|_{H^1(\Omega)},$$

$$\|w\|_{H^1(\Omega)} \leq C_\Omega(-|\Omega|^{1/2}_{L^2} + a|\varphi - v|_2 + b),$$

and we have

$$\frac{1}{C_\Omega}\|w\|_{L^2(\Omega)} \leq C_\Omega(-|\Omega|^{1/2}_{L^2} + a|\varphi - v|_2 + b),$$

$$\|w\|_{L^2(\Omega)} \leq C^2_\Omega(-|\Omega|^{1/2}_{L^2} + a|\varphi - v|_2 + b).$$

$$B(0, R) = \left\{ w \in L^2(\Omega) \ : \ \|w\|_{L^2(\Omega)} \leq R \right\}.$$

2-S is continuous, from L^2 strong to L^2 strong, that means

$$\|\chi_n - \chi\|_{L^2} \longrightarrow 0.$$

So

$$\|S(\chi_n) - S(\chi)\|_{L^2} \longrightarrow 0.$$

we get

$$\int_\Omega \nabla(w_n) \cdot \nabla \xi \, dx = -\int_\Omega H_\varepsilon(\chi_n)\xi_y \, dx + \int_{S_3} \beta(x, \varphi - \upsilon)\cdot \xi \, d\sigma(x).$$

$$\int_\Omega \nabla(w)\cdot\nabla\xi\,dx = -\int_\Omega H_\varepsilon(\chi)\xi_y\,dx + \int_{S_3}\beta(x,\varphi-\upsilon)\cdot\xi\,d\sigma(x).$$

$$\int_\Omega \nabla(w_n-w)\cdot\nabla\xi\,dx = -\int_\Omega H_\varepsilon(\chi_n-\chi)\xi_y\,dx.$$

we find $\xi = w_n - w$ and by **Cauchy - Schwarz** inequality we have

$$|w_n-w|^2_{H^1(\Omega)} = -\int_\Omega \big[H_\varepsilon(\chi_n)-H_\varepsilon(\chi)\big](w_n-w)dx,$$

$$\frac{1}{C_\Omega}|w_n-w|^2_{L^2(\Omega)} \le -\|H_\varepsilon(\chi_n)-H_\varepsilon(\chi)\|_2\|w_n-w\|_2,$$

by **Young's** inequality we have

$$\frac{1}{C_\Omega}|w_n-w|^2_{L^2(\Omega)} \le \frac{(-\|H_\varepsilon(\chi_n)-H_\varepsilon(\chi)\|_2)^2}{2} + \frac{(\|w_n-w\|_2)^2}{2},$$

$$\frac{1}{C_\Omega}|w_n-w|^2_{L^2(\Omega)} + \frac{\|w_n-w\|^2_2}{2} \le \frac{(-\|H_\varepsilon(\chi_n)-H_\varepsilon(\chi)\|_2)^2}{2},$$

$$|w_n-w|^2_{L^2(\Omega)}\Big(\frac{1}{C_\Omega}+\frac{1}{2}\Big) \le \frac{(-\|H_\varepsilon(\chi_n)-H_\varepsilon(\chi)\|_2)^2}{2},$$

$$|w_n-w|^2_{L^2(\Omega)} \le C_\Omega(-\|H_\varepsilon(\chi_n)-H_\varepsilon(\chi)\|_2).$$

using

$$\chi_n \longrightarrow \chi.$$

And likeH_ε continuous we have

$$H_\varepsilon(\chi_n) \longrightarrow H_\varepsilon(\chi).$$

So

$$H_\varepsilon(\chi_n) \longrightarrow H_\varepsilon(\chi) \qquad a.e.\chi \qquad \chi \in L^2(\Omega).$$

And we have Dominating function

$$|H_\varepsilon(\chi_n)| \le 1.$$

By dominated convergence we have

$$\|H_\varepsilon(\chi_n)-H_\varepsilon(\chi)\| \xrightarrow{L^2(\Omega)} 0.$$

So

$$w_n \longrightarrow w.$$

3- $S(\overline{B}(0,R))$ is relatively compact
Let χ_n be a sequence of $\overline{B}(0,R)$ by the estimation 2. The sequence $w_n = T(\chi_n)$ is bounded in $H^1(\Omega)$, which injects into $L^2(\Omega)$ in a way.(Rellich théorem) Consequently, this sequence admits a single converging sequence in $L^2(\Omega)$.

All conditions of the Schauder fixed-point theorem are satisfied alor we have u_ε is a sloution of (P).

This is an consequence of the schauder fixed-point theorem(see the proof above).Set

$$\tau_\varepsilon(\upsilon) = \gamma_0(u_\varepsilon),$$

where γ_0 denotes the trace on S_3. Clearly, τ_ε is a continuous map from $L^2(S_3)$ into itself. Set

$$K = \left\{\upsilon \in L^2(S_3) \,|\, |\upsilon - \varphi|_2 \leq R\right\}.$$

Then K is a closed convex of $L_2(S_3)$ .Taking $\xi = u_\varepsilon - \varphi$ in (2.27) we get

$$\begin{aligned}\int_\Omega |\nabla(u_\varepsilon - \varphi)|^2 dx &= -\int_\Omega \nabla\varphi \cdot \nabla(u_\varepsilon - \varphi) dx \\ &\quad -\int_\Omega H_\varepsilon(u_\varepsilon)\cdot(u_\varepsilon - \varphi)_y dx \\ &\quad +\int_{S_3} \beta(x, \varphi - \upsilon)\cdot(u_\varepsilon - \varphi) d\sigma(x).\end{aligned}$$

(We assume φ extended to Ω into a Lipschitz function see [19]). Applying the Cauchy-Schwarz inquality and (2.26) we have

$$\begin{aligned}\|\nabla(u_\varepsilon - \varphi)\|_2^2 &\leq (\|\nabla\varphi\|_2 + |\Omega|^{1/2})\|\nabla(u_\varepsilon - \varphi)\|_2 \\ &\quad +(a|\upsilon - \varphi|_{2,S_3} + b|S_3|^{1/2})|u_\varepsilon - \varphi|_{2,S_3},\end{aligned}$$

With obvious notation . Now, clearly, for some constant C we have

$$|u_\varepsilon - \varphi|_{2,S_3} \leq C\|\nabla(u_\varepsilon - \varphi)\|_2.$$

(see for instance [22])and we deduce

$$\|\nabla(u_\varepsilon - \varphi)\|_2 \leq \|\nabla\varphi\|_2 + |\Omega|^{1/2} + bC|S_3|^{1/2} + aC|\upsilon - \varphi|_{2,S_3} \tag{2.28}$$

and thus

$$\begin{aligned}|u_\varepsilon - \varphi|_{2,S_3} \leq C\|\nabla(u_\varepsilon - \varphi)\|_2 &\leq C\|\nabla\varphi\|_2 + |\Omega|^{1/2} + bC|S_3|^{1/2} \\ &\quad +aC|\upsilon - \varphi|_{2,S_3}.\end{aligned}$$

If we choose $\upsilon \in K$, then $u_\varepsilon = \tau_\varepsilon(\upsilon) \in K$ provided

$$C\|\nabla\varphi\|_2 + |\Omega|^{1/2} + bC|S_3|^{1/2} + aC^2R \leq R,$$

i.e.,if $aC^2 \leq 1$ provided that

$$R \geq (1-aC^2)^{-1} \cdot C\left\{\|\nabla\varphi\|_2 + |\Omega|^{1/2} + bC|S_3|^{1/2}\right\}. \tag{2.29}$$

Thus, assuming $aC^2 < 1$ and condition (2.29) holds, then τ_ε becomes an operator mapping K into K. Furthermore, according to (2.28), u_ε is uniformly bounded in $H^1(\Omega)$, and τ_ε is compact. The existence of a fixed point for τ_ε is a consequence of the Schauder fixed-point theorem, demonstrating the existence of p_ε. To establish $p_\varepsilon > 0$, we proceed as in the proof of Theorem (2.1).

