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The Duo-Free and Critically Duo-Free Graphs

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★ ★ ★ *Nihad* ★ ★ ★

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Abstract

Let $G(V, A)$ be a digraph. A subset $M \subset V$ is a **module** of G if each point outside M has the same behavior to all points of M. Clearly, the empty set, the singleton and the set of vertices V are **modules** of G known as **trivial modules**. A digraph is **indecomposable** if all its modules are trivial, otherwise it is **decomposable**. The central notion of this work is the **duo** which is a module of cardinality 2. The digraphs which do not admit any duo are called **duo-free digraphs**. An important tool in this work is the notion of **duo-free-minimal digraphs** defined as follows : Let G be a duo-free digraph. G is **duo-free-minimal** for elements $x_1, ..., x_k$ of V if for each proper subset X of V containing $x_1, ..., x_k$, the induced subdigraph $G[X]$ has at least one **duo**. The objective of our study is to present the **characterization** of the **duo-free-minimal tournaments** and **graphs** for one or two vertices. Moreover, we describe the **duo-free minimal** k**-vertex digraph** for one vertex where $k \in \{3, 4, 5\}$.

Résumé

Soit $G = (V, A)$ un digraphe. Un sous-ensemble $M \subset V$ est un **module** de G si chaque sommet extérieur de M a le même comportement en vers tous les sommets du module M. Clairement, l'ensemble vide, les singletons et l'ensemble de tous les sommets V sont des **modules** de G, appelés **modules triviaux**. Un digraphe est dit **indécomposable** si tous ses modules sont triviaux, sinon il est **décomposable**.

La notion centrale de ce travail est le **duo**, qui est un module de cardinalité 2. Les digraphes qui n'admettent aucun duo sont appelés **digraphes sans-duo**.

Un outil important dans ce travail est la notion de **digraphes minimal sans-duo**, définie comme suit : Soit G un digraphe sans-duo. On dit que G est **minimal sans-duo** pour les sommets $x_1, ..., x_k$ de V si pour tout sous-ensemble propre $X \subset V$ contenant $x_1, ..., x_k$, le sous-digraphe induit $G[X]$ admet au moins un **duo**.

L'objectif de notre étude est de présenter une **caractérisation** des **tournois et digraphes minimal sand-duo** pour un ou deux sommets. De plus, nous décrivons le **digraphe minimal sans-duo à k sommets** pour un sommet, où $k \in \{3, 4, 5\}$.

Introduction

Among the various mathematical tools employed to address specific classes of problems in operations research, graph theory emerges as one of the most significant and efficient methodologies. It is concerned with analyzing relationships between entities that are modeled as vertices and the connections between them. This approach encompasses a broad range of problems, all sharing the common characteristic of being representable by a structured diagram known as a graph. A graph consists of a set of nodes (vertices), representing individuals, objects, or situations, which are interconnected by links (edges) that illustrate the relationships or interactions among them.

The key concept at the center of this research is the duo a module of size 2 The duo notion appears to be a specific case of module, and therefore risks to be not interesting. On the other hand, from a heuristic viewpoint, it is quite different. In fact, a digraph having a finite chain module or an arc-complete module or an arc-free one is equivalent to having a duo. This allows us to study many properties of several digraphs basing on the duo instead of the large size modules. For

example, the characterization of some critical tournaments may be given in terms of duo, because they are duo-free and the removal of any vertex of them creates a duo. Notice that the first appearance of this notion was in 2008, in the theory of social psychology due to J.L. Beauvois and G. Lopez [4]. In fact, the concept of module has proven to be insufficient in terms of psychology because it does not have a "symmetric" intuitive, which is not the case for the duo. The concept of the duo has been chiefly the subject of several papers, such as [5, 7, 9, 10, 12]. In 2005, J.F. Culus and B. Jouve [9, 10] characterized the critically duo-free tournaments.

An important tool on the primality is the notion of minimal digraphs defined as follows : Let G be an indecomposable digraph. G is *minimal* for elements $x_1, ..., x_k$ of $V(G)$ if for each proper subset X of $V(G)$ containing $x_1, ..., x_k$ such that $|X| \geqslant 3$, $G[X]$ is decomposable. This notion has attracted considerable attention from many researchers. In 1998, A. Cournier and P. Ille [8] characterized the minimal indecomposable digraphs for one or two vertices. M. Alzohairi and Y. Boudabbous [2] in 2014 characterized the minimal triangle-free graphs for three vertices. Recently, M. Alzohairi [1] described the triangle-free graphs which are minimal for some non-stable 4-vertex subset. The thesis is structured into three main chapters :

Chapter 1 : Basic Concepts and Definitions. This introductory chapter presents the fundamental notions necessary for understanding our work, including digraphs, tournaments, duos, and duo-free graphs. Special attention is given to some classical graphs, such as the tournament C_3, which play a central role in the results

discussed in the following chapters.

Chapter 2 : Minimal Duo-Free Tournaments. This chapter is devoted to the description of the duo-free-minimal tournament. It is organized as follows : Section 1 gives some definitions and notations concerning duo-free digraphs. The characterization of the duo-free-minimal tournament for one vertex or two vertices is presented in Section 2.

Chapter 3 : Minimal Duo-Free Digraphs. In this chapter, we present the characterization of the minimal duo-free digraphs for one or two vertices. We start in Section 1 by citing some important definitions and notations. Section 2 gives a characterization of the duo-free graphs for one or two vertices. In Section 3, we describe the duo-free-minimal k-vertex digraphs for one vertex, where $k \in \{3, 4, 5\}$.

This thesis thus contributes to the characterization of duo-free digraphs and opens promising perspectives for further research in this field.

Chapter 1

Concepts and Basics

This chapter provides the reader with the basics of graph theory necessary for a good understanding of the subjects addressed in the rest of this work. We recall some classical definitions.

1.1 Preliminary definitions

This section is devoted to the definition of the basic notions of our work.

Digraph : A *digraph* $G = (V(G), A(G))$ consists of a non empty *vertex* set $V(G)$ and an *arc* set $A(G)$, where an arc is an ordered pair of distinct vertices. A digraph is said to be *finite* if its vertex set is finite. The *cardinality* of the vertex set $V(G)$ is called the *order* of G, commonly denoted by $|V(G)|$. The *size* of G is the cardinality of $A(G)$, denoted by $|A(G)|$.

Some notations are needed. Let G be a digraph. For every two distinct elements x, y of $V(G)$.

- x—— y signifies that $(x, y), (y, x) \in A(G)$.

- $x \ldots y$ (or $x \quad y$) signifies that $(x, y), (y, x) \notin A(G)$.
- $x \longrightarrow y$ signifies that $(x, y) \in A(G)$, $(y, x) \notin A(G)$.

