RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE
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KHALDI Inas et TERRAS Meroua

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Fubini Polynomials and Generating Series: A Probability Approach

Devant le jury composé de :

Mr. RAHMANI Mourad	Prof	U.S.T.H.B	President
Mr. LAISSAOUI Diffalah	Prof	U.Y.F.M	Supervisor
Mme. GUETTAI Ghania	M.C.A	U.Y.F.M	Examiner
Mr. SEBAOUI Madjid	M.C.A	U.Y.F.M	Examiner

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Abstract

This thesis explores fundamental concepts in combinatorial mathematics and probability theory, bridging combinatorial identities and probabilistic interpretations. The first chapter covers foundational elements such as factorials, generating functions, and Stirling numbers. The second chapter introduces probability theory, discussing key distributions and the moment generating function. The final chapter delves into probabilistic versions of degenerate Fubini polynomials, analyzing their properties and recurrence relations. This work aims to deepen the understanding of combinatorial and probabilistic structures, highlighting their theoretical and practical significance.

Keywords: Generating function, Stirling numbers, Bell polynomials, Fubini polynomials, Probability, Degenerate polynomials.

Résumé

Cette thèse explore des concepts fondamentaux en mathématiques combinatoires et en théorie des probabilités, en établissant des liens entre les identités combinatoires et les interprétations probabilistes. Le premier chapitre couvre les éléments fondamentaux tels que les factorielles, les fonctions génératrices et les nombres de Stirling. Le deuxième chapitre introduit la théorie des probabilités, en discutant des distributions clés et de la fonction génératrice des moments. Le dernier chapitre se penche sur les versions probabilistes des polynômes de Fubini dégénérés, en analysant leurs propriétés et leurs relations de récurrence. Ce travail vise à approfondir la compréhension des structures combinatoires et probabilistes, en soulignant leur importance théorique et pratique.

Mots-clés : Fonction génératrice, Nombres de Stirling, Polynômes de Bell, Polynômes de Fubini, Probabilité, Polynômes dégénérés.

Notations

$\mathbb{C}$ The field of complex numbers.

$\mathbb{R}$ The field of real numbers.

$\mathbb{N}$ The set of natural numbers.

$\mathbb{K}[[t]]$ The ring of formal power series in variable t over a field $\mathbb{K}$.

$\mathrm{x}^{\overline{n}}$ The rising factorial.

$\mathrm{x}^{\underline{n}}$ The falling factorial.

$(x/\lambda)^{\overline{n}}$ The generalized rising factorial of x of order n.

$(x)_{n,\lambda}$ The generalized falling factorial of x of order n.

$\delta_{n,k}$ The Kronecker delta symbol.

$s(n,k)$ Signed Stirling numbers of the first kind.

$|\, s(n,k)\,|$ The unsigned Stirling numbers of the first kind.

$\left\{ {n \atop k} \right\}$ Stirling numbers of the second kind.

$\left\{ {n \atop k} \right\}_{\lambda}$ The degenerate Stirling numbers of the second kind.

$\left\{ {n \atop k} \right\}_{Y,\lambda}$ The degenerate Stirling numbers of the second kind associated with random variable Y.

$\binom{n}{k}$ The binomial coefficient.

B_n Bernoulli numbers.

$B_n(x)$ Bernoulli polynomials.

$\left\lfloor {n \atop k} \right\rfloor$ The Lah numbers (unsigned).

$L_n(x)$ The Lah polynomials.

ϕ_n The Bell numbers.

$\phi_n(x)$ The Bell polynomials.

$\phi_{n,k}(x)$ The Bell partial polynomials.

$\phi_{n,\lambda}(x)$ The degenerate Bell polynomials.

$\phi^{Y}_{n,\lambda}(x)$ The degenerate Bell polynomials associated with random variable Y.

F_n The Fubini numbers.

$F_n(x)$ The Fubini polynomials.

$F_{n,\lambda}(x)$ The degenerate Fubini polynomials.

$F_{n,\lambda}^{(r)}(x)$ The degenerate Fubini polynomials of order r.

$F_{n,\lambda}^{Y}(x)$ The degenerate Fubini polynomials associated with random variable Y.

$F_{n,\lambda}^{(r,Y)}(x)$ The degenerate Fubini polynomials of order r associated with random variable Y.

Ω The sample space.

$\emptyset$ The whole empty.

$\Gamma(x)$ Gamma function.

Introduction

General Introduction

The field of combinatorial mathematics and probability theory is rich with intriguing concepts and applications that span across numerous disciplines. This thesis aims to explore a selection of these fundamental concepts, building a bridge between combinatorial identities and probabilistic interpretations. The three chapters in this thesis are structured to provide a comprehensive journey from foundational elements to specialized research findings, each contributing to a deeper understanding of the mathematical landscape.

In the first chapter, we lay the groundwork by presenting essential elements that will be useful for the rest of the thesis. This includes a review of rising and falling factorials, generating functions, explicit forms, recurrence relations, and Stirling numbers, along with their combinatorial properties. These foundational tools are crucial for the subsequent exploration of more advanced topics. For more precise and detailed concepts, readers are referred to key references in the literature.

The second chapter transitions into the realm of probability theory. We begin by introducing fundamental concepts in probability, followed by a discussion on several discrete and continuous probability distributions. The chapter emphasizes the moment generating function, a vital tool for understanding the distribution of random variables. We provide an overview of basic probability notions and cover essential laws, including both discrete (such as binomial and Poisson) and continuous (such as normal and exponential) distributions. This chapter serves to establish a solid probabilistic foundation that supports the specialized investigations in the final chapter.

The third chapter delves into the intricate study of probabilistic versions of degenerate Fubini polynomials, inspired by the work of Kim. Here, we consider a random variable YY with an existing moment generating function in a neighborhood of the origin. The focus is on exploring the probabilistic degenerate Fubini polynomials associated with Y, along with their higher-order counterparts. We analyze various properties of these polynomials, including explicit expressions, specific identities, and recurrence relations. By examining Kim's findings, we aim to provide a comprehensive overview of the theoretical underpinnings and practical implications of these polynomials, highlighting their significance in the probabilistic context.

Through these chapters, this thesis contributes to the understanding of combinatorial and probabilistic structures, offering both theoretical insights and practical applications. The interconnections between these mathematical domains underscore the richness of the field and the potential for future research.

1

Basics notions

In this chapter, we present some basic elements which will be useful for the rest, including a review of rising and falling factorials, the generating function, the explicit form, the recurrence relation, Stirling numbers and their combinatorial properties. For more precise and detailed concepts the reader can refer to [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 20].

Preliminairies

1.1 Rising factorial and falling factorial

1.1.1 Rising factorial

Definition 1.1.1 *Let x be a real or complex number and n a positive integer, we define the rising factorial of x of order n, denoted as $x^{\overline{n}}$, by :*

$$x^{\overline{n}} = x(x+1)...(x+n-1), \text{ if } n \geq 1 \text{ and } x^{\overline{0}} = 1.$$

Example 1.1

$$\begin{aligned} x^{\overline{1}} &= x \\ x^{\overline{2}} &= x(x+1) = x^2 + x \\ x^{\overline{3}} &= x(x+1)(x+2) = x^3 + 3x^2 + 2x \\ x^{\overline{4}} &= x(x+1)(x+2)(x+3) = x^4 + 6x^3 + 11x^2 + 6x. \end{aligned}$$

Equivalently, we call the generalized rising factorial of x of order n and coefficient λ ($\lambda \in \mathbb{R}$), the polynomial

$$(x/\lambda)^{\overline{n}} = x(x+\lambda)...(x+(n-1)\lambda), \text{ if } n \geq 1 \text{ and } (x/\lambda)^{\overline{0}} = 1.$$

1.1.2 Falling factorial

Definition 1.1.2 *Let x be a real or complex number and n a positive integer, the falling factorial of x of order n, denoted as $x^{\underline{n}}$, by :*

$$x^{\underline{n}} = x(x-1)...(x-n+1), \text{ if } n \geq 1 \text{ and } x^{\underline{0}} = 1.$$

Example 1.2

$$\begin{aligned} x^{\underline{1}} &= x \\ x^{\underline{2}} &= x(x-1) = x^2 - x \\ x^{\underline{3}} &= x(x-1)(x-2) = x^3 - 3x^2 + 2x \\ x^{\underline{4}} &= x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x. \end{aligned}$$

Where, for any $\lambda \in \mathbb{R}$, the generalized falling factorials ara given by

$$(x)_{n,\lambda} = x(x-\lambda)...(x-(n-1)\lambda), \text{ if } n \geq 1 \text{ and } \quad (x)_{0,\lambda} = 1.$$

The relation between the rising factorial and the falling factorial is given by :

$$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}}.$$

The sequences of polynomials $x^{\overline{n}}$ and $x^{\underline{n}}$ form bases of the $\mathbb{R}$-vector space of polynomials with real coefficients, usually denoted as $\mathbb{R}[x]$.

1.2 Binomail coefficients

Definition 1.2.1 *Let n and k be two non-negative integers. The binomial coefficient of k of order n, denoted as $\binom{n}{k}$, is given by :*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!} \quad (0 \leq k \leq n), \tag{1.1}$$

with $\binom{n}{k} = 0$ for $k > n$, and $\binom{n}{k} = 1$ for $k = 0$ or $k = n$.

Definition 1.2.2 *In combinatorics, the binomial coefficient $\binom{n}{k}$ is the number of ways to choose a subset with k elements from a set of n elements.*

Example 1.3 *In the set $[5] = \{a, b, c, d, e\}$, there are ten ways to choose three elements :*

$$\{a,b,c\}; \{a,b,d\}; \{a,b,e\}; \{a,c,d\}; \{a,c,e\}; \{a,d,e\}; \{b,c,d\}; \{b,c,e\}; \{b,d,e\}; \{c,d,e\}.$$

Therefore, $\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{1\times2\times3\times4\times5}{(1\times2\times3)(1\times2\times3)} = 10.$

Newton's binomial formula

Let x and y be two real numbers and n a non-negative integer.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{1.2}$$

Pascal's recurrence relation

The binomial coefficients satisfying the following relation :

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \quad \text{for every} \quad n \geq 1. \tag{1.3}$$

Pascal's triangle

From the recurrence relation (1.3), we can construct Pascal's triangle

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Table 1.1: Pascal's triangle numbers

Property 1.2.1 *The binomial coefficients satisfy nearly an infinite number of identities, including :*

1. *The absorption identity*

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}. \tag{1.4}$$

2. *The symmetry*

$$\binom{n}{k} = \binom{n}{n-k}, \text{ with } n \geqslant k \geqslant 0. \tag{1.5}$$

3. *The top summation*

$$\binom{n+1}{k+1} = \sum_{j=0}^{n} \binom{k}{j}, \text{ with } n, k \geqslant 0. \tag{1.6}$$

4. *The parallel summation*

$$\binom{m+n+1}{n} = \sum_{n \geqslant j} \binom{m+j}{j}. \tag{1.7}$$

5. *The trinomial transformation*

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}. \tag{1.8}$$

6. *The Vandermonde convolution*

$$\binom{n+m}{k} = \sum_{j=0}^{n} \binom{n}{j}\binom{m}{k-j}. \tag{1.9}$$

1.2.1 Generalization of binomial coefficients

A generalization of the binomial coefficient $\binom{n}{k}$ consists in replacing n by a complex number x, and to define $\binom{x}{k}$ by posing, $x \in \mathbb{C}$ and $k \in \mathbb{N}$ by :

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!} = \frac{x^{\underline{k}}}{k!}. \tag{1.10}$$

and having almost the same properties as $\binom{n}{k}$. One has :

$$\binom{-k}{n} = \binom{k+n-1}{n}(-1)^n. \tag{1.11}$$

1.3 The Kronecker symbol

The Kronecker Symbol is given by :

$$\delta_{n,k} = \begin{cases} 1 & \text{if} \quad n = k \\ 0 & \text{otherwise.} \end{cases}$$

1.4 Generating function

Definition 1.4.1 *The field $\mathbb{K}$ will be always the field of real numbers $\mathbb{R}$, or the field of the complex numbers $\mathbb{C}$, with a characteristic of 0.*

Definition 1.4.2 *The notation $\mathbb{K}[[t]]$ represents the set of all generating functions over the field $\mathbb{K}$.*

Definition 1.4.3 *Let $(a_n)_{n\geq 0}$ be a sequence belonging to $\mathbb{K}$ The ordinary generating function (O.G.F) (or the formal power series) of the sequence $(a_n)_{n\geq 0}$ is the series*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots$$

Another type of generating functions is

Definition 1.4.4 *The Exponential generating function (E.G.F)*

$$\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} = a_0 + a_1 \frac{t}{1!} + a_2 \frac{t^2}{2!} + \cdots$$

Example 1.4 *Consider the following generating function :*

$$\sum_{n=0}^{\infty} a^n t^n = 1 + at + (at)^2 + \cdots + (at)^n + \cdots = \frac{1}{1-at}, \ \ a \in \mathbb{C}.$$

Another example deduced from the previous one, by putting $a = 4$, is

$$1 + 4t + 16t^2 + \cdots + 4^n t^n + \cdots = \frac{1}{1-4t}.$$

Example 1.5 *The simplest Exponential generating function is certainly*

$$e^{\alpha.t} = \sum_{n=0}^{\infty} \alpha^n \frac{t^n}{n!}.$$

1.4.1 Operations on generating functions

Let $f, g, h \in \mathbb{K}[[t]]$. Let us write

$$f := f(t) = \sum_{n=0}^{\infty} a_n t^n, g := (t) = \sum_{n=0}^{\infty} b_n t^n, h := h(t) = \sum_{n=0}^{\infty} c_n t^n.$$

We define some operations on $\mathbb{K}[[t]]$.