□

Remark 2.2. Theorem (2.2) applies, for instance, when, for some constants α, γ,

$$|\beta(x,u)| \leq \alpha|u|^{1-\varepsilon} + \gamma \quad \sigma - a.s. \quad x \in S_3, \quad \forall u \in \mathbf{R}. \tag{2.30}$$

Indeed, in this case, it is easy to show that (2.26) holds for a as small as we wish.

We are now able to show

Theorem 2.3. *Assume that β is a function satisfying (2.11) and (2.7)-(2.10) or (2.26) then there exists a solution (p,χ) to problem (P).*

Proof. Let p_ε be the solution to (P_ε). From (2.23), (2.28) we deduce

$$|p_\varepsilon|_{1,2} \leq C.$$

Where C is some constant independent of ε . So, using classical compactness arguments (see [14]) we can extract a subesquence of ε, still denoted by ε, such that, for som $p \in V$,

$$p_\varepsilon \rightharpoonup p \quad \text{in} \quad H^1(\Omega), \quad p_\varepsilon \rightharpoonup p \quad \text{in} \quad L^2(\Omega) \quad \text{and} \quad \text{a.e.on} \quad \Omega, \tag{2.31}$$

$$\gamma_0(p_\varepsilon) \rightharpoonup \gamma_0(p) \quad \text{in } L^2(S_3). \tag{2.32}$$

Since $\{v \in V | v(x) \geq 0 \ \text{ a.e.on } \ \Omega\}$ is closed and convex it is weakly closed and thus p is in this set so that

$$p \geq 0 \quad \text{a.e. on} \quad L^2(S_3). \tag{2.33}$$

Since $H_\varepsilon(p_\varepsilon)$ is uniformly bounded there exists a function χ such that

$$H_\varepsilon(p_\varepsilon) \rightharpoonup \chi \quad \text{in} \quad L^2(\Omega). \tag{2.34}$$

The set

$$\left\{f \in L^\infty(\Omega) | 0 \leq f \leq 1 \ \text{ a.e.in } \ \Omega\right\}.$$

being closed and convex is weakly closed and, by (2.34), we have

$$0 \le \chi \le 1 \quad \text{a.e.in } \Omega. \tag{2.35}$$

By (2.31) on $[p>0]$ we have

$$H_\varepsilon(p_\varepsilon) \to 1 \quad \text{a.e.}$$

and thus, by the Lebesgue theorem,

$$H_\varepsilon(p_\varepsilon) \to 1 \quad \text{in} \quad L^2([p>0]).$$

Since, by (2.34), we also have

$$H_\varepsilon(p_\varepsilon) \rightharpoonup \chi \quad \text{in} \quad L^2([p>0]),$$

we deduce

$$\chi = 1 \quad \text{on} \quad [p>0], \tag{2.36}$$

and thus (P)(i),(ii) follows.

Next, for $\xi H^1(\Omega)$, $\xi \ge 0$ on S_2, we have, for any $\delta > 0$,

$$\xi \frac{p_\varepsilon}{\delta} \in V.$$

Thus inserting this function in (P_ε) we obtain

$$\int \nabla p_\varepsilon \cdot \nabla \left(\xi \hat{} \frac{p_\varepsilon}{\delta} \right) dx + H_\varepsilon(p_\varepsilon) \left(\xi \hat{} \frac{p_\varepsilon}{\delta} \right)_y dx - \int_{S_3} \beta(x, \varphi - p_\varepsilon) \cdot \xi \hat{} \frac{p_\varepsilon}{\delta} d\sigma(x) = 0.$$

Thus , we also have

$$\begin{aligned} &\int_{[\xi \le p_\varepsilon/\delta]} \nabla p_\varepsilon \cdot \nabla \xi \, dx + \int_\Omega H_\varepsilon(p_\varepsilon) \left(\xi \hat{} \frac{p_\varepsilon}{\delta} \right)_y dx - \int_{S_3} \beta(x, \varphi - p_\varepsilon) \cdot \left(\xi \hat{} \frac{p_\varepsilon}{\delta} \right) d\sigma(x) \\ &\qquad = - \int_{[p_\varepsilon/\delta < \xi]} \frac{|\nabla p_\varepsilon|^2}{\delta} dx \le 0. \end{aligned} \tag{2.37}$$

(If f and g are two functions we denote by [f < g] and $[f \le g]$ the sets defined by

$$[f < g] = \left\{ (x,y) \in \Omega | f(x,y) < g(x,y) \right\}.$$

and

$$[f \le g] = \{(x,y) \in \Omega | f(x,y) \le g(x,y)\}.$$

We use this notation subsequently without further notice.) Now, using the divergence theorem, note that

$$\int_\Omega H_\varepsilon(p_\varepsilon)\left(\xi \hat{} \frac{p_\varepsilon}{\delta}\right)_y dx = -\int_\Omega (H_\varepsilon(p_\varepsilon))_y \cdot \xi \hat{} \frac{p_\varepsilon}{\delta} dx + \int_{\partial\Omega} H_\varepsilon(p_\varepsilon) \cdot \nu_y \cdot \xi \hat{} \frac{p_\varepsilon}{\delta} d\sigma(x), \tag{2.38}$$

where ν_y denotes the second entry of the outward normal to Ω. Note that in this formula as well as in (2.2) we use the fact that

$$\gamma_0\left(\xi \hat{} \frac{p_\varepsilon}{\delta}\right) = \gamma_0(\xi) \hat{} \gamma_0\left(\frac{P_\varepsilon}{\delta}\right).$$

Letting $\delta \longrightarrow 0$ in and since

$$\xi \hat{} \frac{p_\varepsilon}{\delta} \longrightarrow \xi \quad \text{a.e.on} \quad p_\varepsilon > 0.$$

We obtain, by the Lebesgue theorem,

$$\begin{aligned} \lim_{\delta \longrightarrow 0} \int_\Omega H_\varepsilon(p_\varepsilon)\left(\xi \hat{} \frac{P_\varepsilon}{\delta}\right)_y dx &= -\int_\Omega H_\varepsilon(p_\varepsilon)_y \cdot \xi \, dx + \int_{\partial\Omega} H_\varepsilon(p_\varepsilon) \cdot \nu_y \cdot \xi \, d\sigma(x) \\ &= \int_\Omega H_\varepsilon(p_\varepsilon) \cdot \xi_y \, dx. \end{aligned} \tag{2.39}$$