For $x \in V(G)$ and $Y \subset V(G) \setminus \{x\}$, $x \longrightarrow Y$ (resp. $Y \longrightarrow x$) signifies that for every $y \in Y$, $x \longrightarrow y$ (resp. $y \longrightarrow x$). We define by the same manner $x \text{—} Y$, $x \ldots Y$. For $X \neq Y \subset V(G)$, $X \longrightarrow Y$ signifies that for every $x \in X$, $x \longrightarrow Y$. $X \text{—} Y$ and $X \ldots Y$ are defined in the same way. Consider an equivalence relation, denoted by $\sim$, between the ordered pairs of distinct vertices of G as follows : for $x \neq y \in V(G)$ and $u \neq v \in V(G)$ said that $(x, y) \sim (u, v)$ if ($x \longrightarrow y$ and $u \longrightarrow v$), or ($y \longrightarrow x$ and $v \longrightarrow u$), or ($x \text{—} y$ and $u \text{—} v$), or ($x \ldots y$ and $u \ldots v$). For $Y \subset V(G)$ and for $x \in V(G) \setminus Y$, $x \sim Y$ signifies that for any $y, y' \in Y$, $(x, y) \sim (x, y')$. For $X, Y \subset V(G)$, $X \sim Y$ signifies that for every $x \in X$, $x \sim Y$. Otherwise, we note $X \nsim Y$.

Subdigraph : Given a digraph G, with each subset X of $V(G)$ associate the *subdigraph* $G[X]$ of G induced by X defined by $V(G[X]) = X$ and $A(G[X]) = A(G) \cap (X \times X)$. For $X \subset V(G)$, the subdigraph $G[V(G) \setminus X]$ is also denoted by $G - X$ and by $G - x$ if $X = \{x\}$.

Symmetric digraph (**or graph**) : A *symmetric digraph* (*or graph*) is a digraph G such that for $x \neq y \in V(G)$, $(x, y) \in A(G)$ if and only if $(y, x) \in A(G)$. Let G be a graph. Given a vertex x of G, the *neighbourhood* of x in G, denoted by $N_G(x)$, is the set $N_G(x) = \{y \in V(G) \setminus \{x\} : (x, y) \in A(G)\}$. The cardinality of $N_G(x)$ is called the *degree* of x and is denoted by $d_G(x)$. When $N_G(x) = \emptyset$, the vertex x is said to be an *isolated* vertex of G. A graph G is *complete* (resp. *empty*) when for all $x \neq y \in V(G)$, $(x, y) \in A(G)$ (resp. $(x, y) \notin A(G)$).

Tournament. Subtournament : A digraph T is a *tournament* if for $x \neq y \in V(T)$,

$(x, y) \in A(T)$ if and only if $(y, x) \notin A(T)$. Each induced subdigraph of a tournament is called a *subtournament*.

Dual. Complement : Given a digraph G, the *dual* (resp. the *complement*) digraph of G is the digraph G^* (resp. $\overline{G}$) defined as follows : let $x \neq y \in V(G)$, $(x, y) \in A(G^*)$ (resp. $(x, y) \in A(\overline{G})$) when $(y, x) \in A(G)$ (resp. $(x, y) \notin A(G)$).

Connectivity : Let G be a graph. *An equivalence relation* $\mathcal{R}$ is defined on $V(G)$ as follows : for all $x \in V(G)$, $x\mathcal{R}x$ and for $x \neq y \in V(G)$, $x\mathcal{R}y$ if there is a sequence $x = x_0, \cdots, x_n = y$ of vertices of G such that for all $i \in \{0, \cdots, n-1\}$, $(x_i, x_{i+1}) \in A(G)$. The equivalence classes of $\mathcal{R}$ are called *connected components* of G. A non trivial connected component of G is a connected component of G which is not reduced to a singleton. Say that G is a *connected graph* if it has a unique connected component.

Given two digraphs G and G' such that $V(G) \cap V(G') = \emptyset$, the disjoint union of the digraphs G and G' is the digraph $G \oplus G' = (V(G) \cup V(G'), A(G) \cup A(G'))$.

Path. Cycle : A *path*, denoted by P_n, is a graph defined on $\mathbb{N}_n = \{0, \ldots, n-1\}$, $n \geqslant 2$ by for $i \neq j \in \mathbb{N}_n$, $(i, j) \in A(P_n)$ if $| i - j | = 1$ (see Figure 1.1). A *cycle*, denoted by C_n, is a graph defined on $\mathbb{N}_n = \{0, \ldots, n-1\}$, $n \geqslant 3$ and it is obtained from P_n by adding the arcs $(0, n-1)$ and $(n-1, 0)$.

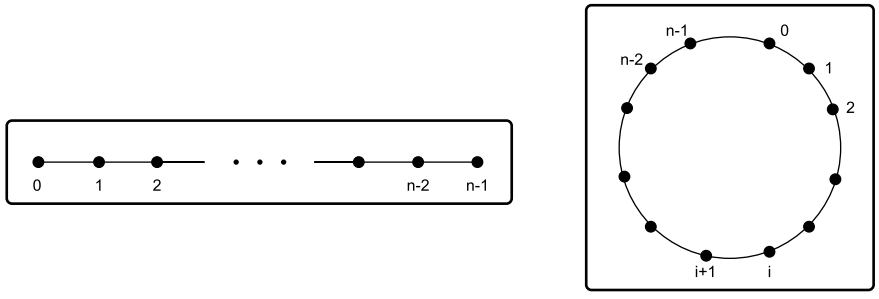


FIGURE 1.1 – Path P_n and cycle C_n.

Poset. Total order : *Poset* is a digraph G such that for every $x, y, z \in V(G)$, if $(x, y) \in A(G)$ and $(y, z) \in A(G)$, then $(x, z) \in A(G)$. For instance, the digraph $N = (\{1, 2, 3, 4\}, \{(2, 1), (2, 3), (4, 3)\})$ is a poset. A poset which is a tournament is called a *total order* or a *chain*. (see Figure 1.2) We denote by $\underline{n}$ the total order whose vertex set is $\{1, ..., n\}$ and whose arcs are the ordered pairs (i, j) such that $1 \leqslant i < j \leqslant n$. Each n-vertex finite chain is called *n-chain*.

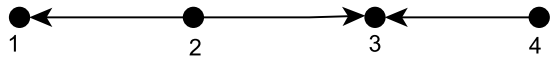


FIGURE 1.2 – Digraph N

Isomorphy : Let G and G' be two digraphs. A bijection f from $V(G)$ onto $V(G')$ is an *isomorphism* from G onto G' provided that for any $x, y \in V(G)$, $(x, y) \in A(G)$ if and only if $(f(x), f(y)) \in A(G')$. The digraphs G and G' are then said to be *isomorphic* which is denoted by $G \simeq G'$. A digraph G' embeds in G if G' is isomorphic to a subdigraph of G. Otherwise, say that G omits G'.