Definition 1.4.5 *(Equality)*
The generating functions f and g are equal equivalence to $a_n = b_n, \forall n \in \mathbb{N}$

Definition 1.4.6 *(Multiplication by a scalar)*
Let $f \in \mathbb{K}[[t]]$ and $\lambda \in \mathbb{K}$, then

$$\lambda f(t) = \lambda \left(\sum_{n=0}^{\infty} a_n t^n \right) = \sum_{n=0}^{\infty} \lambda a_n t^n. \tag{1.12}$$

Definition 1.4.7 *(Sum)*
The sum $(f + g)$ of f and g, is defined by

$$f(t) + g(t) = \sum_{n=0}^{\infty} (a_n + b_n) t^n. \tag{1.13}$$

Example 1.6 *Consider the two generating functions :*

$$f(t) = \sum_{n=0}^{\infty}(2)^n t^n = 1 + 2t + 4t^2 + 8t^3 + \cdots = \frac{1}{1-2t},$$
$$g(t) = \sum_{n=0}^{\infty}(-2)^n t^n = 1 - 2t + 4t^2 - 8t^3 + \cdots = \frac{1}{1+2t},$$

the sum of f and g is then :

$$f(t) + g(t) = 2 + 8t^2 + 36t^4 + \cdots = \frac{1}{1-2t} + \frac{1}{1+2t},$$

after an easy calculation, we obtain :

$$f(t) + g(t) = 2\left(\sum_{n=0}^{\infty}(4)^n t^{2n}\right).$$

Definition 1.4.8 *The subtraction of f and g is defined by :*

$$f(t) - g(t) = \sum_{n=0}^{\infty}(a_n - b_n)t^n. \tag{1.14}$$

Example 1.7 *Let g be defined by :*

$$\begin{aligned} g(x) &= 1 + 3t + 6t^2 + 10t^3 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}t^n \\ &= \frac{1}{(1-t)^3}, \end{aligned}$$

and h defined by

$$\begin{aligned} h(t) &= 1 + 4t + 9t^2 + 16t^3 + 25t^4 + \cdots \\ &= \sum_{n=0}^{\infty}(1+n)^2 t^n \\ &= \frac{(1+t)}{(1-t)^3}. \end{aligned}$$

we have

$$h(t) - g(t) = 0 + t + 3t^2 + 6t^3 + \cdots = \sum_{n=0}^{\infty}(1+n)t^n = \frac{t}{(1-t)^3}.$$

Definition 1.4.9 *(Product)*
The product of f and g is defined by

$$f(t)\cdot g(t) = \left(\sum_{n=0}^{\infty}a_n t^n\right)\left(\sum_{n=0}^{\infty}b_n t^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_k b_{n-k}\right)t^n. \tag{1.15}$$

Example 1.8 *Let the two generating functions be given by*

$$f(t) = \sum_{n=0}^{\infty}3nt^n \quad \text{and} \quad g(t) = \sum_{n=0}^{\infty}6t^n.$$

The product of these two generating functions is then :

$$\begin{aligned} h(t) = f(t)\cdot g(t) &= 3(0 + 1t + 2t^2 + \cdots) \times (1 + 6t + 36t^2 + \cdots) \\ &= (0\times 1)t^0 + (0\times 6 + 3\times 1)t + (0\times 36 + 3\times 6 + 6\times 1)t^2 + \cdots \\ &= 3t + 24t^2 + 153t^3 + \cdots \\ &= \frac{3}{(1-t)^2(1-5t)}. \end{aligned}$$

1.4.2 Important operations with generating functions

Theorem 1.1 *Generating functions are closed under multiplication and addition.*

Theorem 1.2 *The multiplication of generating functions is distributive with respect to the addition that is :*

$$f(t) \cdot (g(t) + h(t)) = f(t) \cdot g(t) + f(t) \cdot h(t) = (g(t) + h(t)) \cdot f(t).$$

Identities and inverses

The set of the generating functions, also have identity elements. First we want to find $I(t)$ such that :

$$f(t) + I(t) = f(t) = I(t) + f(t).$$

Here, obviously, $I(t) = 0$. For the product the identity element $I(t)$ is such that

$$f(t)I(t) = I(t)f(t) = f(t).$$

Here, obviously, $I(t) = 1$.

Example 1.9

$$f(t) = -(1-t) - \frac{(1-t)^2}{2} - \frac{(1-t)^3}{3} - \cdots = \sum_{n=1}^{\infty} -\frac{(1-t)^n}{n}.$$

$$g(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t.$$

We have

$$\begin{aligned} f(g(t)) &= -\frac{\left(1 - \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\right)}{1} - \frac{\left(1 - \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\right)^2}{2} - \cdots \\ &= \ln(e^t) \\ &= t. \end{aligned}$$

Let us define the inverse of f, if it exists by g such that

$$f(t) \cdot g(t) = g(t) \cdot f(t) = 1.$$

The element g is denoted by f^{-1}.

Theorem 1.3 *The generating function*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n,$$

has an inverse if and only if $a_0 \neq 0$.

Derivatives and integration

Definition 1.4.10 *Derivative of a generating function $f(t)$. We define the derivative of the generating function f by*

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty} a_n t^n\right) = \sum_{n=1}^{\infty} n a_n \cdot t^{n-1}. \tag{1.16}$$

The second derivative of f is defined similarly, by considering the generating series $\sum_{n=0}^{\infty} b_n t^n$, where $b_i = i a_i$. The higher derivatives are defined by an obvious way.

We use the notation $\frac{d^k}{dt^k}$ for the k^{th} derivative in respect to t.

Example 1.10

$$f(t) = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + t^3 + t^4 + \cdots = \frac{1}{1-t},$$

then

$$\frac{d}{dt} f(t) = 1 + 2t + 3t^2 + 4t^3 + \cdots = \sum_{n=1}^{\infty} n t^{n-1} = \frac{1}{(1-t)^2}.$$

Definition 1.4.11 *We define the integral of the generating function f as*

$$\int_0^t f(t)dt = \int_0^t \sum_{n=0}^{\infty} a_n t^n dt = \sum_{n=0}^{\infty} \frac{a}{n+1} t^{n+1}$$

Almost all the rules known in Calculus for derivatives and integrals are valid with generating functions too.

Example 1.11 *We have $g(t)$ the generating functions :*

$$g(t) = 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{720}t^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} = \cos(t),$$

so

$$\int_0^t g(t)dt = t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \cdots$$

now we note that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} = \sin(t).$$

Compositions

Definition 1.4.12 *Let be f and g be the two generating functions :*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad g(t) = \sum_{n=0}^{\infty} b_n t^n,$$

the composition of f and g is defined by

$$f(g(t)) = \sum_{n=0}^{\infty} a_n \left(\sum_{m=0}^{\infty} b_m t^m \right)^n. \tag{1.17}$$

Example 1.12 *Consider the two generating functions :*

$$f(t) = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n = \frac{1}{(1-t)},$$
$$g(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = e^t,$$

so

$$f(g(t)) = 1 + \sum_{n=0}^{\infty} \frac{t^n}{n!} + \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \right)^2 + \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \right)^3 + \cdots = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{t^m}{m!} \right)^n,$$

this summation is just

$$\frac{1}{1-e^t}.$$

In the following tables, we provide a compilation of fundamental ordinary generating functions and some operations on it

$G(t)$	Summation Notation	Expanded Notation
$\frac{1}{1-t}$	$\sum_{k=0}^{\infty} t^k$	$1+t+t^2+t^3+\cdots$
$\frac{1}{1+t}$	$\sum_{k=0}^{\infty} (-1)^k t^k$	$1-t+t^2-t^3+\cdots$
$\frac{1}{1-t^n}$	$\sum_{k=0}^{\infty} t^{nk}$	$1+t^n+t^{2n}+t^{3n}+\cdots$
$\frac{1}{1-ct^n}$	$\sum_{k=0}^{\infty} c^k t^{nk}$	$1+ct^n+c^2t^{2n}+c^3t^{3n}+\cdots$
$\frac{1}{(1-t)^n}$	$\sum_{k=0}^{\infty} \binom{n+k-1}{k} t^k$	$1+nt+\binom{n+1}{2}t^2+\binom{n+2}{3}t^3+\cdots$
$\frac{1}{(1-t)^2}$	$\sum_{k=0}^{\infty} (k+1)t^k$	$1+2t+3t^2+4t^3+\cdots$
$(1+t)^c$	$\sum_{k=0}^{\infty} \binom{c}{k} t^k$	$1+ct+\binom{c}{2}t^2+\binom{c}{3}t^3+\cdots$
e^t	$\sum_{k=0}^{\infty} \frac{1}{k!} t^k$	$1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots$

Table 1.2: Table elementary ordinary generating functions

Operation OGF	Squence
$f(t)=\sum_{n\geq 0} a_n t^n$	$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$
$g(t)=\sum_{n\geq 0} b_n t^n$	$b_0, b_1, b_2, b_3, \ldots, b_n, \ldots$
$tf(t)=\sum_{n\geq 1} a_{n-1} t^n$	$a_0, a_1, a_2, a_3, \ldots, a_{n-1}, \ldots$
$\frac{f(t)-a_0}{t}=\sum_{n\geq 0} a_{n+1} t^n$	$a_1, a_2, a_3, \ldots, a_{n+1}, \ldots$
$f'(t)=\sum_{n\geq 0}(n+1)a_{n+1} t^n$	$a_1, 2a_2, \ldots, (n+1)a_{n+1}, \ldots$
$\int_0^t f(t)dt=\sum_{n\geq 1} \frac{a_{n-1}}{n} t^n$	$a_0, \frac{a_1}{2}, \frac{a_2}{3}, \frac{a_3}{4}, \ldots, \frac{a_{n-1}}{n}, \ldots$
$f(\lambda t)=\sum_{n\geq 0} \lambda^n a_n t^n$	$a_0, \lambda a_1, \lambda^2 a_2, \lambda^3 a_3, \ldots, \lambda^n a_n, \ldots$
$f(t)+g(t)=\sum_{n\geq 0}(a_n+b_n)t^n$	$a_0+b_0, a_1+b_1, a_2+b_2, a_3+b_3, \ldots, a_n+b_n, \ldots$
$(1-t)f(t)=a_0+\sum_{n\geq 1}(a_n-a_{n-1})t^n$	$a_0, a_1-a_0, \ldots, a_n-a_{n-1}, \ldots$
$f(t)\cdot g(t)=\sum_{n\geq 0}\left(\sum_{0\leq k\leq n} a_k b_{n-k}\right) t^n$	$a_0b_0, a_1b_0, a_0b_1, \ldots, \sum_{0\leq k\leq n} a_k b_{n-k}, \ldots$

Table 1.3: Operation on ordinary generating functions

1.5 The Stirling numbers

1.5.1 Stirling numbers of the first kind

Definition 1.5.1 *The signed Stirling numbers of the first kind, denoted as $s(n,k)$, are the coefficients of the falling factorial polynomials in the canonical basis x^n, that's to say :*

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)x^k. \tag{1.18}$$

Definition 1.5.2 *The unsigned Stirling numbers of the first kind, denoted as $\mid s(n,k) \mid$, are the coefficients of the rising factorial polynomials in the canonical basis x^n, that's to say :*

$$x^{\overline{n}} = \sum_{k=0}^{n} |s(n,k)|x^k,$$

we have :

$$|s(n,k)| = (-1)^{n-k}s(n,k).$$

Definition 1.5.3 *(Combinatorics) The Stirling numbers of the first kind $\mid s(n,k) \mid$ counts the number of permutations of n objects that consist of exactly k disjoint cycles.*

Example 1.13 *Write $x^{\underline{2}}$ as a polynomial*

$$\begin{aligned} x^{\underline{2}} &= x(x-1) \\ &= x^2 - x. \end{aligned}$$

We can say from this example that :
$s(2,0) = 0$.
$s(2,1) = -1$.
$s(2,2) = 1$.

