People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
University of Médéa
Faculty of Sciences
Department of Mathematics and Computer Science

DISSERTATION

Presented by

Selma BOUHALLA

to obtain Master degree in Mathematics

Speciality : Mathematical Analysis and Applications

Entitled

ON SOME FRACTIONAL p-LAPLACIAN PROBLEMS

Publicly defended on 2024 In front of the jury composed of :

President:	**Noureddine Benaichouche**
Examiner:	**Rezak Souilah**
Supervisor:	**Hocine Ayadi**

Academic Year 2023/2024

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ACKNOWLEDGEMENTS

First, i thank god almighty who has guide me in my academic journey and enabled me to complete this work successfully.

All thanks and appreciation to Mr Hocine AYADI for his guidance and follow-up during my preparation for my graduation thesis. I thank him for his continuous support and encouragement.

My thanks also go to Mr. Noureddine BENAICHOUCHE president of the jury who agreed to reread this work.

I warmly thank Mr Rerak SOUILAH who did me honor by agreeing to examine and judge this work.

I would also like to thank all my professors in the Mathematics and Computer Science department at University of Médéa.

I will never forget to express gratitude and remember with mercy the first person who instilled in me the love for learning and studying, my teacher Boukertebi, may Allah have mercy on him.

DEDICATION

To the source of safety from which I draw my strength, to my dear father.

To the one who was the first support in achieving my ambitions, to my dear mother.

To my brothers Hamza and Zakaria.

ABSTRACT

In this dissertation, we study some fractional elliptic equations. We begin by presenting some facts about the fractional Sobolev spaces and we introduce some definitions and some properties of the fractional Laplacian operator. Then, we prove the existence and the uniqueness of solution of Laplace fractional equation using two methods; Lax-Milgram theorem and Variational Method. We end this work by applying Minty-Browder method to establish the existence and the uniqueness of the weak solution for the p-Laplacian fractional equation.

NOTATIONS

$\mathbb{R}^N$	euclidean space of dimension , n ≥ 1.
$\mathbb{R}^{m\times N}$	the space of real matrices.
Ω	an open bounded set of $\mathbb{R}^N$.
$\partial\Omega$	topological boundary of Ω.
$\|\Omega\|$	Lebesgue measure of the set Ω.
$x=(x_1,\cdots,x_N)$	generic point of $\mathbb{R}^N$.
$\mathrm{d}x=\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_N$	Lebesgue measure on Ω.
$\overline{\Omega}$	the closure of the set Ω.
$\rightharpoonup$	weak convergence.
$\longrightarrow$	strong convergence.
$\hookrightarrow$	Compact injection.
V	a Banach space.
$\langle\cdot,\cdot\rangle_{V'\times V}$	the duality bracket between the space V and its topological dual V'.
$\|.\|_V$	the norm on the Banach space V.
$[\cdot]$	designates the semi-norm.
$(\cdot,\cdot)$	the inner product on the Hilbert space H.
$\|.\|$	the norm on the Hilbert space H.
$\mathscr{C}^0(\mathbb{R})$	the space of continuous functions.
∂_i	The i-th partial derivative; $\partial_i=\frac{\partial}{\partial x_i}$.
∂^α	The derivative of order α; $\partial^\alpha=\frac{\partial^{\|\alpha\|}}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}$.
$\nabla u=\left(\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\cdots,\frac{\partial u}{\partial x_N}\right)$	gradient of $v:\Omega\longrightarrow\mathbb{R}$.
Δu	The Laplacian of u.

$\operatorname{div} v$	The divergence of the vector $v = (v_1, ..., v_N)$. $\operatorname{div} v = \frac{\partial v_1}{\partial x_1} + ... + \frac{\partial v_N}{\partial x_N}$.
$[s]$	denotes the integer part of s.
a.e.	almost everywhere.
p'	The conjugate exponent of p, That is to say $p' = \frac{p}{p-1}$.

INTRODUCTION

The historical development of the Fractional p-Laplacian equation is tightly interwoven with the evolution of mathematical models aimed at elucidating complex physical phenomena across various disciplines. Originating from the realm of partial differential equations, the concept of the p-Laplacian operator emerged in the mid-20th century, notably through the pioneering works of Soviet mathematician Olga Ladyzhenskaya and her research associates. Initially conceived as an extension of the Laplace operator, the p-Laplacian operator provided a platform to explore nonlinear characteristics in diffusion processes and fluid dynamics.

In the latter part of the 20th century, as mathematical inquiries delved deeper into nonlinear phenomena, particularly within the realm of fractional calculus, the amalgamation of these two fields gave rise to the formulation of the p-Laplacian Fractional equation. This innovative equation introduced fractional order derivatives into the traditional framework of the p-Laplacian, thereby enabling a more refined depiction of phenomena such as anomalous diffusion, non-local interactions, and intricate material properties.

The emergence of the p-Laplacian Fractional equation signified a significant milestone in mathematical modeling, furnishing a potent tool for the analysis and prediction of systems characterized by nonlinear dynamics and fractional behavior. Its applicability extends across a diverse spectrum of disciplines including physics, biology, engineering, and economics, where it has facilitated valuable insights into phenomena ranging from anomalous transport in porous media to the dynamics of biological populations. As researchers continue to delve into its theoretical underpinnings and practical implications, the $p-$Laplacian fractional equation persists as a cornerstone of contemporary mathematical research, catalyzing innovation and fostering interdisciplinary collaboration.

The application of the fractional p-Laplacian equation is not confined to a singular domain but extends its reach into diverse fields, highlighting its adaptability and efficacy in capturing intricate phenomena across various disciplines. In the realm of physics, particularly fluid dynamics, the p-Laplacian equation emerges as a fundamental tool for elucidating the behavior of fluids, offering insights into phenomena such as turbulence and complex flow patterns. Its inherent nonlinearity enables the accurate representation of fluid behavior under diverse conditions, contributing significantly to our understanding of natural processes and aiding in the design of efficient engineering systems.

Moreover, in materials science and engineering, the p-Laplacian equation plays a pivotal role in unraveling the mechanical properties of heterogeneous materials. By incorporating nonlinearities, it provides a comprehensive framework for analyzing deformation, fracture mechanics, and phase transitions in materials ranging from metals to composites. This understanding is crucial for optimizing material performance and ensuring structural integrity in various engineering applications, including aerospace, automotive, and civil engineering.