A graph G is *triangle-free* if it does not have any induced subgraph isomorphic to a triangle.

Module. Primality : Let G be a digraph. A subset M of $V(G)$ is a *module* (or *interval*) of G (see Figure 1.3) if each point outside M has the same behavior to all points of M i.e $\forall a, b \in M$ and $\forall x \in V(G) \setminus M$, $(a, x) \in A(G)$ if and only if $(b, x) \in A(G)$, and

$(x, a) \in A(G)$ if and only if $(x, b) \in A(G)$. The *trivial modules* are the empty-set $\emptyset$, the vertex set $V(G)$ and the singletons $(\{x\}, x \in V(G))$. For instance, a chain which is a module is called a *chain module*.

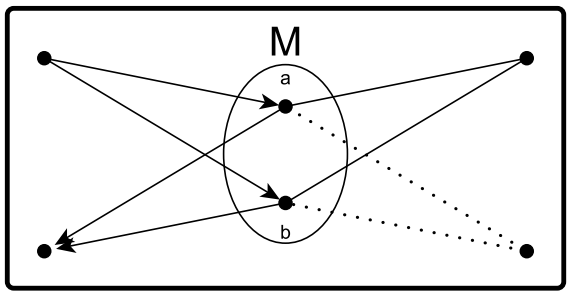


FIGURE 1.3 – Module.

A digraph is *indecomposable* (or *prime*) if all its modules are trivial ; otherwise, it is *decomposable*. For instance, the 5-vertex graph Q_5 defined on $\{1, \ldots, 5\}$ as follows : for $i \neq j \in \{1, \ldots, 5\}$, (i, j) is an arc of Q_5 when $\{i, j\} \in \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{2, 5\}, \{3, 5\}\}$ (see Figure 1.4) is indecomposable.

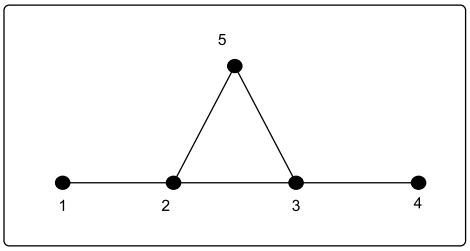


FIGURE 1.4 – Q5.

Modular partition : Let G be a digraph. A partition of $V(G)$ whose all the elements of which are modules of G, is called a *modular partition* of G.

Criticality : Let G be an indecomposable digraph and x be a vertex of G. The vertex x is *critical* if $G - x$ is decomposable. The indecomposable digraph G is

critical if for all $x \in V(G)$, x is a critical vertex.

Duo. Duo-free digraph : Let G be a digraph. Called *duo* each 2-element module of G. If G has no duo, it is said to be a *duo-free digraph*. A tournament T is duo-free if T has no duo. For instance, the 3-cycle, the diamond and the double diamond shown in (Figure 1.5) and defined below are duo-free tournaments. Up to isomorphism, the 3-vertex tournament $T_3 = (\{1,2,3\}, (1,2), (2,3), (3,1))$ is called a *3-cycle*. A *diamond* is a 4-vertex tournament having exactly one 3-cycle. Up to isomorphism, there are two diamonds δ and δ*, where δ is the tournament defined on $\{1,2,3,4\}$ by $\delta[\{1,2,3\}]$ is a 3-cycle and $4 \longleftarrow \{1,2,3\}$. A *double diamond* is a 5-vertex tournament with exactly one 3-cycle. More precisely, up to isomorphism, a double diamond Δ is defined on $\{1,2,3,4,5\}$ by $\Delta[\{2,3,4\}]$ is a 3-cycle, $1 \longrightarrow \{2,3,4,5\}$ and $5 \longleftarrow \{2,3,4\}$.

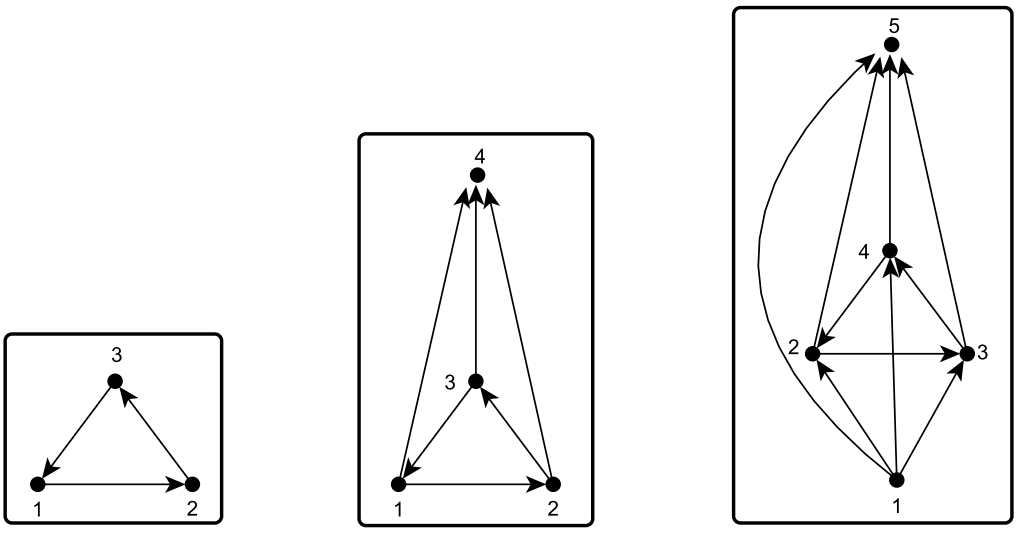


FIGURE 1.5 – 3-cycle, diamond and double diamond.

Critically duo-free digraph : Let G be a duo-free digraph. G is called *critically duo-free digraph* if for all $x \in V(G)$, $G - x$ has at least one duo.

Minimality : Given a duo-free digraph G and vertices $x_1, ..., x_k$ of G, G is *duo-free-minimal* for $x_1, ..., x_k$ if for each proper subset X of $V(G)$ containing $x_1, ..., x_k$, $G[X]$ has at least one duo.

Chapter 2

The Duo Free Tournaments

This chapter is devoted to the description of the duo-free-minimal tournaments. It is organized as follows : Section 1 *gives some definitions and notations concerning duo-free digraphs. The characterization of the duo-free-minimal tournaments for one vertex or two vertices is presented in Section* 2.

2.1 Some preliminary results

We shall needed some definitions and notations in order to formulate the results of this chapter.

Proposition 2.1.1 ([5]). *Given a duo-free digraphs G, there is a subset X of V such that $|X| = 3$ or 4 and $G[X]$ is duo-free.*

Definition 2.1.2 ([5]). *Let G be a digraph and X be a subset of $V(G)$ such that $G[X]$ is duo-free.*

— $Ext_D(X) = \{x \in V(G) \setminus X : G[X \cup \{x\}]$ *is duo-free*$\}$.