The generating function

The exponential generating function of the signed Stirling numbers of the first kind is given by :

$$\sum_{n=k}^{\infty} s(n,k)\frac{t^n}{n!} = \frac{(\ln(1+t))^k}{k!}.$$

The explicit formula

The explicit formula of the signed and unsigned Stirling numbers of the first kind are given respectively by :

$$s(n,k) = \frac{n!}{k!}\sum_{i_1+\cdots+i_k=n}\frac{1}{i_1\cdots i_k},$$

and

$$|s(n,k)| = \sum_{j,h=0}^{n-k}(-1)^{j+h}\binom{h}{j}\binom{n-1+h}{n-k+h}.$$

The recurrence relation

The Stirling numbers of the first kind satisfy the following recurrence relation :

$$\begin{cases} s(n,0) = s(0,k) = 0, \quad \text{except} \quad s(0,0) = 1, \\ s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k), \quad (n,k \geqslant 1). \end{cases} \tag{1.19}$$

Table of the Stirling numbers of the first kind

Using the triangular recurrence relation (1.19), we can calculate the Stirling numbers of the first kind $s(n,k)$:

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	−1	1					
3	0	2	−3	1				
4	0	−6	11	−6	1			
5	0	24	−50	35	−10	1		
6	0	−120	274	−225	85	−15	1	
7	0	720	−1764	1624	−735	175	−21	1

Table 1.4: Stirling numbers of the first kind

1.5.2 Stirling numbers of the second kind

Definition 1.5.4 *Stirling numbers of the second kind, denoted as* $\left\{ {n \atop k} \right\}$ *are the coefficients of the polynomial* x^n*, that's to say :*

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}.$$

Definition 1.5.5 *(Combinatorics) The Stirling numbers of the second kind,* $\left\{ {n \atop k} \right\}$ *counts the number of partitions of the set* $\{1, 2, \ldots, n\}$ *into* k *non-empty subsets (*$0 \leq k \leq n$*).*

Example 1.14 *The set* $[4]$ *contain* 7 *partitions into two non-empty subsets* $\left\{ {4 \atop 2} \right\}$ *represent the partitions into* 2 *subsets :*

$$\begin{gathered}\{\{1\}\{2,3,4\}\}, \quad \{\{2\}\{1,3,4\}\}, \quad \{\{3\}\{1,2,4\}\}, \quad \{\{4\}\{1,2,3\}, \\ \{1,2\}\{3,4\}\}, \quad \{\{1,3\}\{2,4\}\}, \quad \{\{1,4\}\{2,3\}\}.\end{gathered}$$

As a result, $\left\{ {4 \atop 2} \right\} = 7$*.*

The generating function

The ordinary generating function of Stirling numbers of the second kind is given by :

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} t^n = \frac{t^n}{(1-t)(1-2t)\cdots(1-kt)}.$$

The exponential generating function of Stirling numbers of the second kind is given by :

$$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\} \frac{t^n}{n!} = \frac{1}{k!}(e^t - 1)^k.$$

The explicit formula

The Stirling numbers of the second kind are given by the following explicit formula :

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

The recurrence relation

The Stirling numbers of the second kind satisfy the following recurrence relation :

$$\begin{cases} \left\{ {n \atop 0} \right\} = \left\{ {0 \atop k} \right\} = 0, \quad \text{except} \quad \left\{ {0 \atop 0} \right\} = 1, \\ \\ \left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}, \ n \geq 1. \end{cases}$$

Table of Stirling numbers of second kind

$n \diagdown k$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	1					
3	0	1	3	1				
4	0	1	7	6	1			
5	0	1	15	25	10	1		
6	0	1	31	90	65	15	1	
7	0	1	63	301	350	140	21	1

Table 1.5: Stirling numbers of the second kind

1.6 Numbers and Bell polynomials

1.6.1 The Bell numbers

Definition 1.6.1 *The number of partitions of a set with n elements is measured by the Bell numbers, denoted ϕ_n.*

They can also be defined as the sum of the Stirling numbers of the second kind

$$\phi_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}.$$

The generating functions

The ordinary and exponential generating functions of the Bell numbers are given respectively by :

$$\sum_{n\geq 0} \phi_n x^n = \sum_{n\geq 0} \frac{x^n}{(1-x)(1-2x)\cdots(1-nx)}$$

and

$$\sum_{n\geq 0} \phi_n \frac{x^n}{n!} = e^{e^x - 1}.$$

The recurrence relation

Bell numbers can be computed using the following recurrence relation

$$\phi_0 = 1 \quad \text{and} \quad \phi_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \phi_k.$$

The initial values of these numbers are given in the following table

n	1	2	3	4	5	6	7	8
ϕ_n	1	2	5	15	52	203	877	4140

Table 1.6: The first values of Bell numbers

The Dobiniski's formula

We have the following Dobinski's formula :

$$\phi_n = \frac{1}{e}\sum_{k\geq 0}\frac{k^n}{k!}.$$

Another explicit formula defined by B.N. Guo and F. Qi in terms of the Kummer hypergeometric function and the Stirling numbers of the second kind

$$\phi_n = \frac{1}{e}\sum_{k=0}^{n}(-1)^{n-k}k!\left\{{n \atop k}\right\}\ {}_1F_1\left(\begin{matrix}k+1\\2\end{matrix};1\right),$$

where the Kummer hypergeometric function ${}_1F_1$ is defined by :

$${}_1F_1\left(\begin{matrix}a\\b\end{matrix};x\right) = \sum_{n\geq 0}\frac{a^{\overline{n}}}{b^{\overline{n}}}\frac{x^n}{n!}.$$

1.6.2 The Bell polynomials

Definition 1.6.2 *The Bell polynomials $\phi_n(x)$, also known as exponential polynomials, are defined by :*

$$\phi_n(x) = \sum_{k=0}^{n}\left\{{n \atop k}\right\}x^k.$$

The first Bell polynomials are

$$\begin{aligned}\phi_0(x) &= 1\\ \phi_1(x) &= x\\ \phi_2(x) &= x + x^2\\ \phi_3(x) &= x + 3x^2 + x^3\\ \phi_4(x) &= x + 7x^2 + 6x^3 + x^4.\end{aligned}$$

The generating function

The exponential generating function for these polynomials is as follows :

$$\sum_{n\geq 0}\phi_n(x)\frac{z^n}{n!} = e^{x(e^z-1)}.$$

The explicit formula

An explicit formula of Bell polynomials $\phi_n(x)$

$$\phi_n(x) = \sum_{k=0}^{n}(-1)^{n-k}\left\{{n \atop k}\right\}L_k(x).$$

The recurrence relation

The polynomials $\phi_n(x)$ satisfy the following recurrence relation

$$\phi_n(0) = 0,\ \phi_{n+1}(x) = x\sum_{k=0}^{n}s(n,k)\phi_k(x).$$

Dobinski's formula

The polynomials $\phi_n(x)$ are given by :

$$\phi_n(x) = e^{-x} \sum_{k \geq 0} \frac{k^n}{k!} x^k.$$

We notice the Bell numbers satisfy :

$$\phi_n = \phi_n(1).$$

1.6.3 The partial Bell polynomials

Definition 1.6.3 *The (exponential) partial Bell partition polynomials $\phi_{n,k}(x_1, x_2, ...)$ in an infinite number of variables x_j , $(j \geq 1)$, were introduced by Bell as a mathematical tool for representing the n-th derivative of composite functions.*

The generating function

For any integer $k \geq 0$, the partial Bell polynomials are defined by their generating function

$$\frac{1}{k!} \left(\sum_{i=1}^{\infty} x_i \frac{t^i}{i!} \right)^k = \sum_{n=k}^{\infty} \phi_{n,k}(x_1, x_2, ..., x_{n-k+1}) \frac{t^n}{n!}. \tag{1.20}$$

The explicit formula

This polynomials are given explicitly by the formula

$$\phi_{n,k}(x_1, x_2, ..., x_n) = \sum_{\pi(n,k)} \frac{n!}{k_1!...k_n!} \left(\frac{x_1}{1!} \right)^{k_1} \left(\frac{x_2}{2!} \right)^{k_2} ... \left(\frac{x_n}{n!} \right)^{k_n}, \tag{1.21}$$

where

$$\pi(n,k) = \left\{ (k_1, ..., k_n) \in \mathbb{N}^n : \sum_{i=1}^{n} k_i = k, \ \sum_{i=1}^{n} i k_i = n \right\}.$$

It is easy to obtain the following expression :

$$\phi_{n,k}(abx_1, ab^2 x_2, ..., ab^n x_n) = a^k b^n \phi_{n,k}(x_1, x_2, ..., x_n).$$

It is well-known that for appropriate choices of the variables x_J, the (exponential) partial Bell partition polynomials can be reduced to some special combinatorial sequences. We will mention the following special cases

$$\begin{aligned}
s(n,k) &= \phi_{n,k}(0!, -1!, 2!, -3!, ...), \text{ (signed) Stirling numbers of the first kind,} \\
\left\{ n \atop k \right\} &= \phi_{n,k}(1, 1, 1, ...), \text{ Stirling numbers of the second kind,} \\
\left\lfloor n \atop k \right\rfloor &= \phi_{n,k}(1!, 2!, 3!, ...), \text{ (unsigned) Lah numbers.}
\end{aligned}$$

From the generating function of the Bell polynomials, it follows that

$$\phi_{n,k}(x_1, x_2, ..., x_n) = \sum_{k=0}^{n} \phi_{n,k}(x_1, x_2, ..., x_{n-k+1}).$$

1.7 The Bernoulli numbers and polynomials

1.7.1 The Bernoulli numbers

Definition 1.7.1 *For all* $n \geq 0$*, the Bernoulli numbers, denoted* B_n *are defined by :*

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = n+1.$$

with $B_0 = 1$.

The first values of B_n

n	0	1	2	3	4	5	6	7
B_n	1	1/2	1/6	0	-1/30	0	1/42	0

Table 1.7: The first values of Bernoulli numbers

Remark 1 *All the Bernoulli numbers are rational.*

The generating function

The Bernoulli numbers are defined by the generating function

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{te^t}{e^t - 1}.$$

The explicit formula

The explicit formula for the Bernoulli numbers is given by :

$$B_n = (-1)^n \sum_{k=0}^{n} (-1)^k \frac{k!}{k+1} \left\{ n \atop k \right\}.$$

1.7.2 The Bernoulli polynomials

Definition 1.7.2 *For* $n \in \mathbb{N}$*, the Bernoulli polynomials, denoted* $B_n(x)$*, are defined by :*

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$

with $B_n(1) = B_n$.

Here are the first few values of the Bernoulli polynomials

$$\begin{aligned}
B_0(x) &= 1, \\
B_1(x) &= x - \frac{1}{2}, \\
B_2(x) &= x^2 - x + \frac{1}{6}, \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}.
\end{aligned}$$

The generating function

The exponential generating function of these polynomials is given by :

$$\sum_{n\geq 0} B_n(x)\frac{t^n}{n!} = \frac{te^{tx}}{e^t-1}.$$

Property 1.7.1

1. $B_0(x) = 1$.
2. $B_n(0) = B_n(1)$, *with* $n \geq 0$.
3. *For* $n \geq 1$, $B_n'(x) = nB_{n-1}(x)$.
4. $\int_0^1 B_n(x)\,dx = 0, \quad (n \geq 1)$.
5. $B_{2p+1}(0) = B_{2p+1}(1) = 0, \quad (p \in \mathbb{N})$.
6. *For* $n \in \mathbb{N}$ $B_n(x) = (-1)^n B_n(1-x)$.
7. $B_n(x+1) - B_n(x) = nx^{n-1}, \quad (n \in \mathbb{N})$.