Next, passing in the limit in we easily get, for any $\xi \in H^1(\Omega)$, $\xi \ge 0$ on S_2,

$$\int_\Omega \nabla p_\varepsilon \cdot \nabla \xi \, dx + \int_\Omega H_\varepsilon(p_\varepsilon) \xi_y \, dx - \int_{S_3} \beta(x, \varphi - p_\varepsilon) \cdot \xi \, d\sigma(x) \le 0. \tag{2.40}$$

The only difficulty may be to pass to the limit in the last integral of the left-hand side of (2.2). For that note that

$$-\int_{S_3} \beta(x, \varphi - p_\varepsilon) \cdot \left(\xi \hat{} \frac{p_\varepsilon}{\delta}\right) d\sigma(x),$$

$$= -\int_{S_3 \cap [p>0]} \beta(x, \varphi - p_\varepsilon) \cdot \left(\xi \hat{} \frac{p_\varepsilon}{\delta}\right) d\sigma(x) - \int_{S_3 \cap [p=0]} \beta(x, \varphi) \cdot (\xi \hat{} 0) d\sigma(x),$$

$$\ge -\int_{S_3 \cap [p>0]} \beta(x, \varphi - p_\varepsilon) \cdot \left(\xi \hat{} \frac{p_\varepsilon}{\delta}\right) d\sigma(x) - \int_{S_3 \cap [p=0]} \beta(x, \varphi) \cdot \xi \, d\sigma(x).$$

Since $\beta(x, \varphi) \ge 0$. Letting $\delta \longrightarrow 0$, by the Lebesgue theorem, we deduce

$$\lim_{\delta \longrightarrow 0} - \int_{S_3} \beta(x, \varphi - p_\varepsilon) \cdot \left(\xi \hat{} \frac{\varepsilon}{\delta} \right) d\sigma(x) \geq - \int_{S_3} \beta(x, \varphi - p_\varepsilon) \cdot \xi \, d\sigma(x).$$

and (2.40) follows.

Then letting $\varepsilon \longrightarrow 0$ in (2.40) and using (2.31),(2.32),(2.34) we obtian , for any $\xi \in H^1(\Omega)$, $\xi \geq 0$ on S_2,

$$\int_\Omega \nabla p \cdot \xi \, dx + \int_\Omega \chi \xi_y \, dx - \int_{S_3} \beta(x, \varphi - p) \cdot \xi \, d\sigma(x) \leq 0,$$

and (P)(iii) follows □

Chapter 3

Properties of the solutions

Sommaire

3.0.1 Some Properties of the solutions

In this section we describe some useful properties of any solution (p,χ) to (P). Some of these results are similar to the ones in [13] but proved with new methods.

First we have

Proposition 3.1

Let (p,χ) be a pair solution to (P). Then we have in the distributional sense

$$\Delta p + \chi_y = 0 \quad in\ \Omega. \tag{3.1}$$

$$\Delta p \geq 0, \quad \chi_y \leq 0 \quad in\ \Omega. \tag{3.2}$$

Proof. If $\xi \in \mathscr{D}(\Omega)$, is a test functin for (P)(iii)and we get since ξ vanishes on $\partial\Omega = \Gamma$,

$$\int_\Omega \nabla p \cdot \xi + \chi \xi_y \, dy = 0, \qquad \forall \xi \in \mathscr{D}(\Omega),$$

Which is (3.1).

Next, if $\xi \in \mathscr{D}(\Omega)$, $\xi \geq 0$, then $\pm H_\varepsilon(p)\xi$ is a test function for (P)(iii). (H_ε is defined by (2.12).) From (P)(iii) we then deduce

$$\int_\Omega \nabla p \cdot \nabla(H_\varepsilon(p)\xi) + \chi \cdot (H_\varepsilon(p)\xi)_y \, dx = 0.$$

Hence , since , on $[p=0]$, $H_\varepsilon(p)=0$ and , on $[p>0]$,$\chi = 1$,

$$\int_\Omega H_\varepsilon(p)\nabla p \cdot \nabla\xi + (H_\varepsilon(p)\xi)_y \, dx = -\int_\Omega H'_\varepsilon(p)\xi \cdot |\nabla p|^2 \, dx \leq 0.$$

Since $H_\varepsilon(p)\xi = 0$ on $\partial\Omega$, we get, applying the divergence theorem for the second integral
,

$$\int_\Omega H_\varepsilon(p)\nabla p\cdot\nabla\xi\,dx\le 0,\qquad\qquad \forall\xi\in\mathscr{D}(\Omega),\quad \xi\ge 0.$$

Letting $\varepsilon\longrightarrow 0$, by the Lebesgue convergence theorem we obtian

$$\int_\Omega \nabla p\cdot\nabla\xi\,dx\le 0,\qquad\qquad \forall\xi\in\mathscr{D}(\Omega),\quad \xi\ge 0. \tag{3.3}$$

The first inequality of (3.2) follows ,the second resulte from (3.1). □

As consequence we have

Corollary 3.1. Let (p,χ) be a pair solution to (P) then, for any $s>1$,

$$p\in W^{1,s}_{loc}(\Omega).$$

Proof. This is a consequence of (3.1) since $\chi_y\in W^{-1,s}(\Omega)$ for any s and from the usual regularity theory see,for instance, [6]. □

Remark 3.1. As a consequence , our corollary

$$[p>0]=(x,y)\in\Omega|p(x,y)>0.$$

is open in Ω since p is continuous in Ω (see [8] and [20]). In fact p is continuous at any point of $\Omega\cup S_2$.

We also have

Proposition 3.2

Let (p,χ) be a solution to (P). Let $(x_0,y_0)\in\Omega$. If $p(x_0,y_0)>0$ then there exists $\varepsilon>0$ such that the cylinder

$$C_\varepsilon=(x,y)\in\Omega||x-x_0|<\varepsilon, y<y_0+\varepsilon.$$

lies in the set $[p>0]$.

If $p(x_0,y_0)=0,\quad$ then $\quad p(x_0,y)=0,\qquad \forall(x_0,y)\in\Omega, y>y_0.$

Proof. The proof follows [13] but we reproduce it for the reader's convenience . If $p(x_0,y_0)>0$, then , since the set [p>0] is open, there exists some $\varepsilon>0$ such that the square

$$Q_\varepsilon=\left\{(x,y)\in\Omega||x-x_0|<\varepsilon,|y-y_0|<\varepsilon\right\}.$$

Is included in [p > 0]. On Q_ε, by (P)(i) we have $\chi=1$, hence, since by (3.2) χ is nondecreasing in y, we have $\chi=1$ on C_ε. By (3.1) we deduce that p is harmonic on C_ε. If p should vanish on C_ε we would then get a contradiction to the maximum principle. This proves the first part of the proposition. The second part follows directly from the first. □

In the case of Dirichlet boundary conditions and in a case like, for instance, in Figure1 the pressure remains positive below S_3 (see [13]). We would like to show that this is no longer the case in the present situation.

First let us show

Proposition 3.3

Assume that $\beta \equiv 0$. Then every solution to (P) is given by

$$(p,\chi) = (h-y,1), \tag{3.4}$$

on every connected component of $[p>0]$,

$$(p,\chi) = (0,0), \tag{3.5}$$

elsewhere.