— *For any* $u \in X$, $X_D(u) = \{x \in V(G) \setminus X, \{u, x\}$ *is a duo of* $G[X \cup \{x\}]\}$.

Lemma 2.1.3 ([5]). *Given a digraph G, consider a subset X of $V(G)$ such that $G[X]$ is duo-free. The family of the nonempty elements $\mathcal{P}_X = \{Ext_D(X)\} \cup \{X_D(u) : u \in X\}$ is a partition of $V(G) \setminus X$.*

As useful properties of modules, we have.

Proposition 2.1.4. *Let T be a tournament.*

1. *If X and Y are two modules of T, then $X \cap Y$ is a module of T.*
2. *If X and Y are two modules of T such that $X \cap Y \neq \emptyset$, then $X \cup Y$ is a module of T.*
3. *If X and Y are two modules of T such that $X \setminus Y \neq \emptyset$, then $Y \setminus X$ is a module of T.*
4. *If X is a module of T and $Y \subseteq X$ is a module of $T[X]$, then Y is a module of T.*
5. *Let W be a subset of $V(T)$. If X is a module of T, then $X \cap W$ is a module of $T[W]$.*

Remark 2.1.5. *Let G be a digraph. The digraphs G, G^* and $\overline{G}$ have the same duos. Moreover, the following three assertions are equivalent :*

1. *G is duo-free.*
2. *G^* is duo-free.*
3. *$\overline{G}$ is duo-free.*

Lemma 2.1.6 ([6]). *Let T be a tournament and $x \neq y \in V(T)$. x and y do not belong to a chain module of T if and only if there is a subset X of $V(T)$ containing x and y such that one of the following assertions is verified :*

1. *$T[X]$ is isomorphic to Δ.*
2. *$T[X]$ is isomorphic to δ or δ^*.*
3. *$T[X]$ is isomorphic to C_3.*

Remark 2.1.7.

1. *Let T be a duo-free tournament with $|V(T)| \geq 3$. Then there is a subset X of $V(T)$ such that $T[X] \simeq C_3$.*
2. *Up to isomorphism, the two diamonds δ and δ^* are the only 4-vertex duo-free tournaments.*

Proposition 2.1.8 ([5])**.** *Let G be a duo-free digraph with $|V(G)| \geqslant 5$. For each $x \in V(G)$, there is a subset X of $V(G)$ containing x such that $3 \leq |X| \leq 5$ and $G[X]$ is duo-free.*

Let G be an indecomposable digraph. Given vertices $x_1, ..., x_k$ of G. G is *minimal* for $x_1, ..., x_k$ if for each proper subset X of $V(G)$ containing $x_1, ..., x_k$ such that $|X| \geqslant 3$, $G[X]$ is decomposable. In 1998, A. Cournier and P. Ille characterized the minimal indecomposable digraphs for one or two vertices [8]. In 2014, M. Alzohairi and Y. Boudabbous characterized the minimal triangle-free graphs for three vertices [2]. Recently, M. Alzohairi characterized the triangle-free graphs which are minimal for some non-stable 4-vertex subset [1].

In this chapter, we present the characterization of the duo-free-minimal tournaments for one or two vertices.

2.2 The duo-free minimal tournaments

2.2.1 The duo-free-minimal tournaments for one vertex

It can be easily verified C_3 is duo-free-minimal for each of its vertices and that δ and δ^* are duo-free-minimal for 4.

Theorem 2.2.1 ([3]). *Let x be a vertex of duo-free tournament T, with $|V(T)| \geq 3$. The tournament T is duo-free-minimal for x if and only if either T is isomorphic to C_3 or there is an isomorphism f from T onto δ or δ^* such that $f(x) = 4$.*

Proof : Let T be a duo-free-minimal tournament for x, with $x \in V(T)$. By Proposition 2.1.8, there exists a subset X of $V(T)$ including x such that $3 \leq |X| \leq 5$ and $T[X]$ is duo-free. Since T is duo-free-minimal for x, $T[X] = T$. If $|X| = 3$, then $T[X]$ is isomorphic to C_3. Now, we can assume that $|X| \geq 4$. Since $T[X]$ is duo-free, there is a subset Y of X such that $T[Y]$ is isomorphic to C_3. We may assume from what precedes that $x \notin Y$. We now examine the following cases :

Case 1 : Assume that $|X| = 4$. Since $T[X]$ is duo-free, by assertion 2 of Remark 2.1.7, $T[X]$ is isomorphic to δ or δ^*. Let $\{x_1, x_2, x_3\}$ be a subset of X such that $T[\{x_1, x_2, x_3\}]$ is isomorphic to C_3. Since $x \notin \{x_1, x_2, x_3\}$, consider the function f defined on X as follows : $f(x_i) = i$, for $i \in \{1, 2, 3\}$ and $f(x) = 4$. It is clear that, the function f is an isomorphism from $T[X]$ onto δ or δ^* such that $f(x) = 4$.

Case 2 : Assume that $|X| = 5$. If $T[X]$ is indecomposable, then for each $z \in X$, there is a subset Z of X containing z such that $T[Z]$ is isomorphic to C_3. Thus, we

may assume that $T[X]$ is decomposable and duo-free, then $T[X]$ admits a module I of cardinality 3 or 4. We distinguish the following cases :

— Assume that $|I| = 3$, with $I = \{x_1, x_2, x_3\}$. Since $T[X]$ is duo-free, $T[I]$ is isomorphic to C_3. Consider the function f defined on $I \cup \{x\}$ as follows : $f(x_i) = i$, for $i \in \{1, 2, 3\}$ and $f(x) = 4$. The function f is an isomorphism from $T[I \cup \{x\}]$ onto δ or δ^* such that $f(x) = 4$.

— Assume that $|I| = 4$. Since $T[X]$ is duo-free and I is a module of $T[X]$, $T[I]$ is duo-free. By Case 1, we assume that $x \notin I$. There exists a subset $\{x_1, x_2, x_3\}$ of I such that $T[\{x_1, x_2, x_3\}]$ is isomorphic to C_3. Consider the function f defined on $\{x_1, x_2, x_3, x\}$ as follows $f(x_i) = i$, for $i \in \{1, 2, 3\}$ and $f(x) = 4$. The function f is an isomorphism from $T[\{x_1, x_2, x_3, x\}]$ onto δ or δ^* such that $f(x) = 4$.

□

The following corollary is an immediate consequence of Theorem 2.2.1.

Corollary 2.2.2 ([3]). *Let T be a duo-free tournament, with $|V(T)| \geq 3$. For each vertex x of $V(T)$, there is a subset X of $V(T)$ containing x such that one of the following assertions is verified :*

— *$T[X]$ is isomorphic to C_3.*

— *There is an isomorphism f from $T[X]$ onto onto δ or δ^* such that $f(x) = 4$.*

2.2.2 The duo-free-minimal tournaments for two vertices

It is easy to verify that Δ and Δ^* are duo-free-minimal for 1 and 5. Clearly, C_3 is duo-free-minimal for each two vertices of it and for $x \in \{1,2,3\}$, δ and δ* are duo-free-minimal for x and 4.