1.8 The Lah numbers and polynomials

1.8.1 The Lah numbers

Definition 1.8.1 *The Lah numbers (unsigned)* $\left\lfloor {n \atop k} \right\rfloor$ *are coefficients expressing rising factorial in terms of falling factorial*

$$x^{\overline{n}} = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor x^{\underline{n}} \tag{1.22}$$

and

$$x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} \left\lfloor {n \atop k} \right\rfloor x^{\overline{n}}.$$

Definition 1.8.2 *(Combinatorics) These numbers count the number of partitions of a set of* n*elements into exactly* k *ordered lists.*

Example 1.15 *In the set* $[3]$*, there are* 8 *subsets.*

$$\{\emptyset; \{1\}; \{2\}; \{3\}; \{1,\ 2\}; \{1,\ 3\}; \{2,\ 3\}; \{1,\ 2,\ 3\}\}.$$

For $\left\lfloor {3 \atop 2} \right\rfloor$*, there are* 6 *subsets with two elements :*

$$\{1,\ (2,3)\}; \{1,\ (3,2)\}; \{2,\ (1,3)\}; \{2,\ (3,1)\}; \{3,\ (1,2)\}; \{3,\ (2,1)\}.$$

Consequently $\left\lfloor {3 \atop 2} \right\rfloor = 6$.

The generating function

The exponential generating function of Lah numbers is as follows :

$$\sum_{n\geq k} \left\lfloor {n \atop k} \right\rfloor \frac{x^n}{n!} = \frac{1}{k!}\left(\frac{x}{1-x}\right)^k.$$

The explicit formula

The Lah numbers have the following explicit formula :

$$\left\lfloor {n \atop k} \right\rfloor = \frac{n!}{k!}\binom{k-1}{n-1},\ n \geq k \geq 1. \tag{1.23}$$

The recurrence relation

For $n \geq k \geq 1$, we have the recurrence relation :

$$\left\lfloor {n \atop k} \right\rfloor = \left\lfloor {n-1 \atop k-1} \right\rfloor + (n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor, \tag{1.24}$$

and $\left\lfloor {n \atop 0} \right\rfloor = \delta_{n,0}$ for $n \geq 0$, $\left\lfloor {0 \atop 0} \right\rfloor = 1$, $\left\lfloor {n \atop 0} \right\rfloor = \left\lfloor {n \atop k} \right\rfloor$.
Using the recurrence relation, the following table represents the initial values of these numbers

$n \diagdown k$	1	2	3	4	5	6	7
1	1						
2	2	1					
3	6	6	1				
4	24	36	12	1			
5	120	240	120	20	1		
6	720	1800	1200	300	30	1	
7	5040	15120	12600	4200	630	42	1

Table 1.8: The first values of Lah numbers

1.8.2 The Lah polynomials

Definition 1.8.3 *The Lah polynomials, denoted by $L_n(x)$, are defined by*

$$L_n(x) = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor x^n.$$

The first values of Lah polynomials are

$$\begin{aligned}
L_0(x) &= 1 \\
L_1(x) &= x \\
L_2(x) &= x^2 + 2x \\
L_3(x) &= x^3 + 6x^2 + 6x \\
L_4(x) &= x^4 + 12x^3 + 36x^2 + 24x \\
L_5(x) &= x^5 + 20x^4 + 120x^3 + 240x^2 + 120x \\
L_6(x) &= x^6 + 30x5^5 + 300x^4 + 1200x^3 + 1800x^2 + 720x \\
L_7(x) &= x^7 + 42x^6 + 630x^5 + 4200x^4 + 12600x^3 + 15120x^2 + 5040x.
\end{aligned}$$

1.9 The Fubini polynomials and numbers

1.9.1 The Fubini numbers

Definition 1.9.1 *For all $n \in \mathbb{N}$, the Fubini numbers, denoted F_n are defined by :*

$$F_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k!,$$

these numbers count all the possible set partitions of an n element set such that the order of the blocks matters. Besides with this combinatorial property, these numbers are seen in the evaluation of the following series

$$\sum_{k=0}^{\infty} \frac{k^n}{2^k} = 2F_n.$$

The first values of F_n are

n	0	1	2	3	4	5	6	7
F_n	1	1	3	13	75	541	4683	47293

Table 1.9: The first values of Fubini numbers

The generating function

The generating function of the Fubini numbers is given by

$$\sum_{n \geq 0} F_n \frac{t^n}{n!} = \frac{1}{2 - e^t}.$$

The recurrence relation

$$2F_n = \sum_{k=0}^{n} \binom{n}{k} F_k, \quad n > 0,$$

and

$$2 \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_k = (-1)^n F_n + 1, \quad n \geq 0,$$

1.9.2 The Fubini polynomials

Definition 1.9.2 *For $n \geq 0$, The Fubini polynomials are defined by :*

$$F_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! x^k,$$

with $F_n(1) = F_n$, $F_n(-2) = (-1)^n 2F_n$.

The generating function

The exponential generating function for Fubini polynomials is given as :

$$\frac{1}{1 - x\left(e^t - 1\right)} = \sum_{n=0}^{\infty} F_n\left(x\right) \frac{t^n}{n!}$$

and are related to the geometric series

$$\left(x \frac{d}{dx} \right)^n \frac{1}{1-x} = \sum_{k=0}^{\infty} k^n x^k = \frac{1}{1-x} F_n\left(\frac{x}{1-x} \right), \quad |x| < 1.$$

Because of this relation, Fubini polynomials are also called geometric polynomials.

The recurrence relation

The following recurrence relation holds for the Fubini polynomials

$$F_{n+1}(x) = x\frac{d}{dx}\left[F_n(x) + xF_n(x)\right].$$

2

Fundamental concepts in probability and the degenerate numbers and polynomials

In this chapter, we lay the groundwork by introducing fundamental concepts in probability theory, followed by a discussion of several discrete and continuous probability distributions. We then proceed to explore the moment generating function, a crucial tool in probability and statistics for understanding the distribution of a random variable.

The chapter begins with an overview of basic probability notions, providing a solid foundation for the subsequent sections. We cover essential probability laws, including both discrete distributions (such as the binomial and Poisson distributions) and continuous distributions (such as the normal and exponential distributions).

Next, we delve into the moment generating function (M.G.F), explaining its definition, properties, and significance in the study of random variables. The M.G.F is particularly useful for deriving moments and understanding the behavior of distributions.

Finally, we introduce several degenerate polynomials, which play a pivotal role in the probabilistic analysis presented in later chapters. We discuss their definitions, key properties, and relevance to the broader themes of the thesis.

This chapter aims to equip the reader with the necessary probabilistic background and mathematical tools required for a deeper understanding of the advanced topics discussed in subsequent chapters. For more precise and detailed concepts the reader can refer to [8, 12, 16, 19].

2.1 Fundamental concepts in probability

Definition 2.1.1 *A random variable is a function from the sample space to the real numbers*

$$X : \Omega \to \mathbb{R}.$$

If $X(\Omega)$ is at most countable, there are two main types of random variables

2.1.1 Discrete random variables

Definition 2.1.2 *A random variable X takes a finite or countable number of values (its sample space is included in $\mathbb{N}$), it is referred to as a discrete variable.*

Probability mass function (P.M.F)

Definition 2.1.3 *Let X be a discrete random variable with range $X(\Omega) = \{x_1, x_2, x_3, ...\}$ (finite or countable infinite). The function*

$$P_X = P(X = x_k), \text{ for } k = 1,\ 2,\ 3, ...$$

is called the probability mass function (P.M.F) of X.

Remark 2 *$P(\{X = x_i\})$ is equivalent to $P(X = x_i) = f_X(x_i)$).*

Property 2.1.1

1. *$0 \leq P(X = x) \leq 1$ for all x.*
2. *$\sum_{x \in Support(X)} P(X = x) = 1$.*
3. *For all subset $A \subset X(\Omega)$, we have $P(X \in A) = \sum_{x \in A} P_X(x)$.*

Consider n discrete random variables $X_1, X_2, \ldots, X_n$. We say that $X_1, X_2, \ldots, X_n$ are independent if

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P(X_1 = x_1) \times P(X_2 = x_2) \times \cdots \times P(X_n = x_n),$$

for all $x_1, x_2, \ldots, x_n$.

2.1.2 Cumulative distribution function (C.D.F)

Definition 2.1.4 *The cumulative distribution function (C.D.F) of a random variable X is defined as*

$$F_X(x) = P(X \leq x), \quad \text{for all } x \in \mathbb{R}.$$

Property 2.1.2

1. *$\forall x \in \mathbb{R}$, $0 \leq F_X(x) \leq 1$.*
2. *F_X is increasing over $\mathbb{R}$.*
3. *$\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$.*
4. *if $a \leq b$ $P(a \leq X \leq b) = F_X(b) - F_X(a)$.*

2.1.3 Continuous random variables

Definition 2.1.5 *Continuous random variables can take any value within a continuous interval of real numbers. It is denoted as: $X : \Omega \longrightarrow \mathbb{R}$.*

Probability density function (P.D.F)

Definition 2.1.6 *Consider a continuous random variable X with an absolutely continuous (C.D.F) $F_X(x)$. The function $f_X(x)$ defined by*

$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x), \quad \text{if } F_X(x) \text{ is differentiable at } x$$

is called the probability density function (P.D.F) of X.

Property 2.1.3

1. *$f_X(x) \geq 0$ for all $x \in \mathbb{R}$.*
2. *$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$.*
3. *$P(a \leq X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\, dx$.*
4. *More generally, for a set A, $P(X \in A) = \int_A f_X(x)\, dx$.*

2.1.4 Numerical characteristics

The expectation

Definition 2.1.7 *The expectation (or expected value) of a discrete random variable X is defined by*

$$E(X) = \sum_{i=1}^{\infty} x_i P(X = x_i).$$

Property 2.1.4

1. $E(g(X)) = \sum_x g(x) f_X(x)$.
2. $E(X^n) = \sum_x x^n f_X(x)$.*moment of order n.*
3. $E(aX + bY) = aE(X) + bE(Y)$, *a and b are constants.*
4. *If X and Y are independent, then* $E(XY) = E(X) \times E(Y)$.

The variance

Definition 2.1.8 *The variance is the dispersion of possible values around the expected value.*

Definition 2.1.9 *The variance of a random variable X is defined by :*

$$\begin{aligned} Var(X) &= E(X - E(X))^2. \\ &= \sum_x (x - E(X))^2 f_X(x) = E(X^2) - (E(X))^2 \\ &= \sum_x x^2 f_X(x) - \left(\sum_x x f_X(x) \right)^2. \end{aligned}$$

The covariance

Definition 2.1.10 *Let X and Y be two random variables. The covariance of X and Y, denoted as $Cov(X, Y)$, is defined by:*

$$\begin{aligned} Cov(X, Y) &= E((X - E(X))(Y - E(Y))) \\ &= E(XY) - E(X)E(Y). \end{aligned}$$

proposition 1 *Let X and Y be two random variables. Then, we have the following properties:*

1. $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$.
2. $X \perp Y \implies Cov(X, Y) = 0$.
3. $Cov(X, Y) = 0 \not\Rightarrow X \perp Y$.