Proof. If $\beta \equiv 0$ we have

$$\int_\Omega \nabla p \cdot \nabla \xi + \chi \xi_y \, dx \le 0 \qquad \forall \xi \ge 0 \qquad \text{on } S_2.$$

Taking $\xi = \pm p$ we deduce

$$\int_\Omega |\nabla p|^2 + \chi p_y \, dx = 0. \tag{3.6}$$

Taking $\xi = k + y$ for k large enough in such a way that $k + y > 0$ on S_2 we get

$$\int_\Omega p_y + \chi \, dx \le 0.$$

Since $\chi = 1$ on $[p>0]$ and $0 \le \chi \le 1$ this last inequality implies

$$\int_\Omega \chi p_y + \chi^2 \, dx \le 0. \tag{3.7}$$

Adding (3.6) and (3.7) we get

$$\int_\Omega |\nabla p|^2 + 2\chi p_y + \chi^2 \, dx = 0.$$

Hence

$$\int_\Omega p_x^2 + (p_y + \chi)^2 \, dx \le 0. \tag{3.8}$$

Thus we obtain

$$\nabla p = (0, -\chi) \qquad \text{a.e. in } \Omega.$$

In particular, on any connected component of $[p>0]$

$$\nabla p = (0,-1),$$

and thus $p=(h-y)$. Outside

$$(0,0)=\nabla p=(0,-\chi),$$

and the proposition follows. □

Remark 3.2. It is clear that every pair given by (3.4),(3.5) defines a solution to (P) for $\beta=0$. For instance, in the case of Figure 1 the pair

$$(p,\chi)=((k-y)^+,\chi_{[y<k]}),$$

is a solution to (P) for $\beta=0$ provided that K is small enough (see 4.3), Clearly, different values of k are suitable here so that (P) does not have a unique solution. We are encountering "pool" solutions as in the case of the dam problem with the Dirichlet boundary condition–see [13]. We will see that again in Theorem 3.

We are now able to prove

Proposition 3.4

The region below S_3 is not in general saturated, i.e , there is not in general $p>0$ below S_3.

Proof. Consider β satisfying the assumptions of Theorem 2. Then, for any $0<\eta<1$, there exists a (p_η,χ_η) solution of (P) corresponding to $\eta\beta$. In particular (P)(iii) reads

$$\int_\Omega \nabla p_\eta\nabla\xi+\chi_\eta\xi_y\,dx-\int_{S_3}\eta\beta(x,\varphi-p_\eta)\cdot\xi\,d\sigma(x)\le 0, \tag{3.9}$$

for any $\xi\in H^1(\Omega)$, $\xi\ge 0$ on S_2, Taking $\xi=p_\eta-\varphi$ in this inequality we deduce very easily (compare with (2.23) and (2.28)) that

$$|p_\eta|_{1,2}\le C,$$

where C is independent of η. Hence, up to a subsequence,

$$P_\eta\rightharpoonup P \quad \text{in } H^1(\Omega), \qquad P_\eta\longrightarrow P \quad L^2(\Omega), \qquad P_\eta\longrightarrow P \quad L^2(S_3),$$

$$P_\eta\longrightarrow P \qquad \text{a.e. in } \Omega, \qquad \chi_\eta\rightharpoonup\chi \text{ in } L^2(\Omega),$$

when $\eta\longrightarrow 0$. Passing to the limit in (3.9) we obtain

$$\int_\Omega \nabla p\nabla\xi+\chi\xi_y\le 0, \qquad \forall\xi\in H^1(\Omega),\ \xi\ge 0 \text{ on } S_2. \tag{3.10}$$

Moreover, see the proof of Theorem (2.3), we have, clearly,

$$0 \le p, \quad p \in V, \qquad 0 \le \chi \le 1, \qquad \chi = 1 \qquad \text{on } [p > 0],$$

and thus (p, χ) is a solution of (P) for $\beta \equiv 0$. If, for any η, we had $p > 0$ below S_3 we would have $\chi_\eta \equiv 1$ below S_3 and thus at the limit $\chi \equiv 1$ below S_3. This is clearly not the case by Proposition 3. □

Thus, roughly speaking, if β is sufficiently small, an unsaturated region could develop. In Section 5, we will provide examples where $p = 0$ in a vicinity below S_3 and elaborate further on this matter.
Given that the properties of (p, χ) in the model we are studying are notably distinct from those under Dirichlet boundary conditions, it is important to verify that problem (P) constitutes a free boundary problem .
For that we can prove

Theorem 3.1. *Let us assume that β is chosen as in Theorem (2.3). Moreover, let us assume that*

$$\beta(x,u)\cdot u \ge 0, \qquad \forall u \in \boldsymbol{R}, \sigma - a.e. \qquad x \in S_3. \tag{3.11}$$

Let (p,χ) be a solution to (P) and let us denote by h_1 the level of the highest reservoir. Then we have

$$(p,\chi) = (k - y, 1),$$

on any connected component of $[p > 0]$ that intersects $[y > h_1]$. Moreover, outside these connected components we have

$$p \le (h_1 - y)^+. \tag{3.12}$$

Proof. Let (p,χ) be a solution of (P).Taking

$$\xi = (p - (h_1 - y)^+)^+$$

in (P)(iii)we obtain

$$\int_\Omega \nabla p \nabla (p - (h_1 - y)^+)^+ + \chi (p - (h_1 - y)^+)^+_y dx$$

$$-\int_{S_3} \beta(x, \varphi - p)(p - (h_1 - y)^+)^+ d\sigma(x) \le 0$$

We integrate only on $p > (h_1 - y)^+ > \varphi$, so on this set, by (3.11), we have

$$\beta(x, \varphi - p) \le 0.$$

and the above inequality becomes

$$\int_\Omega \nabla p \cdot \nabla (p-(h_1-y)^+)^+ + \chi (p-(h_1-y)^+)^+_y dx \leq 0.$$

Now, on the set $p > (h_1-y)^+$ we have $p>0$ and thus $\chi = 1$ and we obtain

$$\int_\Omega \nabla p \cdot \nabla (p-(h_1-y)^+)^+ + (p-(h_1-y)^+)^+_y dx \leq 0.$$

Which can be written

$$\int_{[y\leq h_1]} |\nabla (p-(h_1-y))^+|^2 dx + \int_{[y>h_1]} |\nabla p|^2 + p_y dx \leq 0. \tag{3.13}$$

Now taking $\xi = (h_1-y)^+$ in (P)(iii) we obtain, since this flmetion vanishes on S_3,

$$\int_{[y>h_1]} p_y + \chi dx \leq 0.$$

Noting that $\chi^2 \leq \chi$ and $\chi p_y = p_y$ a.e. we obtain

$$\int_{[y>h_1]} \chi p_y + \chi^2 dx \leq 0. \tag{3.14}$$

Adding (3.13) and (3.14) we get

$$\int_{[y\leq h_1]} |\nabla (p-(h_1-y))^+|^2 dx + \int_{[y>h_1]} p_x^2 + (p_y+\chi)^2 dx \leq 0. \tag{3.15}$$

From which we deduce

$$\nabla p = (0,-\chi) \qquad \text{on} \quad [y>h_1]. \tag{3.16}$$

Thus, on any connected component C of $[p>0]$ that intersects $[y>h_1]$ we have

$$p = k-y.$$

Indeed, $p=k-y$ on the part of C intersecting $[y>h_1]$. By analytic continuation since, by (3.1), $\nabla p = 0$ in C and thus p is analytic in C, we have $p=k-y$ on C.
Outside of these connected components we have $p=0$ and thus (3.12) or, by (3.15),