Theorem 2.2.3 ([3]). *Let x and y be distinct vertices of a duo-free tournament, with $|V(T)| \geq 3$. The tournament T is duo-free-minimal for x and y if and only if one of the following assertions is verified :*

- *T is isomorphic to C_3.*
- *There is an isomorphism f from T onto δ or δ* such that $f(\{x,y\}) = \{i,4\}$, for $i \in \{1,2,3\}$.*
- *There is an isomorphism f from T onto Δ or Δ^* such that $f(\{x,y\}) = \{1,5\}$.*

Chapter 3

The Minimal Duo-Free-Digraphs

In this chapter, we present the characterization of the minimal duo-free digraphs for one or two vertices. We start in Section 1 *by citing some important definitions and notations. Section* 2 *gives a characterization of the duo-free graphs for one or two vertices. In Section* 3*, we describe the duo-free-minimal k-vertex digraphs for one vertex, where* $k \in \{3, 4, 5\}$*.*

3.1 Some preliminary results

With respect to the primality, we recall the following notation and results.

Let G be a digraph. Given a proper subset X of $V(G)$ such that $|X| \geq 3$ and $G[X]$ is indecomposable. Consider the following subsets of $V(G) \setminus X$:

— $\mathrm{Ext}(X)$ is the set of $x \in V(G) \setminus X$ such that $G[X \cup \{x\}]$ is indecomposable;

— $[X]$ is the set of $x \in V(G) \setminus X$ such that X is a module of $G[X \cup \{x\}]$;

— Given $u \in X$, $X(u)$ is the set of $x \in V(G) \setminus X$ such that $\{u, x\}$ is a module of $G[X \cup \{x\}]$.

The family of the nonempty elements of $[\mathrm{Ext}(X), [X] \cup X(u), u \in X]$ is denoted by $\mathcal{P}_X$.

Lemma 3.1.1 ([11]). *Given a digraph G, consider a subset X of $V(G)$ such that $G[X]$ is indecomposable. The family $\mathcal{P}_X$ constitutes a partition of $V(G) \setminus X$.*

Lemma 3.1.2 ([5]). *Let G be a duo-free graph with $|V(G)| \geq 4$. Then there is a subset X of $V(G)$ such that $G[X] \cong P_4$*

3.2 The duo-free-minimal graphs

3.2.1 The duo-free-minimal digraphs for one vertex

To describe the duo-free-minimal graphs for one vertex, we introduce the following 5-vertex graph called Q_5 (resp. $P_4 \oplus P_1$)(see Figure 3.1) defined on $\{1, ..., 5\}$ as follows : for $i \neq j \in \{1, ..., 5\}$, (i, j) is an arc of Q_5 (resp. $P_4 \oplus P_1$) when $\{i, j\} \in \{\{1,2\}, \{2,3\}, \{3,4\}, \{2,5\}, \{3,5\}\}$ (resp. $\{i, j\} \in \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,5\}, \{2,5\}, \{3,5\}, \{4,5\}\}$).

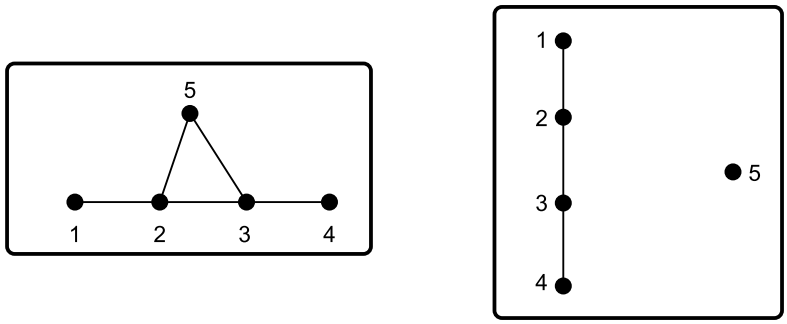


FIGURE 3.1 – Q_5 and $P_4 \oplus P_1$

Remark 3.2.1. *It is clear that Q_5 is isomorphic to its complement.*

Lemma 3.2.2. *The graphs Q_5 is duo-free-minimal for 5.*

Proof :

Let G be graph. To begin, we prove that G is duo-free. Indeed, let $X = \{1, 2, 3, 4\}$. Clearly, G is isomorphic to P_4. Since P_4 is duo-free and P_4 is small duo-free graph. So, $G[X]$ is duo-free. We will use the partition $\mathcal{P}_X$ of $V(G) \setminus X$. Since $1 \dots 3$ and $1 \dots 5$ and 3—5 then $5 \notin X_D(1)$, by interchanging 1 (resp. 3) by 4 (resp. 2), we obtain $5 \notin X_D(4)$. Since 3—5 and $4 \dots 5$ and 3—4 then $5 \notin X_D(3)$, by interchanging 3 (resp. 4) by 2 (resp. 1), we obtain $5 \notin X_D(2)$. We get $5 \notin X_D(i)$, where $i \in \{1, 2, 3, 4\}$, so $5 \in \mathrm{Ext}_D(X)$, thus $Q_5[X \cup \{5\}]$ is duo free.

Now, consider a proper subset Y of $\{1, ..., 5\}$ containing 5 such that $|Y| \geq 3$. Since the graphs with 3 vertices have a duo. We can assume that $|Y| = 4$. First, assume that, $Y - \{1\}$ (resp. $Y - \{4\}$), we obtain $G[\{2, 3, 5\}]$ is a cycle. Hence it is not isomorphic to P_4. So $G[Y - \{1\}]$ (resp. $G[Y - \{4\}]$) has a duo. Second, assume that $Y - \{2\}$ (resp. $Y - \{3\}$) we obtain $G[Y - \{2\}]$ (resp. $G[Y - \{3\}]$) is not connected and thus, G is not isomorphic to P_4, then has a duo. Consequently, G is duo-free-minimal for 5.