Moving into the realm of biology and ecology, the p-Laplacian equation facilitates the modeling of complex ecological systems and population dynamics. By accounting for spatial heterogeneity and non-local interactions, it enables researchers to simulate ecological processes such as species dispersal, competition, and predator-prey dynamics more accurately. This modeling framework aids in ecological conservation efforts, wildlife management, and understanding the impact of environmental changes on biodiversity.

Furthermore, in the realm of image processing and computer vision, the p-Laplacian equation emerges as a versatile tool for various tasks such as edge detection, image denoising, and segmentation. Its ability to preserve important features while suppressing noise makes it particularly valuable in applications where precise image analysis is essential, including medical imaging, remote sensing, and autonomous navigation systems.

In summary, the widespread application of the p-Laplacian equation underscores its indispensability across a multitude of disciplines, from fundamental physics to practical engineering and computational sciences. Its versatility and effectiveness make it an invaluable asset for researchers and practitioners alike, driving advancements in understanding and analyzing complex systems in both natural and engineered environments.

The thesis comprises three chapters, each of which is succinctly outlined below.

Chapter 1:(Preliminaries) It includes a summary of the findings from real and functional analysis that will be frequently referenced.

Chapter 2: It includes important definitions, characteristics, and important theorems in this space.

Chapter 3: It includes the application of random long jumps and definitions and properties of Fractional Laplacian , along with proving the existence of the unique solution in two ways (Lax-Milgram theorem and Variational Method) for the following problem:

$$\begin{cases} (-\Delta)^s u = f, & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

Chapter 4: It entails demonstrating the existence of the unique solution by method (Minty-Browder) to the following problem:

$$\begin{cases} (-\Delta)_p^s u = f & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

CHAPTER 1

Preliminaries

This chapter is meant to provide an overview of the real and functional analysis results that will be used afterwards. Moreover, we present some basic facts concerning the necessary function spaces.

1.1 Lebesgue spaces $L^p(\Omega)$

In this section we recall some basic facts on classical Lebesgue and Sobolev spaces with constant exponent that we will use in the remainder of this dissertation. For further details, we refer to [5].

Definition 1.1 *Let p in $\mathbb{R}$ such as $1 \leq p < \infty$. We define the Lebesgue space $L^p(\Omega)$ as:*

$$L^p(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} : \ u \text{ is measurable and } \int_\Omega \left|u(x)\right|^p \mathrm{d}x < \infty \right\},$$

which is equipped with the norm:

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega \left|u(x)\right|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

Definition 1.2 *For $p = \infty$, we define $L^\infty(\Omega)$ space by:*

$$L^\infty(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} : \ u \text{ is measurable and } \sup_{x\in\Omega} \left|u(x)\right| \leq M \text{ a.e. in } \Omega \right\}.$$

The norm of this space defined by :

$$\|u\|_{L^\infty(\Omega)} = \inf\{c > 0 : \ |u(x)| \leq c \ \text{a.e } x \in \Omega\}.$$

The following inequality named after Otto Hölder, is a fundamental inequality between integrals and an indispensable tool for the study of L^p spaces.

Proposition 1.1 (Holder's inequality) *Let p, q in $[1,+\infty]$ with $\frac{1}{p}+\frac{1}{q}=1$. If u in $L^p(\Omega)$ and v in $L^q(\Omega)$ and we:*

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}.$$

In particular, if f and g are in the Hilbert space $L^2(\Omega)$, then Holder's inequality for $p=q=2$ implies

$$|\langle f,g\rangle| \leq \|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)},$$

where the angle brackets refer to the inner product of $L^2(\Omega)$. This is also called Cauchy-Schwarz inequality.

Theorem 1.1 (Lebesgue's dominated convergence theorem) *Let $p \geq 1$, and let $\{(f_n)_n\} \subset L^p(\Omega)$ and $f \in L^p(\Omega)$. We suppose that $f_n \longrightarrow f$ a.e. in Ω and that there exists g in $L^p(\Omega)$ such that for all n in $\mathbb{N}$ we have : $|f_n| \leq g$ a.e. in Ω. So $f_n \longrightarrow f \in L^p(\Omega)$.*

Lemma 1.1 *Let $a \geq 0$, $b \geq 0$ and let $1 \leq p \leq +\infty$. Then we have:*

$$(a+p)^p \leq 2^{p-1}(a^p+b^p).$$

1.2 Fourier transform

Definition 1.3 *Let f be a function of $L^1(\mathbb{R}^N)$. We call the Fourier transform of f, denoted by $\hat{f}$ or $\mathscr{F}(f)$, the function defined on $\mathbb{R}^N$ by:*

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-ix.\xi} f(x)\mathrm{d}x, \ \forall \xi \in \mathbb{R}^N,$$

where

$$x \cdot \xi = \sum_{j=1}^{N} x_j \xi_j, \ \forall (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N.$$

The Fourier transform can also be calculated using the following formula:

$$\hat{f}(u) = \int_{\mathbb{R}^N} e^{-2\pi ix.\xi} f(x)\mathrm{d}x, \ \forall u \in \mathbb{R}^N.$$

Definition 1.4 (Inverse of Fourier transform) *Let f be a function of $L^1(\mathbb{R}^N)$. We call the*

Fourier transform of f that we note $\overline{F}$ or $\mathscr{F}^{-1}(f)$ the function defined on $\mathbb{R}^N$ by:

$$\overline{F}(u)=\int_{\mathbb{R}^N} e^{-ix.u} f(u)\mathrm{d}u \quad \forall u\in\mathbb{R}^N.$$

1.3 Sobolev space $W^{1,p}(\Omega)$

Definition 1.5 *Let $\Omega\subset\mathbb{R}^N$, p in $[0,+\infty]$. We define the Sobolev space $W^{1,p}(\Omega)$ by:*

$$W^{1,p}(\Omega)=\{u\in L^p(\Omega):\ |\nabla u|\in L^p(\Omega)\},$$

the norm of this space defined by :

i) if $1\le p<+\infty$

$$\|u\|_{W^{1,p}(\Omega)}=\|u\|_{L^p(\Omega)}+\|\nabla u\|_{L^p(\Omega)},$$

ii) if $p=+\infty$

$$\|u\|_{W^{1,\infty}(\Omega)}=\max(\|u\|_{L^\infty(\Omega)}+\|\nabla u\|_{L^\infty(\Omega)}).$$

Proposition 1.2

1) *The space $W^{1,p}(\Omega)$ is a Banach space for $1\le p\le+\infty$.*
2) *The space $W^{1,p}(\Omega)$ is reflexive for $1<p<+\infty$.*
3) *The space $W^{1,p}(\Omega)$ is separable for $1\le p<+\infty$.*

Definition 1.6 *We say a function f is rapidly decreasing if there exist M_N constants such that:*

$$|f(x)|=M_N|x|^{-N} \quad as \quad x\to\infty,$$

for $N=0,1,2,...$

In other words, a rapidly decreasing function $f(x)$ still goes to zero after multiplication by any polynomial $p(x)$ as $x\longrightarrow\infty$.