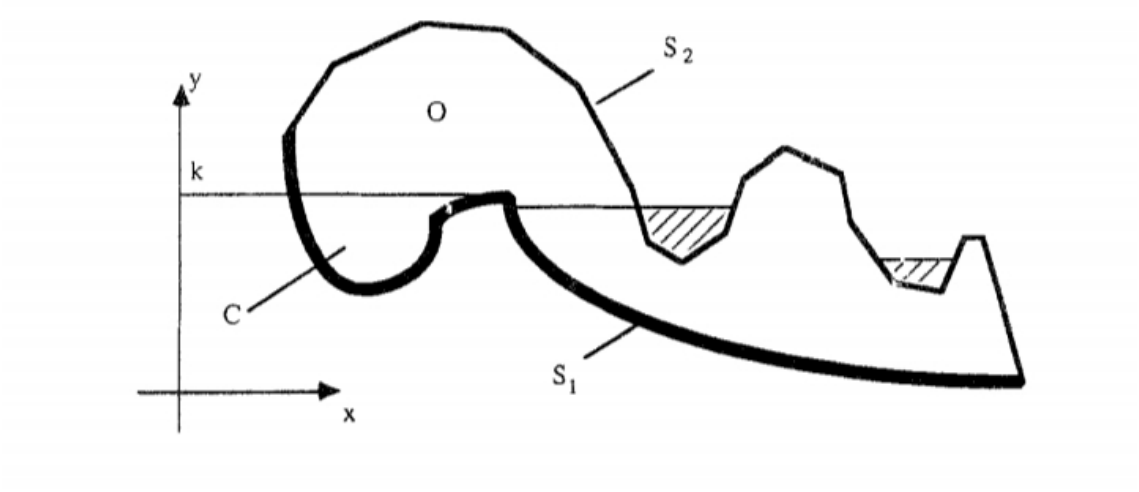
$$\nabla (p-(h_1-y)^+)^+ = 0,$$

when $y<h_1$. Thus, on any connected component of the set $[y<h_1]$ we have

$$(p-(h_1-y)^+)^+ = \text{Cst}.$$

However, any such component touches the line $y=h_1$ somewhere where the constant

is 0 since $p = 0$ (if p was not equal to 0 we would be on a component C of $[p > 0]$ intersecting $[y > h_1]$). If the constant vanishes, then clearly (3.12) holds.
This completes the proof □



Remark 3.3. Consider, for instance, the situation shown in Figure 2. Then for $y > k$ we have $p = 0$. If not, we would have $p = h - y$ for $h > k$ and this would lead to a contradiction to $p = 0$ on S_2. For the same reason we have $(p, \chi) = (0, 0)$ in O (see Figure 2 and (3.16)).

Taking $\xi = \pm p\chi_c$ (χ_c is the characteristic function of C)in (P)(iii) we get

$$\int_c |\nabla p|^2 + \chi p_y \, dx = 0.$$

Taking $\xi = (y - k)\chi_c$ we get

$$\int_c p_y + \chi \, dx \leq 0,$$

and procceeding as above we obtain

$$\nabla p = (0, -\chi) \qquad \text{in} \qquad C,$$

and thus

$$p = (h - y)^+ \qquad \text{in } C,$$

for some $h \leq k$. Conversely, any function $((h-y)^+, \chi_{[y<h]})$ on C extended by (p, χ) outside of C is a solution to (P). Hence, as in the case of Dirichlet boundary conditions, uniqueness results for this problem will only be up to "pools," i.e., functions $((h - y)^+, \chi_{[y<h]})$, see [13] .

We would now like to stress some other differences of this model compared with the classical one, i.e., the one with Dirichlet boundary conditions. We do that through some examples.

Chapter 4

Different examples

Sommaire

4.0.1 Some Particular Examples

The first case we consider is described in Figure 4.1, where we assume that $S_1 = \varnothing$ and β is independent of x. Then we would like to show that in this case when

$$0 \leq \frac{\beta(\varphi)}{\nu_y} \leq 1. \tag{4.1}$$

The only solution to (P) is given by

$$(p, \chi) = \left(0, \left(\frac{\beta(\varphi)}{\nu_y}\right)\chi_I\right),$$

Where χ_I denotes the characteristic function of the region below S_3, denoted by I in Figure 3, and ν_y is the y entry of the unit outward normal to F on S3 (we have denoted by $\beta(\varphi)/\nu_y$ the function independent of y equal to $\beta(\varphi)/\nu_y$ on S_3). Thus this is a particular case where uniqueness holds. However, we see that we cannot, in general, expect χ to be a characteristic function of a set (χ is not if $\beta(\varphi)/\nu_y < 1$).

Moreover, the porous medium is completely unsaturated. So, the situation is quite different to the one in the classical dam problem (see [13]).
So, let us prove

Proposition 4.1

Assume that (4.1) holds and that β is nondecreasing with $\beta(0) = 0$.
Then the problem (P) corresponding to Figure 4.1 has a unique solution given by

$$(p, \chi) = \left(0, \left(\frac{\beta(\varphi)}{\nu_y}\right)\chi_I\right). \tag{4.2}$$

Proof. First let us check that (p, χ) given by (4.2) satisfies (P). We only have to check

(P)(iii).For that note that since $\beta(\varphi)/\nu_y$ is a function of x only

$$\int_\Omega \nabla p\cdot\nabla\xi+\chi\xi_y\,dx-\int_{S_3}\beta(\varphi-p)\cdot\xi\,d\sigma(x)$$

$$=\int_I\left(\frac{\beta(\varphi)}{\nu_y}\xi_y\right)_y dx-\int_{S_3}\beta(\varphi)\cdot\xi\,d\sigma(x).$$

$$=\int_I\frac{\beta(\varphi)}{\nu_y}\xi_y\,dx-\int_{S_3}\beta(\varphi)\cdot\xi\,d\sigma(x).$$

$$=\int_{\partial I\setminus S_3}\frac{\beta(\varphi)}{\nu_y}\cdot n_y\cdot\xi\,d\sigma(x)\le 0. \tag{4.3}$$