□

Theorem 3.2.3 ([3]). *Let x be a vertex of a duo-free graph G, with $|V(G)| \geq 4$. The graph G is duo-free-minimal for x if and only if one of the following assertions is verified :*

— *G is isomorphic to P_4.*

— *There is an isomorphism f from G onto Q_5, $P_4 \oplus P_1$ or $\overline{P_4 \oplus P_1}$ such that $f(x) = 5$.*

Proof : Let G be a duo-free minimal digraph for x ,with $x \in V(G)$. By Proposition 2.1.8, there is a subset X of $V(G)$ containing x such that $3 \leq |X| \leq 5$ and $G[X]$ is duo-free. Since G is duo-free-minimal for x, $G[X] = G$. Moreover, since the graphs with 3 vertices have a duo, $4 \leq |X| \leq 5$. If $|X| = 4$, then $G[X]$ is isomorphic to P_4. Now, we may assume that $|X| = 5$. Since $G[X]$ is duo-free, using Lemma 3.1.2, there is a subset Y of X such that $G[Y]$ is isomorphic to P_4. Let $Y = \{y_1, y_2, y_3, y_4\}$ such that for $i \neq j \in \{1, \ldots, 4\}$, y_i—y_j if and only if $|i - j| = 1$. From what precedes, we may assume that $x \notin Y$. Since $G[Y \cup \{x\}]$ is duo-free, $x \in Ext_D(Y)$. By interchanging G and $\overline{G}$, we can assume that $|N_G(x)| \leq 2$.

First, assume that $N_G(x) = \emptyset$, consider the function f defined on $Y \cup \{x\}$ as follows : $f(y_i) = i$, for $i \in \{1, \ldots, 4\}$ and $f(x) = 5$. The function f is an isomorphism from $G[Y \cup \{x\}]$ onto $P_4 \oplus P_1$ such that $f(x) = 5$.

Second, assume that $|N_G(x)| = 1$. If $N_G(x) = \{y_2\}$ (resp. $N_G(x) = \{y_3\}$), then $\{y_1, x\}$ (resp. $\{y_4, x\}$) is a duo of $G[Y \cup \{x\}]$. If $N_G(x) = \{y_1\}$ (resp. $N_G(x) = \{y_4\}$), then $G[\{y_1, y_2, y_3, x\}]$ (resp. $G[\{y_2, y_3, y_4, x\}]$) is isomorphic to P_4.

Finally, assume that $|N_G(x)| = 2$. If $N_G(x) = \{y_1, y_2\}$ (resp. $N_G(x) = \{y_1, y_3\}$), then $\{y_1, x\}$ (resp. $\{y_2, x\}$) is a duo of $G[Y \cup \{x\}]$. If $N_G(x) = \{y_2, y_3\}$, then consider the function f defined on $Y \cup \{x\}$ as follows : $f(y_i) = i$, for $i \in \{1, \ldots, 4\}$ and $f(x) = 5$. The function f is an isomorphism from $G[Y \cup \{x\}]$ onto Q_5 such that $f(x) = 5$. If $N_G(x) = \{y_1, y_4\}$, then $G[\{y_1, y_2, y_3, x\}]$ is isomorphic to P_4.

□

The following corollary is an immediate consequence of Theorem 3.2.3.

Corollary 3.2.4 ([3]). *Let G be a duo-free graph G, with $|V(G)| \geq 4$, for each vertex x of $V(G)$, there is a subset x of X of $V(G)$ containing x such that one of the following assertions id verified :*

— *G is isomorphic to P_4.*

— *There is an isomorphism f from G onto Q_5, $P_4 \oplus P_1$ or $\overline{P_4 \oplus P_1}$ such that $f(x) = 5$.*

3.2.2 The duo-free minimal graphs for two vertices

To describe the duo-free-minimal graphs for two vertices, we have to introduce the graphs G_1, G_2 and G_3 (see Figure 3.2) defined on $\{1, \ldots, 6\}$ (resp. $\{1, \ldots, 7\}$) as follows :

— For $i \neq j \in \{1, \ldots, 5\}$, (i, j) is an arc of G_1 (resp. G_2) when $\{i, j\} \in \{\{1,2\}, \{2,3\}, \{3,4\}, \{2,5\}, \{3,5\}\}$ (resp. $\{i, j\} \in \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,5\}, \{2,5\}, \{3,5\}, \{4,5\}\}$).

— For $i \neq j \in \{1, \ldots, 7\}$, (i, j) is an arc of G_3 when $\{i, j\} \in \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,7\}, \{2,7\}, \{3,7\}, \{4,7\}, \{5,7\}\}$.

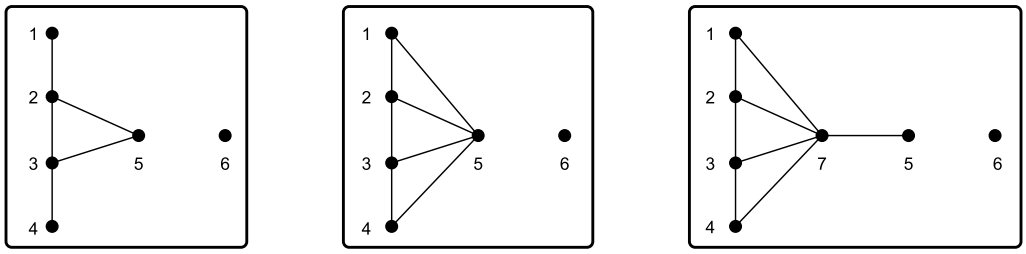


FIGURE 3.2 – G_1,G_2 and G_3.

Lemma 3.2.5 ([3]). *The graphs G_1 and G_2 are duo-free-minimal for 5 and 6.*

Proof : Let G be an element of $\{G_1, G_2\}$. To begin, we prove that G is duo-free.

Indeed, let $X = \{1, 2, 3, 4, 5\}$. Clearly, $G[x]$ is isomorphic to Q_5 (resp. $P_4 \oplus P_1$). Since Q_5 (resp. $P_4 \oplus P_1$) is duo-free, $G[X]$ is duo-free. We will use the partition $\mathcal{P}_X$ of $V(G) \setminus X$, since $6 \ldots \{1, 2, 3, 4\}$, $6 \notin X_D(i)$, for $i \in \{1, 2, 3, 4\}$. Since $2 \ldots 5$ and $2 \ldots 6$, $6 \notin X_D(5)$. Thus, $6 \notin X_D(i)$ for $i \in \{1, ..., 5\}$. Lemma 3.1.2 implies that $6 \in Ext_D(X)$ and hence, G is duo-free.