Definition 1.7 (Schwartz Class) *If a function f in $C^\infty(\mathbb{R}^N)$ and all its partial derivatives are rapidly decreasing then we say f is of Schwartz class. We denote the Schwartz class by $\mathscr{S}(\mathbb{R}^N)$.*

Definition 1.8 (Cauchy principal value) *Depending on the type of singularity in the integrand f, the Cauchy principal value is defined according to the following rules: on the Euclidean space* $\mathbb{R}$ *for a singularity at a finite number b*

$$\lim_{\epsilon\to 0^+}\left[\int_a^{b-\epsilon} f(x)\mathrm{d}x+\int_{b+\epsilon}^{c} f(x)\mathrm{d}x\right]$$

on the Euclidean space $\mathbb{R}^N$ *if K has an isolated singularity at the origin, the principal-value distribution is defined on compactly supported smooth functions by:*

$$[P.v(k)](f)=\lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\epsilon(0)} f(x)k(x)\mathrm{d}x.$$

1.4 Fractional Sobolev spaces

In this chapter, we present the definitions and properties of Sobolev fractional $W^{s,p}$, where $0<s<1$ and $1<p<+\infty$.

1.5 The Fractional Sobolev spaces $W^{s,p}(\Omega)$

Definition 1.9 *Let* $\Omega\subset\mathbb{R}^N$, *s in* $]0,1[$ *and p in* $[1,+\infty[$. *We define the fractional Sobolev space* $W^{s,p}(\Omega)$ *by:*

$$W^{s,p}(\Omega)=\left\{u\in L^p(\Omega): x,y\to\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}}\in L^p(\Omega\times\Omega)\right\},$$

the norm of this space defined by :

$$\|u\|_{W^{s,p}(\Omega)}=\left(\int_\Omega |u(x)|^p\mathrm{d}x+\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}},$$

where the term:

$$[u]_{W^{s,p}(\Omega)}=\left(\int_\Omega\int_\Omega\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}},$$

is the Gagliardo Seminorm of u.

Definition 1.10 *Let* $\Omega\subset\mathbb{R}^N$, *s in* $]0,1[$ *and p in* $[1,+\infty[$. *We define the fractional Sobolev*

space $W_0^{s,p}(\Omega)$ *by:*

$$W_0^{s,p}(\Omega) = \overline{C_0^\infty(\Omega)} \ \ in \ \ W^{s,p}(\Omega),$$

the norm of this space defined by :

$$\|u\|_{W_0^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{p}}.$$

Theorem 1.2 (Poincaré Theorem) *Let* $\Omega \subset \mathbb{R}^N$, s *in* $]0,1[$ *and* p *in* $[1,+\infty[$ *so there exists* λ *such as:*

$$\|u\|_{L^p(\Omega)} \leq \lambda [u]_{\mathbb{W}^{s,p}(\Omega)} \ \forall u \in W_0^{s,p}(\Omega).$$

Therefore if Ω *open bounded set of* $\mathbb{R}^N$ $[.]_{s,p}$ *is a norm of* $W_0^{s,p}$ *equivalent to* $\|.\|_{W^{s,p}(\Omega)}$

Proposition 1.3 *Let* $\Omega \subset \mathbb{R}^N$, s *in* $]0,1[$ *and* p *in* $[1,+\infty[$.*We have*

1) *The space* $W^{s,p}(\Omega)$ *is a Banach space and separable for* $1 \leq p < +\infty$.

2) *The space* $W^{s,p}(\Omega)$ *is reflexive for* $1 < p < +\infty$.

3) *The space* $W^{s,p}(\Omega)$ *is uniformly convex for* $1 < p < +\infty$.

Proposition 1.4 ([8]) 1) *Let* s *in* $]0,1[$ *and* p *in* $[1,+\infty[$, Ω *an open subset of* $\mathbb{R}^N$ *of class* $C^{0,1}$,*boundary bounded we have:*

$$\|u\|_{W^{s,p}(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{s,p}(\Omega),$$

i.e $W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$,

2) *Let* Ω *an open subset of* $\mathbb{R}^N$. *Let* p *in* $[1,+\infty[$ *and* $0 < s \leq s' < 1$, *we have:*

$$\|u\|_{W^{s,p}(\Omega)} \leq C \|u\|_{W^{s',p}(\Omega)} \quad \forall u \in W^{s',p}(\Omega),$$

i.e $W^{s_2,p}(\Omega) \hookrightarrow W^{s_1,p}(\Omega)$, *with* $C = C(N,s) \geq 1$.

3) *For* $0 < s < 1$ *and* p *in* $]1,+\infty[$ *the space* $W^{s,p}(\Omega)$ *is dual of space* $W^{-s,q}(\Omega)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

4) *The space* $W^{s,p}(\Omega)$ *is of local type, that's to say for every* u *in* $W^{s,p}(\Omega)$ *and for every* φ *in* $D(\Omega)$ *the product* φu *belongs to* $W^{s,p}(\Omega)$.

Theorem 1.3 *(see [2]) For $0 < s < 1$ and p in $[1, +\infty[$, the space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$.*

Theorem 1.4 *(see [4]) Let Ω and let s in $]0,1[$, p in $]1, +\infty[$, and $N \geq 1$. So we have:*

1) *If $sp < N$ then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact $\forall q \leq \frac{Np}{N-sp}$.*

2) *If $sp = N$ then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact $\forall q < +\infty$.*

3) *If $sp > N$ then $W^{s,p}(\Omega) \hookrightarrow C_b^{0,\alpha}(\Omega)$ is compact $\forall \alpha \leq s - \frac{N}{p}$.*

1.6 The Fractional Sobolev spaces $H^s(\Omega)$

When $p = 2$, the Fractional Sobolev spaces $W^{s,2}(\Omega)$ it becomes an Hilbert space. That is why we denote it by $H^s(\Omega)$, also We denote $W_0^{s,2}(\Omega)$ by $H_0^s(\Omega)$. can be defined as follows:

Definition 1.11 *The Sobolev space $H^s(\Omega)$ is defined by:*

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) \quad such\ as \quad x, y \to \frac{|u(x) - u(y)|}{|x-y|^{\frac{N}{2}+s}} \in L^2(\Omega \times \Omega) \right\},$$

equipped with the scalar product following:

$$\langle u, v \rangle_{H^s(\Omega)} = \int_\Omega u(x) - v(x) \mathrm{d}x + \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{\left|x - y\right|^{N+2s}} \mathrm{d}x \mathrm{d}y,$$

we can also define this space by using the Fourier transform as follows:

Definition 1.12 *For s in $]0,1[$ we have:*

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : \int_\Omega (1 + |\xi|^{2s}) |\mathscr{F} u(\xi)|^2 \mathrm{d}\xi < +\infty \right\}.$$

Proposition 1.5 *Let s in $]0,1[$ For any u in $H^s(\Omega)$, then:*

$$[u]^2_{H^s(\Omega)} = 2/C \int_\Omega |\xi|^{2s} |\mathscr{F} u(\xi)|^2 \mathrm{d}\xi,$$

that is to say

$$[u]_{H^s(\Omega)} = C \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\Omega)},$$

where $C = C(N,s)$ is a positive constant.

Proposition 1.6 *Let s in $]1/2,1[$ any function u in $H^s(\mathbb{R}^N)$ has a trace v, such that v in $H^{s-\frac{1}{2}}(\mathbb{R}^{N-1})$.*

CHAPTER 2

Fractional Laplacian

In this chapter we study the fractional Laplacian, i.e, $(-\Delta)^s$. We begin by summarizing important properties of this operator and then continue with the motivation that introduces how the fractional Laplacian appeared. After that we present some results concerning the weak solutions and regularity properties of these solutions.

2.1 Modeling via the Fractional Laplacian

The last decade has witnessed great interest in using the fractional Laplacian to model diverse real-world phenomena, such as anomalous diffusion, turbulence, molecular dynamics, and relativistic quantum mechanics of stars. It also has many applications in probability and finance.

In this section we give an example to illustrate how the fractional Laplace operator appears in long jump random walks.

2.1.1 Application in probability and particle motion

Long jump random walks and singular integral kernels:

Let $H:\mathbb{R}^N \to [0,+\infty)$ be even i.e. $H(y)=H(-y)$ for all y in $\mathbb{R}^N$ such that :

$$\sum_{k\in\mathbb{Z}^N} H(k)=1. \tag{2.1}$$

Let $h>0$, we consider a random walk on the lattice $h\mathbb{Z}^N$

We suppose that at τ (unit of time) which depends on h ,defined on any point of $h\mathbb{Z}^N$ to any other point.

We define the probability for which a particle jumps from the point $hk \in h\mathbb{Z}^N$ to the point $h\tilde{k}$ by $H(k-\tilde{k}) = H(\tilde{k}-k)$.

We call $u(x,t)$ the probability that our particle lies at $x \in h\mathbb{Z}^N$ at time $t \in \tau\mathbb{Z}^N$,and we define $u(x, t+\tau)$ by :

$$u(x,t+\tau) = \sum_{k\in\mathbb{Z}^N} H(k)u(x+hk,t).$$

Particularly nice asymptotics are obtained in the case in whic $\tau = h^\alpha$ and H is a homogeneous kernel, say, up to normalization factors

$$H(y) = \left|y\right|^{-(n+\alpha)} \qquad (\text{for} y \neq 0, \text{and } H(0) = 0, \forall \alpha \in (0,2)).$$

Recalling the normalization in (2.1)

$$u(x,t+\tau) - u(x,t) = \sum_{k\in\mathbb{Z}^N} H(k)(u(x+hk,t) - u(x,t)), \tag{2.2}$$

and

$$\frac{H(k)}{\tau} = h^N H(hk), \tag{2.3}$$

according to (2.2) and (2.3) we have:

$$\begin{aligned} \tfrac{u(x,t+\tau)-u(x,t)}{\tau} &= \sum_{k\in\mathbb{Z}^N} \frac{H(k)}{\tau}(u(x+hk,t)-u(x,t)) \\ &= h^N \sum_{k\in\mathbb{Z}^N} H(hk)(u(x+hk,t)-u(x,t)) \\ &= h^N \psi(hk,x,t). \end{aligned} \tag{2.4}$$

Such as

$$\psi(y,x,t) = H(y)\big(u(x+y,t)-u(x,t)\big).$$

Since(2.4) is just the approximating Riemann su

$$\int_{\mathbb{R}^N} \psi(y,x,t)\mathrm{d}y,$$

by send $\tau = h^N \longrightarrow 0^+$and by limit we obtain:

$$\partial_t u(x,t) = \int_{\mathbb{R}^N} \frac{u(x+y,t)-u(x,t)}{|y|^{N+\alpha}} \mathrm{d}y \tag{2.5}$$

$$u(t,y) = \int_{\mathbb{R}^N} \frac{u(x+y)-u(x)}{|y|^{N+\alpha}} \mathrm{d}y. \tag{2.6}$$

This shows that a simple random walk with possibly long jumps produces , in the limit , a singular integral with a homogeneous kernel.

From the long jump random walk to the fractional laplacian:

When $\alpha \in [0,2]$ and u is smooth and bounded, such integral is well defined as principal value, that is as:

$$\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon} \frac{u(x+y)-u(x)}{|y|^{N+\alpha}} \mathrm{d}y.$$

Indeed,$|y|^{N+\alpha}$is integrable at infinity and the function $y/|y|^{N+\alpha}$ is odd, we can define principal value as :

$$\int_{B_1} \frac{\nabla u(x).y}{|y|^{N+\alpha}} \mathrm{d}y = 0.$$

Therefore, we may write the singular integral in (2.6) as principal value near 0 in the form

$$\int_{B_1} \frac{u(x+y)-u(x)-\nabla u(x).y}{|y|^{N+\alpha}} \mathrm{d}y,$$

and the latter is a convergent integral near 0 because

$$\begin{aligned} \frac{|u(x+y)-u(x)-\nabla u(x).y|}{|y|^{N+\alpha}} &\leq \frac{\|D^2 u\|_{L^\infty} |y|^2}{|y|^{N+\alpha}} \\ &= \frac{\|D^2 u\|_{L^\infty}}{|y|^{N+\alpha-2}}, \end{aligned}$$

we replace y by $\tilde{y}$ such as $\tilde{y} = -y$ in (2.6) we have:

$$u(t,\tilde{y}) = \int_{\mathbb{R}^N} \frac{u(x+\tilde{y})-u(x)}{|\tilde{y}|^{N+\alpha}} \mathrm{d}y. \tag{2.7}$$