The standard deviation

Definition 2.1.11 *The standard deviation σ is the square root of the variance, and it is defined by:*

$$\sigma = \sqrt{Var(X)}.$$

2.2 Moment generating functions (M.G.F)

Definition 2.2.1 *The moment generating function (M.G.F) of a random variable X is a function $M_X(t)$ defined as*

$$M_X(t) = E[e^{tX}].$$

Remark 3 *We say that M.G.F of X exists, if there exists a positive constant a such that $M_X(t)$ is finite for all $t \in [-a, a]$.*

Remark 4

1. *The n^{th} moment of a random variable X is defined to be $E[X^n]$.*
2. *The n^{th} central moment of X is defined to be $E[(X - E(X))^n]$.*

2.2.1 Finding moments from (M.G.F)

Recall the Taylor series expansion for e^x: for all $x \in \mathbb{R}$, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Now, we can write

$$e^{tX} = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!} = \sum_{k=0}^{\infty} \frac{X^k t^k}{k!}.$$

Thus, we have

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} E[X^k] \frac{t^k}{k!}, \quad (|t| > 0). \tag{2.1}$$

We conclude that the k-th moment of X is the coefficient of $\frac{t^k}{k!}$ in the Taylor series expansion of $M_X(t)$. Thus, if we have the Taylor series of $M_X(t)$, we can obtain all moments of X. We can derive all moments of X^k from its M.G.F

$$M_X(t) = \sum_{k=0}^{\infty} E[X^k] \frac{t^k}{k!},$$

$$E[X^k] = \frac{d^k}{dt^k} M_X(t)\Big|_{t=0}.$$

2.2.2 Sum of independent random variables

Suppose X_1, X_2, ..., X_n are n independent random variables, and the random variable Y is defined by

$$Y = X_1 + X_2 + ... + X_n.$$

Then

$$\begin{aligned}
M_Y(t) &= E[e^{tY}] \\
&= E[e^{t(X_1+X_2+...+X_n)}] \\
&= E[e^{tX_1} e^{tX_2} ... e^{tX_n}] \\
&= E[e^{tX_1}] \times E[e^{tX_2}] \times ... \times E[e^{tX_n}] \ \text{ (since the } X_i \text{ 's are independent)} \\
&= M_{X_1}(t) \times M_{X_2}(t) \times ... \times M_{X_n}(t).
\end{aligned}$$

2.3 Discrete distributions

2.3.1 Bernoulli distribution

A random variable X is said to be a Bernoulli random variable with parameter $p \in [0,1]$, denoted as $X \sim B(p)$, if its P.M.F is given by

$$P(X = k) = p^k q^{1-k},$$

where $k \in \{0,1\}$ and $q = 1 - p$.

Subsequently, X has an expected value and a variance given by :

$$E(X) = p \quad \text{and} \quad V(X) = p(1-p) = pq.$$

2.3.2 Binomial distribution

It is said that a random variable $X : (\Omega, P) \to \mathbb{R}$ follows a binomial distribution with parameters $n \geq 1$ and $p \in [0,1]$, denoted as $X \sim B(n,p)$, if for any $k \in \{0, \ldots, n\}$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

X possesses an expected mean and a variance as follows :

$$E(X) = np \quad \text{and} \quad V(X) = np(1-p).$$

2.3.3 Poisson distribution

Let $\lambda > 0$. It is said that a random variable X follows the Poisson distribution with parameter λ, denoted as $X \sim P(\lambda)$, if $X(\Omega) = \mathbb{N}$, and its P.M.F is given by

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \text{ for } k \in X(\Omega).$$

Then, X has an expected value (mean) and variance, which are given by :

$$E(X) = \lambda \quad \text{and} \quad V(X) = \lambda.$$

2.3.4 Geometric distribution

It is said that a random variable X follows the geometric distribution with parameter $p \in [0,1]$, denoted as $X \sim G(p)$, if $X(\Omega) = \mathbb{N}^*$

$$P(X = k) = \begin{cases} p(1-p)^{k-1} & \text{for } k \in X(\Omega), \\ 0 & \text{otherwise.} \end{cases}$$

X then has an expected value (mean) and a variance given by :

$$E(X) = \frac{1}{p} \quad \text{and} \quad V(X) = \frac{1-p}{p^2}.$$

2.4 Continuous distributions

2.4.1 Exponential distribution

A continuous random variable X is said to have an exponential distribution with parameter $\lambda > 0$, denoted as $X \hookrightarrow \varepsilon(\lambda)$, if its P.D.F is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $X \hookrightarrow \varepsilon(\lambda)$, then

$$E(X) = \frac{1}{\lambda} \text{ and } var(X) = \frac{1}{\lambda^2}.$$

If X is exponential with parameter $\lambda > 0$, then X is a memoryless random variable, that is

$$P(X > x + a \mid X > a) = P(X > x), \quad \text{for } a,\ x \geq 0.$$

2.4.2 Normal distribution

It is said that a random variable X follows the normal distribution with parameters $m \in \mathbb{R}$ and σ^2, where $\sigma > 0$, denoted as $X \hookrightarrow N(m,\, \sigma^2)$, if it is continuous and has the probability density function

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right).$$

This is equivalent to saying that $\left(\frac{X-m}{\sigma}\right)$ follows a standard normal distribution $N(0,1)$. Note that if X follows a normal distribution $N(m, \sigma^2)$, as a result, X entails an expected mean and a variance outlined as :

$$E(X) = m \quad \text{and} \quad V(X) = \sigma^2.$$

2.4.3 Continuous Uniform distribution

A continuous random variable X is said to have a Uniform distribution over the interval $[a, b]$ (a is different from b), denoted as $X \sim \text{Uniform}(a, b)$, if its PDF is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

In this case, X has an expected value (mean) and a variance given by :

$$E(X) = \frac{a+b}{2} \quad \text{and} \quad V(X) = \frac{(b-a)^2}{12}.$$

2.4.4 Gamma distribution

It is said that X follows the gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$, denoted $\Gamma(\alpha, \lambda)$, if it has the density function

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} e^{-\lambda x} x^{\alpha-1} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha > 0$, the gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx.$$

Note that

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha),$$

$$\Gamma(n) = (n-1)! \quad \text{for } n \in \mathbb{N}^*.$$

A random variable X with this distribution has an expected value and a variance given by

$$E(X) = \frac{\alpha}{\lambda}, \quad \text{and} \quad V(X) = \frac{\alpha}{\lambda^2}.$$

2.5 The degenerate numbers and polynomials

2.5.1 The degenerate exponential

For any $\lambda \in \mathbb{R}$, the degenerate exponential are defined by :

$$e_\lambda^x(t) = (1+\lambda t)^{\frac{x}{\lambda}}, \quad e_\lambda^1(t) = e_\lambda(t). \tag{2.2}$$

From (2.2), we note that

$$e_\lambda^x(t) = \sum_{k=0}^{\infty} \frac{(x)_{n,\lambda}}{n!} t^n, \quad \lim_{\lambda \to 0} e_\lambda^x(t) = e^{xt}, \tag{2.3}$$

with $(x)_{n,\lambda}$ falling factorial of x of order n and coefficient λ.

2.5.2 The degenerate Stirling numbers of the second kind

1. ***Generating function*** : The degenerate Stirling number of the second kind is defined by the generating function

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \frac{1}{k!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^k = \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\}_\lambda \frac{t^n}{n!} \quad (k \geq 0).$$

2. ***Explicit formula*** : The explicitly given by

$$\left\{ {n \atop k} \right\}_\lambda = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j)_{n,\lambda}.$$

3. ***Recurrence relation*** : For $n, k \geq 0$, the degenerate Stirling numbers satisfy the recurrence relations

$$\left\{ {n+1 \atop k+1} \right\}_\lambda = \sum_{i=0}^{n-k} \binom{n}{i} (1)_{i+1,\lambda} \left\{ {n-i \atop k} \right\}_\lambda$$

and

$$\left\{ {n+1 \atop k+1} \right\}_\lambda = \frac{1}{k+1} \sum_{i=0}^{n-k} \binom{n+1}{i+1} (1)_{i+1,\lambda} \left\{ {n-i \atop k} \right\}_\lambda.$$

Substituting $\lambda = -1$, we get the following recurrence relations for Lah numbers. We have

$$\left[{n+1 \atop k+1} \right] = \binom{n}{i} (i+1)! \left[{n-i \atop k} \right]$$

and

$$\left[{n+1 \atop k+1} \right] = \frac{1}{k+1} \sum_{i=0}^{n-k} \binom{n}{i} (i+1)! \left[{n-i \atop k} \right].$$

2.5.3 The degenerate Bell polynomials

1. ***Explicit formula*** : The degenerate Bell polynomials $\phi_{n,\lambda}(x)$ are defined by

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_\lambda x^k, \tag{2.4}$$

we have another relation between this polynomials and Bell polynomials given by

$$\phi_{n,\lambda}(x) = \sum_{k=0}^{n} s(n,k) \phi_k(x) \lambda^{n-k}.$$

When $x = 1$, $\phi_{n,\lambda} = \phi_{n,\lambda}(1)$ are called the degenerate Bell numbers.

Note that

$$\lim_{\lambda \to 0} \phi_{n,\lambda} = \phi_n, \quad (n \geq 0).$$

2. ***Generating function*** : The generating function of the degenerate Bell polynomials is defined by :

$$e^{x(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.5}$$

3. ***Recurrence relation*** : For $n \geq 0$, the degenerate Bell polynomials $\phi_{n,\lambda}(x)$ satisfy the recurrence relation

$$\phi_{n+1,\lambda}(x) = x\sum_{k=0}^{n}\binom{n}{k}(1)_{k+1,\lambda}\phi_{n-k,\lambda}(x).$$

By setting $\lambda = -1$,we obtain

$$L_{n+1}(x) = x\sum_{k=0}^{n}(k+1)!L_{n-k}(x).$$

2.5.4 The degenerate Fubini polynomials

1. ***Explicit formula*** : The degenerate Fubini polynomials are defined by

$$F_{n,\lambda}(x) = \sum_{k=0}^{n}\left\{ n \atop k \right\}_{\lambda} k!x^k, \quad (n \geq 0). \tag{2.6}$$

2. ***Generating function*** : Thus, by (2.6), we get

$$\frac{1}{1-x\left(e_\lambda(t)-1\right)} = \sum_{n=0}^{\infty} F_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.7}$$

From (2.7), we note that

$$\frac{1}{1-x}F_{n,\lambda}\left(\frac{x}{1-x}\right) = \left(x\frac{d}{dx}\right)_{n,\lambda}\frac{1}{1-x} = \sum_{k=0}^{\infty}(k)_{n,\lambda}x^k. \tag{2.8}$$

The degenerate Fubini polynomials of order r

1. ***Generating function*** : For $r \in N$, the degenerate Fubini polynomials of order r are defined by generating function

$$\left(\frac{1}{1-x(e_\lambda(t)-1)}\right)^r = \sum_{n=0}^{\infty} F_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \tag{2.9}$$

2. ***Explicit formula*** : By (2.9), we get

$$F_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n}\binom{k+r-1}{k}x^k\left\{ n \atop k \right\}_{\lambda} k!. \tag{2.10}$$

From (2.9), we have

$$\left(\frac{1}{1-x}\right)^{r+1}F_{n,\lambda}^{(r+1)}\left(\frac{x}{1-x}\right) = \left(x\frac{d}{dx}\right)_{n,\lambda}\left(\frac{1}{1-x}\right)^{r+1} = \sum_{k=0}^{\infty}\binom{k+r}{r}(k)_{n,\lambda}x^k, \tag{2.11}$$

where n, r are nonnegative integers.

2.6 Degenerate function associated with random variable Y

2.6.1 Probabilistic degenerate Stirling numbers of the second kind associated with random variable Y

Let Y be a random variable, assume that $(Y_j)_j \geq 1$ is a sequence of mutually independent copies of Y and $S_k = Y_1 + Y_2 + \cdots + Y_k, (k \geq 1)$ with $S_0 = 0$. The probabilistic degenerate Stirling numbers of the second kind associated with random variable Y are defined by :

$$\left\{ {n \atop k} \right\}_{Y,\lambda} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} E\left[(S_j)_{n,\lambda}\right], \ (n \geq k \geq 0) \tag{2.12}$$

by binomial inversion, the equation (2.12) is equivalent to

$$E\left[(S_k)_{n,\lambda}\right] = \sum_{j=0}^{k} \binom{k}{j} j! \left\{ {n \atop j} \right\}_{Y,\lambda}, \tag{2.13}$$

From (2.12), we note that

$$\frac{1}{k!} \left(E[e_\lambda^Y(t)] - 1\right)^k = \sum_{n=k}^{\infty} \left\{ {n \atop k} \right\}_{Y,\lambda} \frac{t^n}{n!}, \ (k \geq 0). \tag{2.14}$$

2.6.2 Probabilistic degenerate Bell polynomials associated with random variable Y

In view of (2.4), the probabilistic degenerate Bell polynomials associated with Y are defined by

$$\phi_{n,\lambda}^Y(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{Y,\lambda} x^k, \quad (n \geq 0). \tag{2.15}$$

When $Y = 1$, we have $\phi_{n,\lambda}^Y(x) = \phi_{n,\lambda}(x)$. By (2.15), we get

$$e^{x(E[e_\lambda^Y(t)]-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}^Y(x) \frac{t^n}{n!}. \tag{2.16}$$

3

PROBABILISTIC DEGENERATE FUBINI POLYNOMIALS ASSOCIATED WITH RANDOM VARIABLES

In this chapter, we delve into the article by Kim and al [16], which focuses on the study of probabilistic versions of the degenerate Fubini polynomials and their higher-order counterparts. Let Y be a random variable such that its moment generating function exists in a neighborhood of the origin. Kim's work aims to investigate the probabilistic degenerate Fubini polynomials associated with Y, as well as the probabilistic degenerate Fubini polynomials of order r associated with Y.