Now, consider a proper subset Y of $\{1, ..., 6\}$ containing 5 and 6 such that $|Y| \geq 3$. Since the graphs with 3 vertices have a duo, we can assume that $|Y| \geq 4$. First, assume that $|Y| = 4$. Since $6 \ldots X$, $G[Y]$ is not connected and thus, G is not isomorphic to P_4. Hence, Lemma 3.1.2 implies that $G[Y]$ has a duo. Second, assume that $|Y| = 5$. Since $G[X]$ is duo-free-minimal for 5, $G[X \setminus \{i\}]$, for $i \in \{1, ..., 4\}$ has a duo I. Moreover, since $6 \ldots X$, $6 \ldots I$. Hence, I is a duo of $G[Y]$. Consequently, G is duo-free-minimal for 5 and 6. □

Lemma 3.2.6 ([3]). *The graph G_3 is duo-free-minimal for 5 and 6.*

Proof : To begin, we prove that G_3 is duo-free. Indeed, let $X = \{1, 2, 3, 4, 6, 7\}$. Clearly, $G_3[X]$ is isomorphic to G_2 so that, $G_3[X]$ is duo-free. We will use the partition $\mathcal{P}_X$ of $V(G) \setminus X$, since $5 \ldots \{1, 2, 3, 4\}$, $5 \notin X_D(i)$, for $i \in \{1, ..., 4\}$. Moreover, since $1 \ldots 7$ and $1 \ldots 5$ (resp. $7 \ldots 5$ and $7 \ldots 6$), $5 \notin X_D(7)$ (resp. $5 \notin X_D(6)$). Thus, $5 \notin X_D(i)$, for $i \in \{1, 2, 3, 4, 6, 7\}$. Lemma 3.1.2 implies that $5 \in Ext_D(X)$ and hence, G_3 is duo-free.

Now, consider a proper subset Y of $\{1, ..., 7\}$ containing 5 and 6 such that $|Y| \geq 3$. Since the graphs with 3 vertices have a duo, we can assume that $|Y| \geq 4$. First, assume that $|Y| = 4$. Since $6 \ldots \{1, 2, 3, 4, 5, 7\}$, $G_3[Y]$ is not connected and thus, $G_3[Y]$ is not isomorphic to P_4. Hence, $G_3[Y]$ has a duo.

Second, assume that $|Y| = 5$ and let $W = \{1, 2, 3, 4, 7\}$. Clearly, $G_3[W] \simeq P_4 \oplus P_1$. For $i \neq j \in W$, $G_3[W \setminus \{i, j\}]$ is 3-vertex graph has a duo. Thus, the fact that $7 \sim \{1, 2, 3, 4\}$, $G_3[W \setminus \{i, j\}]$ has a duo $\{l, k\}$, where $l \neq k \in \{1, ..., 4\}$. Moreover, since $\{5, 6\} \ldots \{1, 2, 3, 4\}$, $\{5, 6\} \ldots \{l, k\}$. Hence, $\{l, k\}$ is a duo of $G_3[Y]$. Finally, assume that $|Y| = 6$ and let i be an element of $V(G) \setminus Y$. If $i \in \{1, 2\}$ (resp. $i \in \{3, 4\}$), then $\{4, i + 1\}$ (resp. $\{1, i - 1\}$) is a duo of $G_3[Y]$. Moreover, if $i = 7$, then $5, 6$ is a duo of $G_3[Y]$. Consequently, G_3 is duo-free-minimal for 5 and 6. □

The following lemma is immediate.

Lemma 3.2.7 ([3]). *P_5 is duo-free-minimal for 1 and 5.*

We make the following remark.

Remark 3.2.8. *If a graph G is duo-free-minimal for a vertex x, then it is duo-free-minimal for x and each other vertex.*

According to the characterization of the the duo-free-minimal for two vertex, we have the following corollary :

Corollary 3.2.9 ([3]). *P_4 is duo-free-minimal for each two vertices of it and for each $x \in \{1, \ldots, 4\}$, Q_5, $P_4 \oplus P_1$, are duo-free-minimal for x and 5.*

Theorem 3.2.10 ([3]). *Let x and y be distinct vertices of a duo-free graph G, with $|V(G)| \geq 4$. The graph G is duo-free-minimal for x and y if and only if one of the following assertions is verified :*

— *G is isomorphic to P_4.*

— *There is an isomorphism f from G onto P_5 or $\overline{P}_5$ such that $f(\{x, y\}) = \{1, 5\}$.*

— *There is an isomorphism f from G onto Q_5 such that $f(\{x, y\}) = \{5, i\}$, for $x \in \{1, \ldots, 4\}$.*

— *There is an isomorphism f from G onto G_1, G_2, G_3, $\overline{G_1}$, $\overline{G_2}$ or $\overline{G_3}$ such that $f(\{x, y\}) = \{5, 6\}$.*

— *There is an isomorphism f from G onto $P_4 \oplus P_1$ or $\overline{P_4 \oplus P_1}$ such that $f(\{x, y\}) = \{i, 5\}$, for $i \in \{1, ..., 4\}$.*

The following corollary is an immediate consequence of Theorem 3.2.10.

Corollary 3.2.11 ([3])**.** *Let G be a duo-free graph with $|V(G)| \geq 4$. For any distinct vertices x and y of $V(G)$, there is a subset X of $V(G)$ containing x and y such that one of the following assertions is satisfied :*

— *$G[X]$ is isomorphic to P_4.*

— *There is an isomorphism f from $G[X]$ onto $\overline{P_5}$ or P_5 such that $f(\{x, y\}) = \{1, 5\}$.*

— *There is an isomorphism f from $G[X]$ onto Q_5 such that $f(\{x, y\}) = \{5, i\}$, for $i \in \{1, \ldots, 4\}$.*

— *There is an isomorphism f from $G[X]$ onto G_1, G_2, G_3, $\overline{G_1}$, $\overline{G_2}$ or $\overline{G_3}$ such that $f(\{x, y\}) = \{5, 6\}$.*

— *There is an isomorphism f from $G[X]$ onto $P_4 \oplus P_1$ or $\overline{P_4 \oplus P_1}$ such that $f(\{x, y\}) = \{i, 5\}$, for $i \in \{1, \ldots, 4\}$.*

3.3 The duo-free-minimal k-vertex digraphs for one vertex, where $k \in \{3, 4, 5\}$

3.3.1 The duo-free-minimal k-vertex digraphs for one vertex, where $k \in \{3, 4\}$

In order to state the following remarks and lemma, we introduce the following 3-vertex digraphs A_1 and A_2 (see Figure 3.3) defined as follows :

$$A_1 = (\{1,2,3\}, \{(1,2),(2,3)\}) \quad \text{and} \quad A_2 = (\{1,2,3\}, \{(1,2),(2,3),(3,2)\}).$$

We call *3-consecutivity* (resp. *flag*) every digraph isomorphic to A_1 or $\overline{A_1}$ (resp. A_2 or A_2^*).

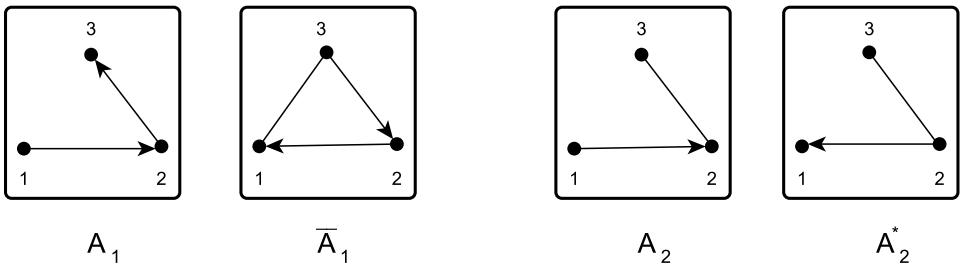


FIGURE 3.3 – Flag and 3-consecutivity.