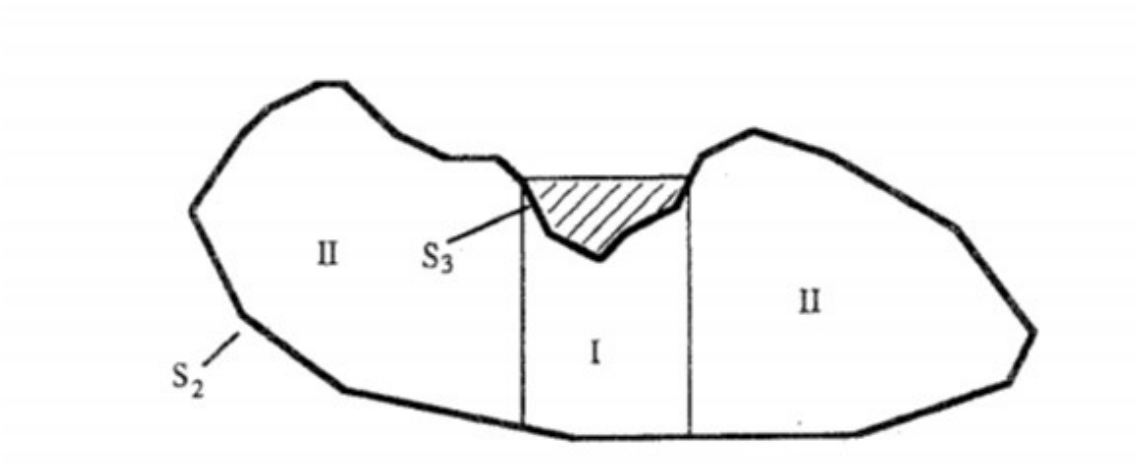


Figure 4.1

For any $\xi\ge 0$ on S_2, n_y denotes the y entry of the outward unit normal to $\partial I\setminus S_3$ and thus $n_y\le 0$ (see Figure 4.3). ∂I denotes the boundary of ∂I . So,(p,χ) given by (4.2) is a solution to (P).Note that from (4.3) we easily deduce that

$$\int_\Omega\chi\cdot\xi_y\,dx-\int_{S_3}\beta(\varphi)\xi\,d\sigma(x)=0,\qquad\forall\xi\in H^1(\Omega),\quad\xi=0\qquad\text{on}\qquad\partial I\cap S_2. \tag{4.4}$$

Let us now denote by (p',χ') another solution to (P).Thus, we have

$$\int_\Omega\nabla p'\cdot\nabla\xi+\chi'\xi_y\,dx-\int_{S_3}\beta(\varphi-p')\cdot\xi\,d\sigma(x)\le 0,$$

$$\forall\xi\in H^1(\Omega),\quad\xi\ge 0\ \text{ on } S_2. \tag{4.5}$$

Taking $\xi=p'$ in (4.5) and $\xi=-p'$ in (4.4) and adding we get

$$\int_\Omega|\nabla p'|^2+(\chi'-\chi)\cdot p'_y\,dx-\int_{S_3}\big(\beta(\varphi-p')-\beta(\varphi)\big)\cdot p'\,d\sigma(x)\le 0. \tag{4.6}$$

However

$$\begin{aligned}\int_\Omega (\chi'-\chi)p'_y\,dx &= \int_\Omega (1-\chi)p'_y\,dx\\ &= \int_{\mathbf{I}}\left(1-\frac{\beta(\varphi)}{v_y}\right)p'(y)dx+\int_{\mathbf{II}}p'(y)\\ &= \int_{\mathbf{I}}\left(\left(1-\frac{\beta(\varphi)}{v_y}\right)p'\right)_y dx\\ &= \int_{S_3}\left(1-\frac{\beta(\varphi)}{v_y}\right)\cdot v_y\cdot p'\,d\sigma(x)\geq 0. \end{aligned} \tag{4.7}$$

Moreover, from the fact that β is nondecreasing we have

$$-\int_{S_3}(\beta(\varphi-p')-\beta(\varphi))\cdot p'\,d\sigma(x)\geq 0. \tag{4.8}$$

Combining(4.6), (4.7), and (4.8)we deduce

$$\int_\Omega |\nabla p'|^2\,dx\leq 0,$$

And thus $p'=p=0$. However, now (4.5) reads

$$\int_\Omega \chi'\xi_y\,dx-\int_{S_3}\beta(\varphi)\cdot\xi\,d\sigma(x)\leq 0,\quad \forall\xi\in H^1(\Omega),\qquad \xi\geq 0 \text{ on } S_2. \tag{4.9}$$

Taking $\xi\in\mathscr{D}(\Omega)$, or using (3.1), we deduce

$$\chi'_y=0\qquad\text{or}\qquad \chi'=\chi'(x).$$

Then if we denote by T and B respectively the upper and the bottom part of Γ we deduce from (4.9), for $\xi\in H^1(\Omega)$, $\xi\geq 0$ on S_2,

$$\begin{aligned}&\int_\Omega(\chi'\xi)_y\,dx-\int_{S_3}\beta(\varphi)\cdot\xi\,d\sigma(x)\geq 0\\ &=\int_T\chi'\cdot\xi\cdot v_y\,d\sigma(x)+\int_B\chi'\cdot\xi\,v_y\,d\sigma(x)-\int_{S_3}\beta(\varphi)\cdot\xi\,d\sigma(x).\end{aligned} \tag{4.10}$$

Taking in (4.10) any ξ that vanishes on S_2 we get

$$\int_{S_3}(\chi' v_y-\beta(\varphi))\cdot\xi\,d\sigma(x)=0.$$

For such a ξ . Hence $\chi'=\beta(\varphi)/v_y$ on S_3 . Then (4.10) becomes

$$0\geq\int_{T\setminus S_3}\chi'\cdot\xi\cdot v_y\,d\sigma(x),\qquad\forall\xi\in H^1(\Omega),\qquad\xi\geq 0 \text{ on } S_2.$$

Taking $\xi \geq 0$ on T, $\xi = 0$ on B we deduce

$$0 \geq \int_{T \setminus S_3} \chi' \cdot \xi \cdot \nu_y \, d\sigma(x),$$

Hence $\chi' = 0$ on $T \setminus S_3$ and the result follows. (We have assumed $\nu_Y > 0$ on T,) This completes the proof of the proposition.

□

Remark 4.1. In general, we do not know if uniqueness of a solution to (P) holds modulo "pools". However, the above example and the one below seem to indicate that this is the case

Remark 4.2. If we assume that a portion of B is impermeable, then $p = 0$ ceases to be a solution. Indeed, if it were the case, we would arrive at (4.9) and consequently (4.2), as shown in the preceding proof. However, it becomes evident that we would lack condition (P)(iii). Therefore, in this case, the solution becomes saturated, meaning that the set $[p > 0]$ has a positive measure.
The same happens when $\beta(\varphi)/\nu_y > 1$ on a set of positive measure.

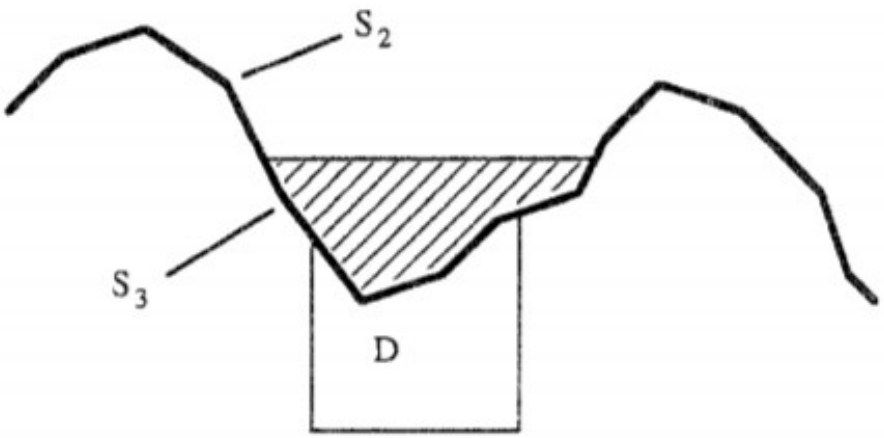


Figure 4.2

Remark 4.3. If $p = 0$ on some "rectangle" D below S_3 as shown in Figure 4.2, then we have necessarily

$$\chi = \frac{\beta(\varphi)}{\nu_y} \qquad \text{on} \qquad D. \tag{4.11}$$

In particulartt due to (P)(i),this situation is impossible (see also(2.6))when

$$\frac{\beta(\varphi)}{\nu_y} > 1 \qquad \text{on} \qquad \partial D \cap S_3.$$