Throughout the chapter, we explore various properties of these polynomials, including explicit expressions, specific identities, and recurrence relations. We present a detailed analysis of the probabilistic degenerate Fubini polynomials, highlighting their significance and the mathematical insights they offer. Additionally, the chapter discusses the derivation of these properties and how they contribute to the broader understanding of the polynomials in a probabilistic context.

By examining Kim's findings, we provide a comprehensive overview of the theoretical underpinnings and practical implications of the degenerate Fubini polynomials, contributing to the field's existing body of knowledge. For more precise and detailed concepts the reader can refer to [16].

3.1 The probabilistic degenerate Fubini polynomials associated with random variable

Definition 3.1.1 *The probabilistic degenerate Fubini polynomials associated with random variable Y are defined by*

$$\frac{1}{1-x\left(E\left[e_{\lambda}^{Y}(t)\right]-1\right)}=\sum_{n=0}^{\infty}F_{n,\lambda}^{Y}(x)\frac{t^{n}}{n!}. \tag{3.1}$$

For $Y=1$, we have $F_{n,\lambda}^{Y}(x)=F_{n,\lambda}(x)$ for $n\geq 0$. When $x=1$, $F_{n,\lambda}^{Y}(x)=F_{n,\lambda}^{Y}(1)$ are called the probabilistic degenerate Fubini numbers associated with random variable Y.

Theorem 3.1 *For $n\geq 0$, we have*

$$F_{n,\lambda}^{Y}(x)=\sum_{k=0}^{n}\left\{ n\atop k\right\}_{Y,\lambda}k!x^{k}.$$

Proof. From (3.1), we note that

$$\begin{aligned}\frac{1}{1-x\left(E\left[e_{\lambda}^{Y}(t)\right]-1\right)}&=\sum_{k=0}^{\infty}x^{k}\left(E\left[e_{\lambda}^{Y}(t)\right]-1\right)^{k} \\ &=\sum_{k=0}^{\infty}x^{k}k!\frac{1}{k!}\left(E\left[e_{\lambda}^{Y}(t)\right]-1\right)^{k}\end{aligned} \tag{3.2}$$

by (2.14)

$$= \sum_{k=0}^{\infty} x^k k! \sum_{n=k}^{\infty} \left\{ n \atop k \right\}_{Y,\lambda} \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} x^k k! \left\{ n \atop k \right\}_{Y,\lambda} \frac{t^n}{n!}.$$

Therefore, by comparing the coefficients on both sides of (3.2). we obtain

$$F_{n,\lambda}^{Y}(x) = \sum_{k=0}^{n} \left\{ n \atop k \right\}_{Y,\lambda} k! x^k.$$

■

Theorem 3.2 *For $n \geq 0$, we have*

$$F_{n,\lambda}^{Y}(x) = \frac{1}{1+x} \sum_{k=0}^{\infty} \left(\frac{x}{1+x} \right)^k E[(S_k)_{n,\lambda}].$$

In particular, for $Y = 1$, we have

$$F_{n,\lambda}(x) = \frac{1}{1+x} \sum_{k=0}^{\infty} \left(\frac{x}{1+x} \right)^k (k)_{n,\lambda}.$$

Proof. *By (3.2), we get*

$$\begin{aligned} \sum_{n=0}^{\infty} F_{n,\lambda}^{Y}(x) \frac{t^n}{n!} &= \frac{1}{1 - x(E[e_{\lambda}^{Y}(t)] - 1)} = \frac{1}{1 + x - xE[e_{\lambda}^{Y}(t)]} \\ &= \frac{1}{1+x} \frac{1}{1 - \frac{x}{1+x} E[e_{\lambda}^{Y}(t)]} \\ &= \frac{1}{1+x} \sum_{k=0}^{\infty} \left(\frac{x}{1+x} \right)^k (E[e_{\lambda}^{Y}(t)])^k \ \textit{(moment generating function of } Y\textit{)} \\ &= \sum_{n=0}^{\infty} \frac{1}{1+x} \sum_{k=0}^{\infty} \left(\frac{x}{1+x} \right)^k E(Y_1 + Y_2 + \ldots + Y_k)_{n,\lambda} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{1+x} \sum_{k=0}^{\infty} \left(\frac{x}{1+x} \right)^k E[(S_k)_{n,\lambda}] \frac{t^n}{n!}. \end{aligned} \tag{3.3}$$

Therefore, by comparing the coefficients on both sides of (3.3). we obtain the result

Theorem 3.3 *Let $Y \sim \Gamma(1, \alpha)$. Then we have*

$$F_{n,\lambda}^{Y}(x) = \sum_{i=0}^{n} \sum_{l=0}^{i} \sum_{k=0}^{l} (\alpha x)^k k! \left| {l \atop k} \right| \frac{1}{(l-i)!} (1-\alpha)^{l-i} \lambda^{n-i} s(n,i), \ (n \geq 0).$$

In particular, for $\alpha = 1$, we have

$$F_{n,\lambda}^{Y}(x) = \sum_{l=0}^{n} \sum_{k=0}^{l} k! \lambda^{n-l} \left| {l \atop k} \right| s(n,l) x^k, \ (n \geq 0).$$

Proof. Let $Y \sim \Gamma(1,\alpha)$. Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{n,\lambda}^{Y}(x)\frac{t^n}{n!} &= \frac{1}{1-x(E[e_\lambda^Y(t)]-1)} = \sum_{k=0}^{\infty} x^k (E[e_\lambda^Y(t)]-1)^k \qquad (3.4)\\
&= \sum_{k=0}^{\infty} x^k \left(\int_0^\infty e_\lambda^y(t)\alpha e^{-\alpha y}dy - 1\right)^k \\
&= \sum_{k=0}^{\infty} x^k \left(\alpha\int_0^\infty e^{y(\frac{1}{\lambda}\log(1+\lambda t)-\alpha)}dy - 1\right)^k \\
&= \sum_{k=0}^{\infty} (\alpha x)^k \left(\frac{1}{\alpha - \frac{1}{\lambda}\log(1+\lambda t)} - 1\right)^k \\
&= \sum_{k=0}^{\infty} k!(\alpha x)^k \frac{1}{k!}\left(\frac{1-\alpha+\frac{1}{\lambda}\log(1+\lambda t)}{1-\left(1-\alpha+\frac{1}{\lambda}\log(1+\lambda t)\right)}\right)^k \text{ (generating function of the Lah numbers)} \\
&= \sum_{k=0}^{\infty} (\alpha x)^k k! \sum_{l=k}^{\infty} \left[\begin{matrix} l \\ k \end{matrix}\right] \frac{1}{l!}\left(1-\alpha+\frac{1}{\lambda}\log(1+\lambda t)\right)^l \\
&= \sum_{k=0}^{\infty} (\alpha x)^k k! \sum_{l=k}^{\infty} \left[\begin{matrix} l \\ k \end{matrix}\right] \frac{1}{l!}\sum_{i=0}^{l}\binom{l}{i}(1-\alpha)^{l-i}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^i \\
&= \sum_{k=0}^{\infty} (\alpha x)^k k! \sum_{l=k}^{\infty} \left[\begin{matrix} l \\ k \end{matrix}\right] \sum_{i=0}^{l}\frac{1}{(l-i)!}(1-\alpha)^{l-i}\frac{1}{i!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^i \\
&= \sum_{k=0}^{\infty} (\alpha x)^k k! \sum_{l=k}^{\infty} \left[\begin{matrix} l \\ k \end{matrix}\right] \sum_{i=0}^{l}\frac{1}{(l-i)!}(1-\alpha)^{l-i}\sum_{n=i}^{\infty}\lambda^{n-i}s(n,i)\frac{t^n}{n!} \\
&= \sum_{l=0}^{\infty}\sum_{k=0}^{l} (\alpha x)^k k! \left[\begin{matrix} l \\ k \end{matrix}\right] \sum_{i=0}^{l}\frac{1}{(l-i)!}(1-\alpha)^{l-i}\sum_{n=i}^{\infty}\lambda^{j-i}s(n,i)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{l=0}^{i}\sum_{k=0}^{l} (\alpha x)^k k! \left[\begin{matrix} l \\ k \end{matrix}\right] \frac{1}{(l-i)!}(1-\alpha)^{l-i}\lambda^{n-i}s(n,i)\frac{t^n}{n!},
\end{aligned}$$

where $s(n,i)$ are the Stirling numbers of the first kind. Here we should observe that, for all t with $|t|$ small, we have

$$\left|\frac{1}{\lambda}\log(1+\lambda t)\right| < \alpha$$

and ($\alpha > 0$), since $\left|\frac{\log(1+x)}{x}\right|$ is bounded on $(0,\infty)$. Therefore, by comparing the coefficients on both sides of (3.4), we obtain

$$F_{n,\lambda}^{Y}(x) = \sum_{i=0}^{n}\sum_{l=0}^{i}\sum_{k=0}^{l} (\alpha x)^k k! \left[\begin{matrix} l \\ k \end{matrix}\right] \frac{1}{(l-i)!}(1-\alpha)^{l-i}\lambda^{n-i}s(n,i). \qquad (3.5)$$

■

Theorem 3.4 *For $n \geq 0$, we have*

$$\int_0^\infty \phi_{n,\lambda}^{Y}(xy)e^{-y}dy = F_{n,\lambda}^{Y}(x).$$

Proof. Now, we observe from (2.15) and Theorem 3.1

$$\begin{aligned}\int_0^\infty \phi_{n,\lambda}^Y(xy)e^{-y}dy &= \sum_{k=0}^n \left\{ n \atop k \right\}_{Y,\lambda} x^k \int_0^\infty y^k e^{-y}dy \\ &= \sum_{k=0}^n \left\{ n \atop k \right\}_{Y,\lambda} x^k \Gamma(k+1) \\ &= \sum_{k=0}^n \left\{ n \atop k \right\}_{Y,\lambda} x^k k! \\ &= F_{n,\lambda}^Y(x), \quad (n \geq 0).\end{aligned} \tag{3.6}$$

■

Theorem 3.5 *For $n \geq 0$, we have*

$$F_{n,\lambda}^Y = \sum_{k=0}^n k! B_{n,k}\left(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], ..., E[(Y)_{n-k+1,\lambda}]\right).$$

Proof. From (2.14), we note that

$$\begin{aligned}\sum_{n=k}^\infty \left\{ n \atop k \right\}_{Y,\lambda} \frac{t^n}{n!} &= \frac{1}{k!}\left(E[e_\lambda^Y(t)] - 1\right)^k = \frac{1}{k!}\left(\sum_{i=1}^\infty E[(Y)_{i,\lambda}]\frac{t^i}{i!}\right)^k \\ &= \sum_{n=k}^\infty B_{n,k}\left(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], ..., E[(Y)_{n-k+1,\lambda}]\right)\frac{t^n}{n!}.\end{aligned} \tag{3.7}$$

Thus by (3.7), we get

$$\left\{ n \atop k \right\}_{Y,\lambda} = B_{n,k}\left(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], ..., E[(Y)_{n-k+1,\lambda}]\right), \ (n \geq k \geq 0). \tag{3.8}$$

Hence

$$B_{n,\lambda}^Y(y) = \sum_{k=0}^n \left\{ n \atop k \right\}_{Y,\lambda} y^k = \sum_{k=0}^n B_{n,k}\left(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], ..., E[(Y)_{n-k+1,\lambda}]\right) y^k. \tag{3.9}$$

By (3.6) and (3.9), we get

$$\begin{aligned}F_{n,\lambda}^Y &= \sum_{k=0}^n B_{n,k}\left(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], ..., E[(Y)_{n-k+1,\lambda}]\right)\int_0^\infty y^k e^{-y}dy \\ &= \sum_{k=0}^n B_{n,k}\left(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], ..., E[(Y)_{n-k+1,\lambda}]\right)\Gamma(k+1) \\ &= \sum_{k=0}^n k! B_{n,k}\left(E[(Y)_{1,\lambda}], E[(Y)_{2,\lambda}], ..., E[(Y)_{n-k+1,\lambda}]\right), \ (n \geq 0).\end{aligned} \tag{3.10}$$

■

Definition 3.1.2 *For $r \in \mathbb{N}$, the probabilistic degenerate Fubini polynomials of order r associated with random variable Y are defined by*

$$\left(\frac{1}{1 - x\left(E[e_\lambda^Y(t)] - 1\right)}\right)^r = \sum_{n=0}^\infty F_{n,\lambda}^{(r,Y)}(x)\frac{t^n}{n!}. \tag{3.11}$$

When $Y = 1$, we note then $F_{n,\lambda}^{(r,Y)}(x) = F_{n,\lambda}^{(r)}(x)$, $(n \geq 0)$.