Remarks 3.3.1.

1. *Given a digraph G, consider a subset X of $V(G)$ such that $G[X]$ is indecomposable. If there is $x \in V(G)$ such that $x \in [X]$, then $G[X \cup \{x\}]$ is duo-free.*

2. *Up to isomorphism, N and $\overline{N}$ are duo-free digraphs.*
3. *3-vertex digraph is indecomposable if and only if it is a flag, a 3-consecutivity or a 3-cycle.*

The following lemma is immediate :

Lemma 3.3.2.

1. *Up to isomorphism, a flag, a 3-consecutivity and a 3-cycle are duo-free-minimal for each of its vertices.*
2. *Up to isomorphism,P_4, N and $\overline{N}$ are duo-free-minimal for each of its vertices.*
3. *The duo-free minimal 4-vertex digraphs for vertex x is the family of the digraph such that $G \setminus \{x\}$ is either a flag, a 3-consecutivity or a 3-cycle.*

3.3.2 The duo-free-minimal 5-vertex digraphs for one vertex

To describe the duo-free-minimal 5-vertex digraphs for one vertex, we have to introduce the following 5-vertex digraph called $\overrightarrow{Q_5}$ (resp. $\overrightarrow{P_4 \oplus P_1}$) (see Figure 3.4) defined on $\{1, ..., 5\}$ as follows : $A(\overrightarrow{Q_5}) = \{(2,1),(2,3),(4,3),(2,5),(5,3)\}$ (resp. $A(\overrightarrow{P_4 \oplus P_1}) = \{\{1,2\},\{2,3\},\{3,4\},(1,5),(2,5),(3,5),(4,5)\}$).

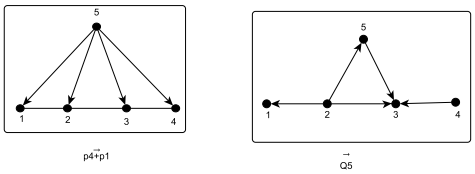


FIGURE 3.4 – 5-vertex digraphs

Lemma 3.3.3. *The digraph $\overrightarrow{P_1 \oplus P_4}$ is duo-free-minimal for* 5.

Proof : To begin, we prove that $\overrightarrow{P_1 \oplus P_4}$ is duo-free. Indeed, let $X = \{1, 2, 3, 4\}$. Since $5 \in [X]$, by Assertion 1 of Remarks 3.3.1, $\overrightarrow{P_1 \oplus P_4}$ is duo-free.

Now, consider a proper subset Y of $\{1, 2, 3, 4, 5\}$ containing 5 such that $|Y| \geq 3$. First, assume that $|Y| = 3$. Since, $5 \in [X]$, then $\overrightarrow{P_1 \oplus P_4}[Y]$ is neither a flag, a 3-consecutivity nor 3-cycle. Thus, by Assertion 3 of Remarks 3.3.1, $\overrightarrow{P_1 \oplus P_4}[Y]$ has a duo. Second, assume that $|Y| = 4$. Since $\overrightarrow{P_1 \oplus P_4}[Y \setminus \{5\}]$ is not isomorphic to P_4, then $\overrightarrow{P_1 \oplus P_4}[Y] \setminus \{5\}$ has a duo. Moreover, as $5 \in [X]$, then I is a duo of $\overrightarrow{P_1 \oplus P_4}[Y]$.

□

Lemma 3.3.4. *The digraph $\overrightarrow{Q_5}$ is duo-free-minimal for* 5.

Proof : To begin, we prove that G is duo-free. Indeed, let $X = \{1, 2, 3, 4\}$. Clearly, $\overrightarrow{Q_5}[X]$ is isomorphic to N. As N is duo-free, $\overrightarrow{Q_5}[X]$ is duo-free. Since $1 \dots 3$ and $3 \longleftarrow 5$, then $5 \notin X_D(1)$. By interchanging 1(resp. 3) by 4 (resp. 2), we obtain $5 \notin X_D(4)$. Moreover, as $2 \longrightarrow 1$ and $1 \dots 5$, then $5 \notin X_D(2)$, since $4 \dots 2$ and $2 \longleftarrow 5$, then $5 \notin X_D(3)$. Thus, $5 \notin X_D(i)$ for $i \in \{1, 2, 3, 4\}$. Lemma implies that $5 \in Ext_D(X)$, and hence, $\overrightarrow{Q_5}$ is duo-free.

Now, consider a proper subset Y of $\{1, ..., 5\}$ containing 5 and such that $|Y| \geqslant 3$. First, assume that $Y = 3$, we distinguish the following cases :

— $\overrightarrow{Q_5}[Y \setminus \{1, 2\}]$ (resp. $\overrightarrow{Q_5}[Y \setminus \{2, 3\}]$), since $5 \longrightarrow 3$ and $3 \longleftarrow 4$ (resp. $3 \dots 5$ and $5 \dots 4$), we obtain $\{4, 5\}$) is a duo.

— $\overrightarrow{Q_5}[Y \setminus \{1, 3\}]$ (resp. $\overrightarrow{Q_5}[Y \setminus \{1, 4\}]$), since $2 \dots 4$ and $4 \dots 5$ (resp. $5 \longrightarrow 3$ and $3 \longleftarrow 2$), we obtain $\{2, 5\}$ is a duo.

— $\overrightarrow{Q_5}[Y \setminus \{2,4\}]$ (resp. $\overrightarrow{Q_5}[Y \setminus \{3,4\}]$), since $5 \dots 1$ and $1 \dots 3$ (resp. $1 \longleftarrow 2$ and $2 \longrightarrow 5$), we obtain $\{5,3\}$ (resp. $\{1,5\}$) is a duo.

Second, we assume that $Y = 4$, we have the following cases :

— $\overrightarrow{Q_5}[Y \setminus \{1\}]$, since $5 \longrightarrow 3 \longleftarrow 2$ and $2 \dots 4 \dots 5$, then $\{2,5\}$ a is duo.

— $\overrightarrow{Q_5}[Y \setminus \{2\}]$, since $4 \longrightarrow 3 \longleftarrow 5$ and $4 \dots 1 \dots 5$, then $\{4,5\}$ a is duo.

— $\overrightarrow{Q_5}[Y \setminus \{3\}]$, since $5 \longleftarrow 2 \longrightarrow 1$ and $5 \dots 4 \dots 1$, then $\{1,5\}$ a is duo.

— $\overrightarrow{Q_5}[Y \setminus \{4\}]$, since $3 \longleftarrow 2 \longrightarrow 5$ and $5 \dots 1 \dots 3$, then $\{5,3\}$ a is duo.

□

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