Therefore, relabeling $\tilde{y}$ as y in(2.7),we have that:

$$\begin{aligned} & 2\int_{\mathbb{R}^N} \frac{u(x+y)-u(x)}{|y|^{N+\alpha}}\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \frac{u(x+y)-u(x)}{|y|^{N+\alpha}}\mathrm{d}y + \int_{\mathbb{R}^N} \frac{u(x+y)-u(x)}{|y|^{N+\alpha}}\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+\alpha}}\mathrm{d}y, \end{aligned} \tag{2.8}$$

which is integrable near 0

It is known that the singular integral in(2.7) is related to the fractional Laplacian $(-\Delta)^{\alpha/2}$

It is also interesting to write the displacement of the above random walk at time$N\tau$, for any $N \in \mathbb{N}$ Namely, if $h\epsilon_j \in h\mathbb{Z}^N$ is the jump performed at time $j\tau$ the above discussed random walk is made in such a way that the probability that ϵ_j equals k is $H(k)$.

The displacement at time τ is then the sum of these innovations, that is

$$\sum_{j=1}^{N} h\epsilon_j.$$

The β-moment associated to this process is then

$$\sum_{k\in\mathbb{Z}^N} |k|^{\beta} H(k) = \sum_{k\in\mathbb{Z}^N\setminus 0} |k|^{\beta-N-\alpha},$$

which is finite if and only if $\beta < \alpha$.

Kernels and Fourier symbols:

Given a nice function u (say, smooth and with rapid decay, for simplicity) the long jump random walk of "Long jump random walks and singular integral kernels" has lead us to the study of integrals of the type

$$\int_{\mathbb{R}^N} \big((u(x+y)+u(x-y)-2u(x))\big) H(y)\mathrm{d}y. \tag{2.9}$$

Let be $\mathscr{L}u$ the integral in (2.9) such as $\mathscr{L}$ a linear operator and look for it symbol in Fourier space . Suppose a function $\mathscr{S}:\mathbb{R}^N \longrightarrow \mathbb{R}$.

The important thing is H and $\mathscr{S}$ are related as follows :

$$\mathscr{S}(\xi)=\int_{\mathbb{R}^N}\big(\cos(\xi.y)-1\big)h(y)\mathrm{d}y. \tag{2.10}$$

To validate (2.10) Fourier transforms the variable x into ξ in:

$$\mathscr{L}u(x)=\mathscr{F}^{-1}(\mathscr{S}(\mathscr{F})),$$

where $\mathscr{F}$ denotes the Fourier transform one obtains:

$$\begin{aligned}
\mathscr{S}(\xi)(\mathscr{F}(u))(\xi) &= \mathscr{F}(\mathscr{L})u \\
&= \mathscr{F}\left(\int_{\mathbb{R}^N}(u(x+y)+u(x-y)-2u(x))H(y)\mathrm{d}y\right) \\
&= \int_{\mathbb{R}^N}\big(\mathscr{F}((u(x+y)+u(x-y)-2u(x)))\big)H(y)\mathrm{d}y \\
&= \int_{\mathbb{R}^N}(e^{i\xi.y}+e^{-i\xi.y}-2)(\mathscr{F}u)(\xi)H(y)\mathrm{d}y \\
&= \int_{\mathbb{R}^N}(e^{i\xi.y}+e^{-i\xi.y}-2)H(y)\mathrm{d}y(\mathscr{F}u)(\xi) \\
&= 2\int_{\mathbb{R}^N}(cos(\xi.y))H(y)\mathrm{d}y(\mathscr{F}u)(\xi).
\end{aligned}$$

The fractional Laplacian

The fractional Laplacian can be introduced naturally in the Fourier space .
In fact, we have this:

$$\partial_j u=\mathscr{F}^{-1}(i\xi_j(\mathscr{F}u)),$$

so:

$$-\Delta u=\mathscr{F}^{-1}(\left|\xi\right|^2(\mathscr{F}u)).$$

Also,we can know for $\alpha\in[0,2]$

$$(-\Delta)^{\alpha/2}u=\mathscr{F}^{-1}(\left|\xi\right|^{\alpha}(\mathscr{F}u)), \tag{2.11}$$

and, we can define principal value of singular integral of the fractional Laplacian as :

$$(-\Delta)^{\alpha/2}u = \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{\left|x-y\right|^{N+\alpha}}\mathrm{d}y, \tag{2.12}$$

by (2.8), one can also write (2.12) as:

$$(-\Delta)^{\alpha/2}u = \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{\left|y\right|^{N+\alpha}}\mathrm{d}y,$$

we can accept that: (2.11) is equivalent to (2.12).

Finally, we note from (2.12), the probability density of the limit long jump random walk in (2.5) may be written as :

$$\partial_t u = -(-\Delta)^{\alpha/2}u.$$

2.2 Some definitions of the fractional Laplacian

In all the following definitions, s represents the fractional power of the fractional Laplacian operator, and $C(N,s)$ is a positive constant depending on the dimension N and the fractional power s.

Definition 2.1 *The fractional Laplace operator is defined by:*

$$(-\Delta)^s u(x) = C(N,s)\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y, \tag{2.13}$$

we note that in the case of $x = y$, the fractional Laplacian operator has a singularity That is why it is known as an integro-differential operator. Therefore, this singularity must be removed.