Theorem 3.6 *For $n \geq 0$, we have*

$$F_{n,\lambda}^{(r,Y)}(x) = \sum_{i=0}^{n} \binom{r+i-1}{i} i! \left\{ {n \atop i} \right\}_{Y,\lambda} x^i.$$

Proof. From (2.14) and (3.11), we note that

$$\begin{aligned} \left(\frac{1}{1 - x(E[e_\lambda^Y(t)] - 1)} \right)^r &= \sum_{i=0}^{\infty} \binom{-r}{i} (-1)^i x^i \left(E[e_\lambda^Y(t)] - 1 \right)^i \\ &= \sum_{i=0}^{\infty} \binom{r+i-1}{i} i! x^i \frac{1}{i!} \left(E[e_\lambda^Y(t)] - 1 \right)^i \\ &= \sum_{i=0}^{\infty} \binom{r+i-1}{i} i! x^i \sum_{n=i}^{\infty} \left\{ {n \atop i} \right\}_{Y,\lambda} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{r+i-1}{i} i! x^i \left\{ {n \atop i} \right\}_{Y,\lambda} \frac{t^n}{n!}. \end{aligned} \tag{3.12}$$

We should know

$$\begin{aligned} \binom{r+i-1}{i} &= \frac{(r+i-1)(r+i-2)...r}{i!} \\ &= \frac{(-r)(-r-1)...(-r-i+1)(-1)^i}{i!} \\ &= \binom{-r}{i}. \end{aligned}$$

Therefore, by (3.11) and (3.12), we obtain the result.

Theorem 3.7 *For $n \geq 1$, we have*

$$F_{n,\lambda}^{Y}(x) = x \sum_{k=1}^{n} \binom{n}{k} E[(Y)_{k,\lambda}] F_{n-k,\lambda}^{Y}(x).$$

Proof. By (3.1), we get

$$\begin{aligned} \sum_{n=1}^{\infty} F_{n,\lambda}^{Y}(x) \frac{t^n}{n!} &= \frac{1}{1 - x(E[e_\lambda^Y(t)] - 1)} - 1 = \frac{x(E[e_\lambda^Y(t)] - 1)}{1 - x(E[e_\lambda^Y(t)] - 1)} \\ &= \frac{xE[e_\lambda^Y(t)]}{1 - x(E[e_\lambda^Y(t)] - 1)} - \frac{x}{1 - x(E[e_\lambda^Y(t)] - 1)} \end{aligned} \tag{3.13}$$

by (2.7)

$$\begin{aligned} &= x \sum_{k=0}^{\infty} E[(Y)_{k,\lambda}] \frac{t^k}{k!} \sum_{l=0}^{\infty} F_{l,\lambda}^{Y}(x) \frac{t^l}{l!} - x \sum_{n=0}^{\infty} F_{n,\lambda}^{Y}(x) \frac{t^n}{n!} \\ &= \left(\sum_{k=0}^{\infty} E[(Y)_{k,\lambda}] \frac{t^k}{k!} - 1 \right) x \sum_{l=0}^{\infty} F_{l,\lambda}^{Y}(x) \frac{t^l}{l!} \\ &= x \sum_{k=1}^{\infty} E[(Y)_{k,\lambda}] \frac{t^k}{k!} \sum_{l=0}^{\infty} F_{l,\lambda}^{Y}(x) \frac{t^l}{l!} \\ &= x \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{t^{k+l}}{k!l!} E[(Y)_{k,\lambda}] F_{l,\lambda}^{Y}(x) \\ &= x \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{t^n}{k!(n-k)} E[(Y)_{k,\lambda}] F_{n-k,\lambda}^{Y}(x) \\ &= \sum_{n=1}^{\infty} x \sum_{k=1}^{n} \binom{n}{k} E[(Y)_{k,\lambda}] F_{n-k,\lambda}^{Y}(x) \frac{t^n}{n!}. \end{aligned}$$

Thus, by equating the coefficients from both sides of equation (3.13), we obtain the result. ■

Theorem 3.8 *For $n \geq 0$ we have*

$$F^Y_{n+1,\lambda}(x) = x\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{k}{i}\binom{n}{k}F^Y_{i,\lambda}(x)F^Y_{k-i,\lambda}(x)E[(Y)_{n-k+1,\lambda}].$$

Proof. From 3.1, we note that

$$\begin{aligned}\sum_{n=0}^{\infty}F^Y_{n+1,\lambda}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}F^Y_{n,\lambda}(x)\frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}F^Y_{n,\lambda}(x)\frac{nt^{n-1}}{n!} = \sum_{n=0}^{\infty}F^Y_{n,\lambda}(x)\frac{d}{dt}\left(\frac{t^n}{n!}\right) \\ &= \frac{d}{dt}\sum_{n=0}^{\infty}F^Y_{n,\lambda}(x)\frac{t^n}{n!} = \frac{d}{dt}\left(\frac{1}{1-x(E[e^Y_\lambda(t)]-1)}\right) \\ &= \frac{xE[e^{Y-\lambda}_\lambda(t)]}{(1-x(E[e^Y_\lambda(t)]-1))^2} = \frac{x}{1-x(E[e^Y_\lambda(t)]-1)}\frac{E[e^{Y-\lambda}_\lambda(t)]}{1-x(E[e^Y_\lambda(t)]-1)} \\ &= x\sum_{i=0}^{\infty}F^Y_{i,\lambda}(x)\frac{t^i}{i!}\sum_{j=0}^{\infty}F^Y_{j,\lambda}(x)\frac{t^j}{j!}\sum_{m=0}^{\infty}E[Y(Y-\lambda)_{m,\lambda}]\frac{t^m}{m!} \\ &= x\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i}F^Y_{i,\lambda}(x)F^Y_{k-i,\lambda}(x)\right)\frac{t^k}{k!}\sum_{m=0}^{\infty}E[(Y)_{m+1,\lambda}]\frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty}x\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{k}{i}\binom{n}{k}F^Y_{i,\lambda}(x)F^Y_{k-i,\lambda}(x)E[(Y)_{n-k+1,\lambda}]\frac{t^n}{n!}.\end{aligned} \tag{3.14}$$

Therefore, upon comparing the coefficients from both sides of (3.4), we find the result.

Theorem 3.9 *For r, $n \geq 0$, we have*

$$\frac{d^r}{dx^r}F^Y_{n,\lambda}(x) = r!\sum_{i=0}^{n}F^{(r+1,Y)}_{i,\lambda}(x)E[(S_r)_{n-i,\lambda}]\binom{n}{i}$$

Proof. Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{d^r}{dx^r}F^Y_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{d^r}{dx^r}\left(\frac{1}{1-x(E[e^Y_\lambda(t)]-1)}\right) = r!\frac{\left(E[e^{Y-\lambda}_\lambda(t)]\right)^r}{(1-x(E[e^Y_\lambda(t)]-1))^{r+1}} \\ &= r!\sum_{i=0}^{\infty}F^{(r+1,Y)}_{i,\lambda}(x)\frac{t^i}{i!}\sum_{k=0}^{\infty}E\left[(Y_1+Y_2+\ldots+Y_r)_{k,\lambda}\right]\frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty}\left(r!\sum_{i=0}^{n}F^{(r+1,Y)}_{i,\lambda}(x)E[(S_r)_{n-i,\lambda}]\binom{n}{i}\right)\frac{t^n}{n!}.\end{aligned} \tag{3.15}$$

By (3.16), we get the result.

Theorem 3.10 *For $n \geq 0$, we have*

$$\frac{1}{1-x}F^Y_{n,\lambda}\left(\frac{x}{1-x}\right) = \sum_{i=0}^{\infty}E[(S_i)_{n,\lambda}]x^i.$$

Taking $x = \frac{1}{2}$, we get

$$\sum_{i=0}^{\infty}E[(S_i)_{n,\lambda}]\left(\frac{1}{2}\right)^i = 2F^Y_{n,\lambda},\ (n \geq 0).$$

Proof. From (2.13), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{\infty}E[(S_i)_{n,\lambda}]x^i\right)\frac{t^n}{n!} &= \sum_{i=0}^{\infty}x^i\sum_{n=0}^{\infty}E[(S_i)_{n,\lambda}]\frac{t^n}{n!} \\ &= \sum_{i=0}^{\infty}x^i\sum_{n=0}^{\infty}E\left[S_i\left(S_i-\lambda\right)\left(S_i-2\lambda\right)...\left(S_i-(n-1)\lambda\right)\right]\frac{t^n}{n!} \\ &= \sum_{i=0}^{\infty}x^i\sum_{n=0}^{\infty}E\left(S_i\left(\frac{S_i}{\lambda}-1\right)\left(\frac{S_i}{\lambda}-2\right)...\left(\frac{S_i}{\lambda}-(n-1)\right)\right)\frac{t^n}{n!} \\ &= \sum_{i=0}^{\infty}x^i\sum_{n=0}^{\infty}E\left(\lambda^n\frac{S_i}{\lambda}\left(\frac{S_i}{\lambda}-1\right)\right)\frac{t^n}{n!} \\ &= \sum_{i=0}^{\infty}x^iE\left(\sum_{n=0}^{\infty}\left(\frac{S_i}{\lambda}\right)_n\frac{(\lambda t)^n}{n!}\right).\end{aligned} \tag{3.16}$$

Now, using the Taylor series expansion of the exponential function

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}\left(\frac{x}{\lambda}\right)_n\frac{(\lambda t)^n}{n!}.$$

Then

$$\begin{aligned}&\sum_{i=0}^{\infty}x^i\sum_{n=0}^{\infty}E\left[(S_i)_{n,\lambda}\right]\frac{t^n}{n!} \\ &= \sum_{i=0}^{\infty}x^iE\left[e_\lambda^{S_i}(t)\right] \\ &= \sum_{i=0}^{\infty}x^i(E[e_\lambda^Y(t)])^i \\ &= \frac{1}{1-xE[e_\lambda^Y(t)]} = \frac{1}{1-x}\frac{1}{1-\frac{x}{1-x}\left(E\left[e_\lambda^Y(t)\right]-1\right)} \\ &= \frac{1}{1-x}\sum_{n=0}^{\infty}F_{n,\lambda}^Y\left(\frac{x}{1-x}\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by comparing the coefficients on both sides of (3.16), we find the result.

Theorem 3.11 *Let Y be the Poisson random variable with parameter $\alpha>0$. Then we have*

$$F_{n,\lambda}^Y(x)=\sum_{i=0}^{n}F_i\left(x\right)\left\{\begin{matrix}n\\i\end{matrix}\right\}_\lambda\alpha^i,\quad n\geq 0.$$

Proof. Let Y be the Poisson random variable with parameter $(\alpha>0)$. Then we have

$$\begin{aligned}E[e_\lambda^Y(t)] &= \sum_{n=0}^{\infty}e_\lambda^n(t)\frac{\alpha^n}{n!}e^{-\alpha} \\ &= e^{-\alpha}\sum_{n=0}^{\infty}\frac{(e_\lambda\alpha)^n}{n!} \\ &= e^{-\alpha}e^{e_\lambda\alpha} \\ &= e^{\alpha(e_\lambda(t)-1)}.\end{aligned} \tag{3.17}$$

From (3.17), we have

$$\sum_{n=0}^{\infty} F_{n,\lambda}^{Y}(x)\frac{t^n}{n!} = \frac{1}{1-x(E[e_\lambda^Y(t)]-1)} = \frac{1}{1-x(e^{\alpha(e_\lambda(t)-1)}-1)} \tag{3.18}$$
$$= \sum_{i=0}^{\infty} F_i(x)\alpha^i \frac{1}{i!}(e_\lambda(t)-1)^i$$
$$= \sum_{i=0}^{\infty} F_i(x)\alpha^i \sum_{n=i}^{\infty} \left\{ {n \atop i} \right\}_\lambda \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} F_i(x) \left\{ {n \atop i} \right\}_\lambda \alpha^i \right) \frac{t^n}{n!}.$$

where $F_i(x)$ are the Fubini polynomials. Therefore, by comparing the coefficients on both sides of (3.18), we find the result.