To prove (4.11) note that if ξ is a function vanishing on $\partial D \setminus S_3$ and if we extend this

function by 0 outside D we obtain

$$\int_D \chi \cdot \xi_y - \int_{S_3 \cap \partial D} \beta(\varphi) \cdot \xi \, d\sigma(x) = 0. \tag{4.12}$$

Thus

$$\chi_y = 0 \quad \text{on } D \qquad \text{or} \qquad \chi = \chi(x) \quad \text{on } D.$$

Then (4.12) reads

$$\int_{S_3 \cap \partial D} (\chi \nu_y - \beta(\varphi)) \cdot \xi \, d\sigma(x) = 0.$$

For any ξ vanishing on $S_3 \cap \partial D$ and the result follows.

We want to demonstrate that the situation outlined in Remark 8, wher $p = 0$ below S_3, could occur even when the lower section of Γ is impermeable (as stated in Remark 7). Essentially, saturation depends heavily on β . In this case, we will also establish that the solution to problem (P) is unique and is connected through the Baiocchi transform (see [18]) to he solution of a specific variational inequality.

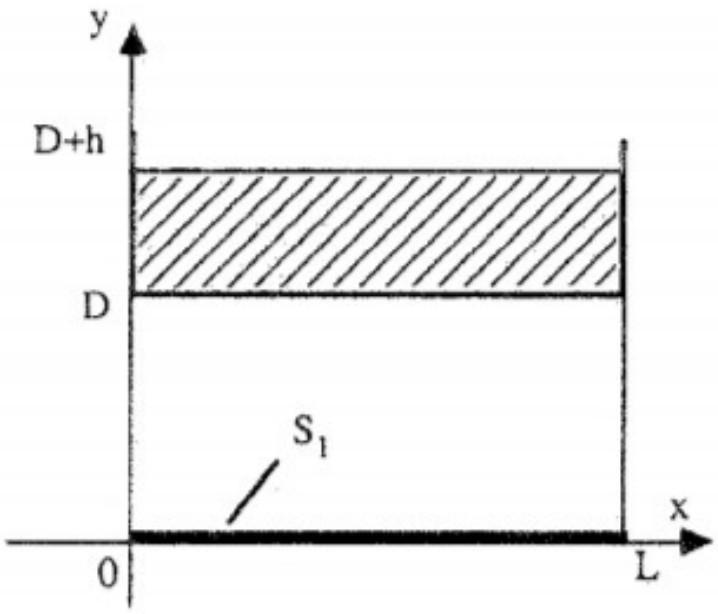


Figure 4.3

We consider the porous medium depicted in Figure (4.3), where a rectangle is described, with its bottom assumed to be impervious, its top covered with water, and its lateral sides contact with the atmosphere . We also assume that β is independent of x and nondecreasing. The level of water is denoted by h, D is the thickness of the porous

medium, and L is its horizontal size .

Remark that if $p=0$ on S_3, then $\beta(\varphi-p)=\beta(h)$ on S_3.So,consider K the closed convex set of $H^1(\Omega)$ defined by

$$K=\left\{v\in H^1(\Omega)|\, v\geq 0, v=0 \text{ on } S_2\cup S_2,\ v=\frac{\beta(h)}{2}\cdot x(L-x) \text{ on } S_1\right\} \tag{4.13}$$

and u the solution of the variational inequality

$$u\in K \qquad \int_\Omega \nabla u\cdot\nabla(v-u)dx\geq\int_\Omega(\beta(h)-1)\cdot(v-u)dx, \qquad \forall v\in K. \tag{4.14}$$

It is evident that there exists a unique solution to this variational inequality (refer to [20]). Furthermore, the boundary data of u admits a $C^\infty(\Omega)$ extension. By combining well-established techniques of regularity for variational inequalities (see [16]) and for elliptic problems in domains with corners (see [21]), we can demonstrate that, for any $p\geq 1$ and any $\alpha\in(0,1)$, we obtain:

$$u\in W^{2,p}(\Omega)\cap C^{1,\alpha}(\overline{\Omega}). \tag{4.15}$$

Moreover, we have

Lemma 4.1. *Let u be the solution of the variational inequality (4.13), (4.14). Assume that*

$$\beta(h)=\beta<\frac{4D^2}{4D^2+L^2}. \tag{4.16}$$

Then

$$0\leq u\leq\frac{1-\beta}{2}\left\{\left(\frac{L}{2}\sqrt{\frac{\beta}{1-\beta}}-y\right)^+\right\}^2 \qquad \text{a.e.in } \Omega. \tag{4.17}$$

In particular $u(x,y)$ vanishes when $y\in[(L/2)\sqrt{\beta/(1-\beta)},D]$.

Proof. First note that (4.16) is equivalent to

$$\frac{L}{2}\sqrt{\frac{\beta}{1-\beta}}<D.$$

For $K\in((L/2)\sqrt{\beta/(1-\beta)},D)$ consider the function

$$z=\frac{1-\beta}{2}\cdot(K-y)^2.$$

We have

$$-\Delta z=\beta-1.$$

and thus z is the solution of the variational inequality

$$z \in K' = \left\{ v \in H^1(\Omega) | v \geq 0\,,\ v = \frac{1-\beta}{2} \cdot (k-y)^2 \text{ on } \Gamma \right\}.$$

$$\int_\Omega \nabla z \cdot \nabla (v-z) dx \geq \int_\Omega (\beta(h)-1) \cdot (v-z) dx \qquad \forall v \in K'.$$

Moreover on S_1 we have

$$z = \frac{1-\beta}{2} \cdot K^2 \geq \frac{\beta}{2} \cdot \frac{L^2}{4} \geq \frac{\beta}{2} \cdot x(x-L) = u.$$

Since such variational inequality has a solution that varies monotonically with respect to the data (see [16] and [7]) we have

$$0 \leq u \leq z = \frac{1-\beta}{2} \cdot (k-y)^2 \qquad \text{in } \Omega.$$

and the result follows since the inequality holds for any $K \in [(L/2)\sqrt{\beta/(1-\beta)}, D]$. □

Then we can prove

Theorem 4.1. *Assume that we are in the case shown in Figure (4.3) and that (4.16) holds.*
Let (p, χ) be a solution to (P). If we set

$$u(x,y) = \int_y^D p(x,t) dt, \tag{4.18}$$

then u is the solution of the variational inequality (4.13) (4.14).