The singularity can be removed when s in $]0,1/2[$ and is defined as a fraction Laplace makes sense. But when s in $[1/2,1[$ the integral in the definition need to be understood in the principal value as follows:

Singular Operator: Let s in $[1/2,1]$ and u in $\mathscr{S}(\mathbb{R}^N)$. We get :

$$(-\Delta)^s u(x) = C(N,s)\text{P.v.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y = C(N,s)\lim_{r\to 0^+}\int_{\mathbb{R}^N\setminus B_r(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y, \tag{2.14}$$

where:

$$C(N,s)=\frac{2^{2s}s\Gamma(s+\frac{N}{2})}{\pi^{\frac{N}{2}}|\Gamma(1-s)|}.$$

In the case $0<s<1/2$, the integral in the definition of the fractional Laplace operator is not really singular near x. Indeed for any u in $\mathscr{S}$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|}{|x-y|^{N+2s}}\mathrm{d}y &\leq \|u'(x)\|_{L^\infty(\mathbb{R}^N)}\int_{B_R}\frac{|x-y|}{|x-y|^{N+2s}}\mathrm{d}y+\|u(x)\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N/B_R}\frac{1}{|x-y|^{N+2s}}\mathrm{d}y\\ &= C\left(\int_{B_R}\frac{1}{|x-y|^{N+2s-1}}\mathrm{d}y+\int_{\mathbb{R}^N/B_R}\frac{1}{|x-y|^{N+2s}}\mathrm{d}y\right)\\ &= C\left(\int_0^R\frac{1}{\rho^{2s}}\mathrm{d}\rho+\int_R^\infty\frac{1}{\rho^{2s+1}}\mathrm{d}\rho\right)<\infty.\end{aligned}$$

Definition 2.2 *Fractional Laplacian can be defined in several ways. One of these methods if we take for example $z=x-y$ in (2.13) the singularity appears in the original, and the definition becomes as follows:*

$$(-\Delta)^s u(x)=C(N,s)\int_{\mathbb{R}^N}\frac{u(x)-u(x+z)}{|z|^{N+2s}}\mathrm{d}z,$$

in this definition we can use symmetry to $|z|^{-N-2s}$, after calculations we get the following definition:

$$(-\Delta)^s u(x)=-\frac{1}{2}C(N,s)\int_{\mathbb{R}^N}\frac{u(x+z)+u(x-z)-2u(x)}{|z|^{N+2s}}\mathrm{d}z, \tag{2.15}$$

the right-hand side is integrable near the origin because, for any smooth enough u in $C^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$,a second-order Taylor expansion yields:

$$\int_{\mathbb{R}^N}\frac{u(x+z)+u(x-z)-2u(x)}{|z|^{N+2s}}\mathrm{d}z\leq\frac{\|D^2u\|_{L^\infty(\mathbb{R}^N)}}{|z|^{N+2s-2}}.$$

Definition 2.3 (Fourier Definition) *Let s in $]0,1[$ and u in $\mathscr{S}(\mathbb{R}^n)$ we get :*

$$(-\Delta)^s u=\mathscr{F}^{-1}(S(\mathscr{F}u)),\ \forall x\in\mathbb{R}^N.$$

Where:

S the function defined on $\mathbb{R}^N\longrightarrow\mathbb{R}$ it is called a the Fourier symbol or multiplier, and Fourier symbol of $(-\Delta)^s$ is $|\xi|^{2s}$ for all ξ in $\mathbb{R}^N$, so the definition becomes as follows:

$$(-\Delta)^s u(x)=\mathscr{F}^{-1}(|\xi|^{2s}(\mathscr{F}u)(\xi))(x). \tag{2.16}$$

In view of the definition (2.15) we remind for s in $]0,1[$ that:

$$(-\Delta)^s u(x) = -\frac{1}{2} C(N,s) \int_{\mathbb{R}^N} \frac{u(x+z)+u(x-z)-2u(x)}{|z|^{N+2s}} \mathrm{d}z,$$

but the difference between the two definitions is the constant $C(N,s)$ where in definition, it is called normalization constant and is defined as follows:

$$C(N,s) = \left(\int_{\mathbb{R}^N} \frac{1-cos\eta}{|\eta|^{N+2s}} \mathrm{d}\eta \right)^{-1}. \tag{2.17}$$

Corollary 2.1 *For any $N > 1$ let $C(N,s)$ be defined by (2.17). The following statements hold:*

1) $\lim\limits_{s\to 1^-} \frac{C(N,s)}{s(1-s)} = \frac{4N}{\alpha_{N-1}}$,

2) $\lim\limits_{s\to 0^+} \frac{C(N,s)}{s(1-s)} = \frac{2}{\alpha_{N-1}}$,

where α_{N-1} denotes the $(N-1)$-dimensional measure of the unit sphere $\mathscr{S}_{N-1}$.

2.3 Some properties of the fractional Laplacian

Theorem 2.1 *1) For any $0 < s_1 < 1$ and $0 < s_2 < 1$, we have:*

$$(-\Delta)^{s_1} \circ (-\Delta)^{s_2} u = (-\Delta)^{s_2} \circ (-\Delta)^{s_1} u.$$

2) $(-\Delta)^s$ is linear operator.

3) $\lim\limits_{s\to 0^+} (-\Delta)^s u = u$ For any u in $C^\infty(\mathbb{R}^N)$.

4) $\lim\limits_{s\to 0^+} (-\Delta)^s u = u$.

Proof.

1. **Proof of property 1.**

$$\begin{aligned}
(-\Delta)^{s_1} \circ (-\Delta)^{s_2} u &= (-\Delta)^{s_1} \left(\mathscr{F}^{-1}(\|\xi\|^{2s_2}(\mathscr{F}u)(\xi))(x) \right) \\
&= \mathscr{F}^{-1}(\|\xi\|^{2s_1}(\mathscr{F}(\mathscr{F}^{-1}(\|\xi\|^{2s_2}(\mathscr{F}u)(\xi))(x)))(\xi))(x) \\
&= \mathscr{F}^{-1}(\|\xi\|^{2s_1}\|\xi\|^{2s_2}(\mathscr{F}u)(\xi))(x)
\end{aligned}$$

$$(-\Delta)^{s_1} \circ (-\Delta)^{s_2} u = \mathscr{F}^{-1}(\|\xi\|^{2(s_1+s_2)}(\mathscr{F}u)(\xi))(x), \tag{2.18}$$

if we replace s_1 by s_2 and s_2 by s_1 in (2.18),we find:

$$(-\Delta)^{s_2}\circ(-\Delta)^{s_1}u=\mathscr{F}^{-1}(\|\xi\|^{2(s_2+s_1)}(\mathscr{F}u)(\xi))(x), \tag{2.19}$$

so according to (2.18) and (2.19) we find :

$$(-\Delta)^{s_1}\circ(-\Delta)^{s_2}u=(-\Delta)^{s_2}\circ(-\Delta)^{s_1}u.$$

2. **Proof of property 2.** $\forall s\in]0,1[$, $u,v\in\mathscr{S}(\mathbb{R}^N)$ and $\lambda\in\mathbb{R}$ we have :

$$\begin{aligned}(-\Delta)^s(\lambda u+v) &= C(N,s)\int_{\mathbb{R}^N}\frac{(\lambda u(x)+v(x))-(\lambda u(y)+v(y))}{|x-y|^{N+2s}}\mathrm{d}y\\ &= C(N,s)\int_{\mathbb{R}^N}\frac{\lambda(u(x)-u(y))}{|x-y|^{N+2s}}\mathrm{d}y+C(N,s)\int_{\mathbb{R}^N}\frac{v(x)-v(y)}{|x-y|^{N+2s}}\mathrm{d}y\\ &= \lambda(-\Delta)^s u+(-\Delta)^s v.\end{aligned}$$

So $(-\Delta)^s$ is linear operator.