■

Theorem 3.12 *Let Y be the Poisson random variable with parameter $(\alpha > 0)$. For $n \geq 0$, we have*

$$\sum_{k=0}^{\infty} \phi_{n,\lambda}(k\alpha)\, x^k = \sum_{k=0}^{\infty} E\left[(S_k)_{n,\lambda}\right] x^k = \frac{1}{1-x} F_{n,\lambda}^{Y}\left(\frac{x}{1-x}\right)$$

Proof. Let Y be the Poisson random variable with parameter $(\alpha > 0)$. we have

$$(E[e_\lambda^Y(t)])^k = e^{k\alpha(e_\lambda(t)-1)} = \sum_{n=0}^{\infty} \phi_{n,\lambda}(k\alpha)\frac{t^n}{n!}. \tag{3.19}$$

and

$$(E[e_\lambda^Y(t)])^k = \sum_{n=0}^{\infty} E\left[(S_k)_{n,\lambda}\right] \frac{t^n}{n!}. \tag{3.20}$$

Thus, by (3.19) and (3.20), we get

$$E[(S_k)_{n,\lambda}] = \phi_{n,\lambda}(k\alpha), \quad (n \geq 0). \tag{3.21}$$

From Theorem 3.10 and (3.21), we have

$$\sum_{k=0}^{\infty} \phi_{n,\lambda}(k\alpha)x^k = \sum_{k=0}^{\infty} E\left[(S_k)_{n,\lambda}\right] x^k = \frac{1}{1-x} F_{n,\lambda}^{Y}\left(\frac{x}{1-x}\right). \tag{3.22}$$

Therefore, by (3.22), we obtain the result.

Theorem 3.13 *For $n \geq 0$, we have*

$$\left(\frac{1}{1-x}\right)^{r+1} F_{n,\lambda}^{(r+1,Y)}\left(\frac{x}{1-x}\right) = \sum_{k=0}^{\infty} \binom{k+r}{k} x^k E\left[(S_k)_{n,\lambda}\right].$$

When $Y = 1$, we have

$$\left(\frac{1}{1-x}\right)^{r+1} F_{n,\lambda}^{(r+1)}\left(\frac{x}{1-x}\right) = \sum_{k=0}^{\infty} \binom{k+r}{k} x^k (k)_{n,\lambda}.$$

Proof. By using Theorem 3.6 and (2.13), we note that

$$\left(\frac{1}{1-x}\right)^{r+1} F_{n,\lambda}^{(r+1,Y)}\left(\frac{x}{1-x}\right) \tag{3.23}$$

$$= \left(\frac{1}{1-x}\right)^{r+1} \sum_{l=0}^{n} \left\{ n \atop l \right\}_{Y,\lambda} \binom{l+r}{l} l! \left(\frac{x}{1-x}\right)^{l} \tag{3.24}$$

$$= \sum_{l=0}^{n} \left\{ n \atop l \right\}_{Y,\lambda} l! \binom{l+r}{l} x^{l} \left(\frac{1}{1-x}\right)^{l+r+1}$$

$$= \sum_{l=0}^{n} \sum_{k=0}^{\infty} \left\{ n \atop l \right\}_{Y,\lambda} l! \binom{l+r}{l} \binom{k+l+r}{l} x^{k+l}$$

$$= \sum_{l=0}^{n} \left(\sum_{k=l}^{\infty} \left\{ n \atop l \right\}_{Y,\lambda} l! \binom{l+r}{l} \binom{k+r}{k-l} \right) x^{k}$$

$$= \sum_{k=0}^{n} \binom{k+r}{k} x^{k} \sum_{l=0}^{k} \left\{ n \atop l \right\}_{Y,\lambda} (k)_{l} + \sum_{k=n+1}^{\infty} \binom{k+r}{k} x^{k} \sum_{l=0}^{n} \left\{ n \atop l \right\}_{Y,\lambda} (k)_{l}$$

$$= \sum_{k=0}^{n} \binom{k+r}{k} x^{k} \sum_{l=0}^{k} \left\{ n \atop l \right\}_{Y,\lambda} (k)_{l} + \sum_{k=n+1}^{\infty} \binom{k+r}{k} x^{k} \sum_{l=0}^{k} \left\{ n \atop l \right\}_{Y,\lambda} (k)_{l}$$

$$= \sum_{k=0}^{\infty} \binom{k+r}{k} x^{k} \sum_{l=0}^{k} \left\{ n \atop l \right\}_{Y,\lambda} (k)_{l}$$

$$= \sum_{k=0}^{\infty} \binom{k+r}{k} x^{k} E\left[(S_k)_{n,\lambda}\right].$$

Then, by (3.24), we get

$$\left(\frac{1}{1-x}\right)^{r+1} F_{n,\lambda}^{(r+1,Y)}\left(\frac{x}{1-x}\right) = \sum_{k=0}^{\infty} \binom{k+r}{k} x^{k} E\left[(S_k)_{n,\lambda}\right].$$

■

Theorem 3.14 *For $n \geq 0$ and $r \geq 1$, we have*

$$F_{n,\lambda}^{(r,Y)}(x) = \frac{1}{\Gamma(r)} \int_{0}^{\infty} y^{r-1} \phi_{n,\lambda}^{Y}(xy) e^{-y} dy.$$

Proof. Now, we observe from (2.15) and Theorem 3.6 that

$$\int_{0}^{\infty} y^{r-1} \phi_{n,\lambda}^{Y}(xy) e^{-y} dy = \sum_{k=0}^{n} \left\{ n \atop k \right\}_{Y,\lambda} x^{k} \int_{0}^{\infty} y^{r+k-1} e^{-y} dy \tag{3.25}$$

$$= \sum_{k=0}^{n} \left\{ n \atop k \right\}_{Y,\lambda} x^{k} \Gamma(r+k)$$

$$= \Gamma(r) \sum_{k=0}^{n} \left\{ n \atop k \right\}_{Y,\lambda} x^{k} \binom{r+k-1}{k} k!$$

$$= \Gamma(r) F_{n,\lambda}^{(r,Y)}(x), \quad (r \in \mathbb{N}).$$

Therefore, by (3.25), we obtain the result.

Theorem 3.15 *For $n \geq 0$, we have*

$$F_{n+1,\lambda}^{(r,Y)}(x) = rx \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} F_{n-k,\lambda}^{(r,Y)}(x) F_{k-j,\lambda}^{Y}(x) E[(Y)_{j+1,\lambda}].$$

Proof. From (3.11) and Theorem 3.14, we have

$$\left(\frac{1}{1-x(E[e_\lambda^Y(t)]-1)}\right)^r = \sum_{n=0}^{\infty} F_{n,\lambda}^{(r,Y)}(x)\frac{t^n}{n!}.$$

and we have

$$F_{n,\lambda}^Y(x) = \int_0^\infty \phi_{n,\lambda}^Y(xy)e^{-y}dy.$$

then

$$\begin{aligned}\left(\frac{1}{1-x(E[e_\lambda^Y(t)]-1)}\right)^r &= \frac{1}{\Gamma(r)}\int_0^\infty y^{r-1}\sum_{n=0}^{\infty}\phi_{n,\lambda}^Y(xy)\frac{t^n}{n!}e^{-y}dy \\ &= \frac{1}{\Gamma(r)}\int_0^\infty y^{r-1}e^{xy(E[e_\lambda^Y(t)]-1)}e^{-y}dy \\ &= \frac{1}{\Gamma(r)}\int_0^\infty y^{r-1}e^{y(xE[e_\lambda^Y(t)]-1-x)}dy.\end{aligned}$$

Where r is a positive integer.

We observe that

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n+1,\lambda}^{(r,Y)}(x)\frac{t^n}{n!} &= \frac{d}{dt}\sum_{n=0}^{\infty} F_{n,\lambda}^Y(x)\frac{t^n}{n!} \\ &= \frac{d}{dt}\left(\frac{1}{1-x(E[e_\lambda^Y(t)]-1)}\right)^r \\ &= \frac{rxE[e_\lambda^{Y-\lambda}(t)Y]}{(1-x(E[e_\lambda^Y(t)]-1))^{r+1}} \\ &= \frac{rx}{(1-x(E[e_\lambda^Y(t)]-1))^r}\frac{1}{1-x(E[e_\lambda^Y(t)]-1)}E[e_\lambda^{Y-\lambda}(t)Y] \\ &= rx\sum_{l=0}^{\infty} F_{l,\lambda}^{(r,Y)}(x)\frac{t^l}{l!}\sum_{i=0}^{\infty} F_{i,\lambda}^Y(x)\frac{t^i}{i!}\sum_{j=0}^{\infty} E[(Y)_{j+1,\lambda}]\frac{t^j}{j!} \\ &= rx\sum_{l=0}^{\infty} F_{l,\lambda}^{(r,Y)}(x)\frac{t^l}{l!}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}F_{k-j,\lambda}^Y(x)E[(Y)_{j+1,\lambda}]\right)\frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty} rx\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}F_{n-k,\lambda}^{(r,Y)}(x)F_{k-j,\lambda}^Y(x)E[(Y)_{j+1,\lambda}]\frac{t^n}{n!}.\end{aligned} \tag{3.26}$$

Therefore, by (3.26), we obtain the result.

Theorem 3.16 *Let Y be the Bernoulli random variable with probability of success p. For $n \geq 0$, we have*

$$F_{n,\lambda}^Y(x) = F_{n,\lambda}(xp).$$

Proof. Let Y be the Bernoulli random variable with probability of success p. Then we have

$$\begin{aligned}E[e_\lambda^Y(t)] &= \sum_{k=0}^{1} e_\lambda^k(t)p(k) = e_\lambda^0(t)(1-p) + e_\lambda^1(t)p \\ &= 1-p+pe_\lambda(t) = 1+p(e_\lambda(t)-1).\end{aligned} \tag{3.27}$$

By (3.27), we get

$$\begin{aligned}\sum_{n=0}^{\infty} F_{n,\lambda}^Y(x)\frac{t^n}{n!} &= \frac{1}{1-x(E[e_\lambda^Y(t)]-1)} = \frac{1}{1-xp(e_\lambda(t)-1)} \\ &= \sum_{n=0}^{\infty} F_{n,\lambda}(xp)\frac{t^n}{n!}.\end{aligned} \tag{3.28}$$

Therefore, by comparing the coefficients on both sides of (3.28), we get

$$F_{n,\lambda}^{Y}(x) = F_{n,\lambda}(xp).$$

. ■

3.2 Conclusion

In conclusion, this thesis has explored the intricate interplay between combinatorial mathematics and probability theory, shedding light on both foundational elements and specialized research findings. Each chapter has contributed uniquely to our overall understanding, providing a comprehensive perspective on the topics at hand.

In Chapter 1, we established the basic combinatorial tools and concepts that form the bedrock of our study. The review of rising and falling factorials, generating functions, explicit forms, recurrence relations, and Stirling numbers has equipped us with the necessary mathematical apparatus to delve deeper into more complex analyses. These elements are crucial for understanding the subsequent probabilistic and combinatorial constructs discussed in later chapters.

Chapter 2 extended our exploration into the domain of probability theory. By introducing fundamental concepts and various discrete and continuous probability distributions, we set a solid foundation for understanding random variables and their distributions. The emphasis on the moment generating function highlighted its importance as a tool for grasping the distributional properties of random variables, thus bridging the gap between theoretical probability and practical application.

In Chapter 3, we focused on the study of probabilistic degenerate Fubini polynomials as inspired by Kim's work. This chapter brought to light the rich structure and properties of these polynomials, including their explicit expressions, identities, and recurrence relations. Our detailed analysis underscored the significance of these polynomials in a probabilistic context, contributing valuable insights to the field and showcasing the depth of interconnectedness between combinatorial and probabilistic theories.

Overall, this thesis has provided a thorough examination of key mathematical constructs, offering both theoretical and practical contributions. The exploration of rising and falling factorials, probability distributions, and degenerate Fubini polynomials demonstrates the versatility and applicability of combinatorial and probabilistic methods. As the mathematical community continues to advance, the insights and methodologies presented in this thesis will hopefully inspire further research and exploration in these fascinating areas.

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