Proof. Let us denote provisionally by u' the integral in (4.18) and by u the solution of the variational inequality (4.13) (4.14) . First, we cleary have

$$u' = 0 \qquad \text{on} \qquad S_2 \cup S_3. \tag{4.19}$$

Next taking $\xi = \psi(x)$ where $\psi(x) \in \mathscr{D}(0, L)$ in (P)(iii) we deduce

$$\int_\Omega p_x \psi_x dx - \int_{S_3} \beta(h - p(x,D)) \psi d\sigma(x) = 0, \tag{4.20}$$

Which can also be written as

$$\int_0^L \left\{ \left(\int_0^D p(x,t) dt \right)_x \cdot \psi_x - \beta(h - p(x,D)) \right\} \cdot \psi dx = 0, \qquad \forall \psi \in \mathscr{D}(0,L).$$

Thus in the distributional sense we have

$$-u'(x,0)_{xx} = \beta(h - p(x,D)) \leq \beta(h) = -u(x,0)_{xx}. \tag{4.21}$$

And thus, by maximum principle,

$$u'(x,0) \leq u(x,0) \qquad \text{on} \quad S_1. \tag{4.22}$$

(Recall that both $u'(x,0)$ and $u(x,0)$ vanish at the endpoints of (0,L)).

If follows from (4.19),(4.22) that

$$u' \leq u \qquad \text{on } \partial\Omega. \tag{4.23}$$

Next, when ζ is a smooth function vanishing on S_2,

$$\xi = \int_0^y \zeta(x,t)dt.$$

is a suitable test function for (P)(iii) .So, noting that

$$p = -u'_y.$$

We deduce

$$\int_\Omega -\nabla u'_y \cdot \nabla \int_0^y \zeta(x,t)dt + \chi\zeta dx - \int_{S_3} \beta(h - p(x,D)) \cdot \int_0^D \zeta(x,t)dt\, d\sigma(x) = 0.$$

This also reads

$$\int_\Omega -\nabla u'_y \cdot \nabla \int_0^y \zeta(x,t)dt + (\chi - \beta(h - p(x,D))) \cdot \zeta dx = 0. \tag{4.24}$$

(dx is the lebesgue measure on $\mathbf{R}^2$). Note that

$$-\nabla u'_y \cdot \nabla \int_0^y \zeta(x,t)dt = \left(-\nabla u' \cdot \nabla \int_0^y \zeta(x,t)dt\right)_y + \nabla u' \cdot \nabla\zeta. \tag{4.25}$$

Then, (4.23) becomes

$$\begin{aligned}
&\int_\Omega \nabla u' \cdot \nabla\zeta + (\chi - \beta(h - p(x,D))) \cdot \zeta dx \\
&= \int_\Omega \left(\nabla u' \cdot \nabla \int_0^y \zeta(x,t)dt\right)_y dx \\
&= -\int_\Omega (p \cdot \zeta)_y dx + \int_\Omega \left(u'_x \cdot \int_0^y \zeta_x(x,t)dt\right)_y dx.
\end{aligned}$$

Note that the function

$$\left(u'_x \cdot \int_0^y \zeta_x(x,t)dt\right)_y \in L^2(\Omega),$$

and that $u'_x = 0$ on S_3 since $u' = 0$ there.Moreover, $\int_0^y \zeta_x(x,t)dx = 0$ on S_1. So by the divergence theorem we deduce

$$\int_\Omega \nabla u' \cdot \nabla\zeta + (\chi - \beta(h - p(x,D))) \cdot \zeta dx = \int_{S_1} P\zeta d\sigma(x) - \int_{S_3} P\zeta d\sigma(x). \tag{4.26}$$

For every smooth ζ vanishing on S_2 . By an easy density argu=ment, (4.26) holds for every $\zeta \in H^1(\Omega), \zeta = 0$ on S_2.

Let us set

$$K' = \left\{v \in H^1(\Omega) | v \geq 0, v = u' \text{on } \partial\Omega\right\}.$$

Then, clearly, taking $\zeta = v - u'$ in (4.26) we deduce

$$\int_\Omega \nabla u' \cdot \nabla(v - u')dx = \int_\Omega (\beta(h - p(x,D)) - \chi) \cdot (v \cdot u')dx, \qquad \forall v \in K'. \tag{4.27}$$

Now, we have

$$[u' > 0] = [p > 0]. \tag{4.28}$$

Indeed, if $u'(x_0, y_0) > 0$, then $p(x,y) > 0$ for some $(x,y) \in \Omega$ such that $y > y_0$ and thus, by Proposition 2, $p(x_0, y_0) > 0$. Conversely, if $p(x_0, y_0) > 0$, then $p(x,y) > 0$ in a neighborhood of (x_0, y_0) and, by (4.18), $u'(x_0, y_0) > 0$. This proves (4.28). Then, from (4.28), we deduce that $\chi u' = u'$ and thus

$$\chi(v - u) = \chi v - u' \leq v - u'.$$

By (4.27) this implies that u' satisfies

$$u' \in K', \quad \int_\Omega \nabla u' \cdot \nabla(v - u')dx \geq \int_\Omega (\beta(h - p(x,D)) - 1) \cdot (v \cdot u')dx, \qquad \forall v \in K'. \tag{4.29}$$

Since

$$(\beta(h - p(x,D)) - 1) \leq \beta(h) = \beta, \qquad u' \leq u \quad \text{on} \quad \partial\Omega,$$

We deduce from the monotonicity of the solution of variational inequalities with respect to the data (see [7]) that

$$u' \leq u \qquad \text{on } \Omega$$

Then, by Lemma 1, $u' = 0$ on a neighborhood of S_3 and so does p. From (4.21) we now have

$$-u'_{xx} = \beta = -u_{xx},$$

Hence $u' = u$ on $\partial\Omega$ and $K' = K$. Moreover, (4.29) becomes equivalent to (4.13), (4.14) and $u' = u$. The result follows. □

We also have

Lemma 4.2. *There exists a smooth function $\varphi : (0, L) \longrightarrow \boldsymbol{R}^+$ such that*

$$[u > 0] = \left\{(x, y) \in \Omega | 0 < y < \varphi(x)\right\}.$$

Proof. Since u is continuous we know that u is positive in a neighborhood of any point of S_1 . Morever, by (4.18) u is decreasing in y so that if $u(x_0, y_0) > 0$, then $u(x_0, y_0) > 0$ for any $y < y_0$. Then result then follows by setting

$$\varphi(x) = \sup y | u(x, y) > 0.$$

The smoothness of φ follows from the well-Known results on variational in-equalities (see [9] and [10]).

Then we can show

Corollary 4.1. Assume that we are in the case shown in Figure (4.3) and that (4.16) holds.Then the solution (p, χ) to (P) is unique and is given by

$$p = -u_y. \tag{4.30}$$

$$\chi = \begin{cases} \beta & \text{on} \quad [u = 0]. \\ 1 & \text{on} \quad [u > 0]. \end{cases} \tag{4.31}$$

Where u is the solution of the variational inequality (4.13), (4.14).

Proof. If (p, χ) is a solution to (P) , and we know that such a solution exists, then, by Theorem (4.1), we have

$$p = -u_y.$$

Next by Lemma 2 and Remark 8 we deduce that (4.31)holds. □

Conclusion

This work was an important applied study of a complex mathematical problem. After presenting the method for deriving the mathematical model of the phenomenon, we proved the existence and uniqueness of the solution under real conditions for dam-related problems. Additionally, we identified many properties of the solution, which strengthens the mathematical work in studying physical and biological phenomena and opens up prospects for finding solutions to problems we face in everyday life.

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