3. **Proof of property 3.** Fix $x\in\mathbb{R}^N$, $R_0>0$ such that $\operatorname{supp}u\subseteq B_{R_0}$ and set $R=R_0+|x|+1$.First,

$$\begin{aligned}\int_{B_R}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\mathrm{d}y &\leq \|u\|_{C^2(\mathbb{R}^N)}\int_{B_R}\frac{\|y\|^2}{\|y\|^{N+2s}}\mathrm{d}y\\ &\leq \alpha_{N-1}\|u\|_{C^2(\mathbb{R}^N)}\int_0^R\frac{1}{\rho^{2s-1}}\mathrm{d}\rho\\ &= \frac{\alpha_{N-1}\|u\|_{C^2(\mathbb{R}^N)}R^{2-2s}}{2(1-s)}.\end{aligned} \tag{2.20}$$

For $|y|>R$ we observe that $|x\pm y|\geq|y|-|x|\geq R-|x|>R$ and consequently $u(x\pm y)=0$ for $|y|>R$. Therefore,

$$\begin{aligned}-\frac{1}{2}\int_{\mathbb{R}^N/B_R}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\mathrm{d}y &= u(x)\int_{\mathbb{R}^N/B_R}\frac{1}{|y|^{N+2s}}\mathrm{d}y\\ &= \alpha_{N-1}u(x)\int_R^\infty\frac{1}{\rho^{2s-1}}\mathrm{d}\rho\\ &= \frac{\alpha_{N-1}R^{-2s}}{2s}u(x).\end{aligned} \tag{2.21}$$

Then by (2.20) and Corollary (2.1), we have

$$\lim_{s\to 0^+}\left[-\frac{C(N,s)}{2}\int_{\mathbb{R}^N/B_R}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\mathrm{d}y\right]=0, \tag{2.22}$$

and so we get,

$$\begin{aligned}\lim_{s\to 0^+}(-\Delta)^s u &= \lim_{s\to 0^+}\left[-\tfrac{C(N,s)}{2}\int_{\mathbb{R}^N/B_R}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\mathrm{d}y\right]\\ &= \lim_{s\to 0^+}\left[-\tfrac{C(N,s)\alpha_{N-1}R^{-2s}}{2s}u(x)\right]\\ &= u(x),\end{aligned}$$

so

$$\lim_{s\to 0^+}(-\Delta)^s u = u.$$

4. **Proof of property 4.** In a similar way with the proof of the previous property

$$\begin{aligned}\left|\int_{\mathbb{R}^N/B_1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\mathrm{d}y\right| &\le 4\|u\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N/B_1}\frac{1}{|y|^{N+2s}}\mathrm{d}y\\ &\le 4\alpha_{N-1}\|u\|_{L^\infty(\mathbb{R}^N)}\int_1^\infty\frac{1}{|\rho|^{2s+1}}\mathrm{d}\rho\\ &= \tfrac{2\alpha_{N-1}}{s}\|u\|_{L^\infty(\mathbb{R}^N)}.\end{aligned}$$

Also we have

$$\lim_{s\to 1^-}\left[-\frac{C_{N,s}}{2}\int_{\mathbb{R}^N/B_1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\mathrm{d}y\right]=0. \tag{2.23}$$

On the other hand, we get

$$\begin{aligned}\left|\int_{B_1}\frac{u(x+y)+u(x-y)-2u(x)-D^2u(x)y.y}{|y|^{N+2s}}\mathrm{d}y\right| &\le \|u\|_{C^3(\mathbb{R}^N)}\int_{B_1}\frac{|y|^3}{|y|^{N+2s}}\mathrm{d}y\\ &\le \alpha_{N-1}\|u\|_{C^3(\mathbb{R}^N)}\int_0^1\frac{1}{|\rho|^{2s-2}}\mathrm{d}\rho\\ &= \tfrac{\alpha_{N-1}\|u\|_{C^3(\mathbb{R}^N)}}{3-2s}.\end{aligned}$$

And that means

$$\lim_{s\to 1^-}-\frac{C_{N,s}}{2}\int_{B_1}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\mathrm{d}y=\lim_{s\to 1^-}-\frac{C_{N,s}}{2}\int_{B_1}\frac{D^2u(x)y.y}{|y|^{N+2s}}\mathrm{d}y. \tag{2.24}$$

Notice that if $i \neq j$ then

$$\int_{B_1} \partial^2_{ij} u(x) y_i . y_j \mathrm{d}y = -\int_{B_1} \partial^2_{ij} u(x) \tilde{y}_i . \tilde{y}_j \mathrm{d}\tilde{y}, \tag{2.25}$$

where $\tilde{y}_k = y_k$ for any $k \neq j$,and $\tilde{y}_j = -y_j$,so

$$\int_{B_1} \partial^2_{ij} u(x) y_i . y_j \mathrm{d}y = 0. \tag{2.26}$$

Also for any fixed i, we have

$$\begin{aligned}
\int_{B_1} \frac{\partial^2_{ij} u(x) y_i^2}{|y|^{N+2s}} \mathrm{d}y &= \partial^2_{ij} u(x) \int_{B_1} \frac{y_i^2}{|y|^{N+2s}} \mathrm{d}y \\
&= \partial^2_{ij} u(x) \int_{B_1} \frac{y_1^2}{|y|^{N+2s}} \mathrm{d}y \\
&= \frac{\partial^2_{ij} u(x)}{N} \sum_{j=1}^{N} \int_{B_1} \frac{y_j^2}{|y|^{N+2s}} \mathrm{d}y \\
&= \frac{\partial^2_{ij} u(x)}{N} \int_{B_1} \frac{|y|^2}{|y|^{N+2s}} \mathrm{d}y \\
&= \frac{\partial^2_{ij} u(x) \alpha_{N-1}}{2N(1-s)}.
\end{aligned} \tag{2.27}$$

Finally, combining (2.23), (2.24), (2.25), (2.26),(2.27) and Corollary (2.1), we conclude that

$$\begin{aligned}
\lim_{s\to 1^-} (-\Delta)^s &= \lim_{s\to 1^-} \left[-\frac{C_{N,s}}{2} \int_{B_1} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} \mathrm{d}y \right] \\
&= \lim_{s\to 1^-} \left[-\frac{C_{N,s}}{2} \int_{B_1} \frac{D^2 u(x) y . y}{|y|^{N+2s}} \mathrm{d}y \right] \\
&= \lim_{s\to 1^-} \left[\frac{C_{N,s}}{2} \sum_{j=1}^{N} \int_{B_1} \frac{\partial^2_{ij} u(x) y_i^2}{|y|^{N+2s}} \mathrm{d}y \right] \\
&= \lim_{s\to 1^-} \left[-\frac{C_{N,s} \alpha_{N-1}}{4N(1-s)} \sum_{i=1}^{N} \partial^2_{ij} u(x) \right] \\
&= -(\Delta) u(x).
\end{aligned}$$

■

2.4 Weak formulation

Let s in $]0,1[$ and $\Omega \subset \mathbb{R}^N$, N in $\mathbb{N}$, be a Lipschitz domain. In this section, we focus on the existence and the uniqueness of a weak solution to the following fractional Poisson problem

$$\begin{cases} (-\Delta)^s u = f, & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.28}$$

where $f \in L^2(\Omega)$ and

$$(-\Delta)^s u = C(N,s)\,\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \mathrm{d}y \text{ with } C(N,s) = \frac{2^{2s} s\Gamma(s+\frac{N}{2})}{\pi^{\frac{N}{2}}\Gamma(1-s)}, \tag{2.29}$$

where P.V. stands for the Cauchy principal value and $C(n,s)$ is a normalization constant. Note that identity (2.29) illustrates the nonlocal structure of the fractional Laplacian. It uses information about u far from x.

We define the fractional Sobolev space $\mathbb{X}_2^s(\Omega)$ where the solutions of (2.28) live as follow

$$\mathbb{X}_2^s(\Omega) = \left\{ u \in H^s(\mathbb{R}^N) :\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

According to [8, Proposition 3.7, page 31], the space $\mathbb{X}_2^s(\Omega)$ coincides with the completion of $\mathscr{C}_0^\infty(\Omega)$ with respect to the norm $[.]_{s,p,\mathbb{R}^N}$.

Now, we give the definition of the weak solutions to Problem (2.28).

Definition 2.4 *We say that u in $\mathbb{X}_2^s(\Omega)$ is a weak solution to the problem (2.28) if*

$$\frac{C(N,s)}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int_\Omega f(x)v(x)\mathrm{d}x,\ \forall v \in \mathbb{X}_2^s(\Omega). \tag{2.30}$$

Notice that the integral in (2.29) is well defined because u in $H^s(\mathbb{R}^N)$. By multiplying (2.28) by the arbitrary test function v in $\mathbb{X}_2^s(\Omega)$ and integrating over $\mathbb{R}^N$, which yields

$$\int_{\mathbb{R}^N} (-\Delta)^s u(x)v(x)\mathrm{d}x = \int_\Omega f(x)v(x)\mathrm{d}x. \tag{2.31}$$

Since

$$\begin{aligned}\int_{\mathbb{R}^N}(-\Delta)^s u(x)v(x)\mathrm{d}x &= \int_{\mathbb{R}^N}\left(C(N,s)\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{\left|x-y\right|^{N+2s}}\mathrm{d}y\right)v(x)\mathrm{d}x\\ &= C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))v(x)}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y\\ &= C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y\\ &\quad +C(N,s)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))v(y)}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where we have used Fubini's Theorem. By changing the role of x and y, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))v(y)}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y=-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))v(x)}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

Therefore,

$$\int_{\mathbb{R}^N}(-\Delta)^s u(x)v(x)\mathrm{d}x=\frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

This and (2.31) yield (2.30).

Theorem 2.2 Let f in $L^2(\Omega)$, there exists a unique weak solution in $\mathbb{X}_2^s(\Omega)$ for the Dirichlet problem (2.28).

We provide two proofs to this Theorem. The first relies on the abstract Lax-Milgram theorem, while the second uses a variational method.

2.4.1 The first method : Lax-Milgram theorem

Theorem 2.3 *Let H be a Hilbert space, let $a : H\times H\to\mathbb{R}$ be a continuous bilinear form, satisfying the coercivity condition*

$$\exists\alpha>0:\ a(v,v)\geq\alpha\|v\|_V^2,\ \ \forall v\in V.$$

Let L be a continuous linear form on V. Then there exists a unique u in H such as:

$$\forall v\in H,\ a(u,v)=L(v).$$

We apply Lax-Milgram's theorem in the Hilbert space $\mathbb{X}_2^s(\Omega)$ with the bilinear form

$$a(u,v)=\frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y,$$

and the linear form

$$L(v)=\int_\Omega f(x)v(x)\mathrm{d}x.$$

1. **Continuity of a.** For all u,v in $\mathbb{X}_2^s(\Omega)$, we have:

$$\begin{aligned}\left|a(u,v)\right| &= \left|\tfrac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y\right| \\ &\leq \tfrac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|\frac{(u(x)-u(y))(v(x)-v(y))}{\left|x-y\right|^{N+2s}}\right|\mathrm{d}x\mathrm{d}y \\ &\leq \tfrac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u(x)-u(y)\right|\left|v(x)-v(y)\right|}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &= \tfrac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u(x)-u(y)\right|\left|v(x)-v(y)\right|}{|x-y|^{\frac{1}{2}(n+2s)}|x-y|^{\frac{1}{2}(n+2s)}}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

   Using Cauchy-Schwartz inequality, we get

$$\begin{aligned}\left|a(u,v)\right| &\leq C\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u(x)-u(y)\right|^2}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|v(x)-v(y)\right|^2}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \\ &\leq C\|u\|_{\mathbb{X}_2^s(\Omega)}\|v\|_{\mathbb{X}_2^s(\Omega)},\end{aligned}$$

   where C is a constant depending only on N, s and Ω. Therefore a is continuous.

2. **Coerciveness of a.** For all u in $\mathbb{X}_2^s(\Omega)$, we have:

$$\begin{aligned}a(u,u) &= \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))^2}{\left|x-y\right|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &\geq \alpha\|u\|^2_{\mathbb{X}_2^s(\Omega)},\end{aligned}$$

   where α is a positive constant depending only on N, s and Ω. Hence a is coercive.

3. **Continuity of L.** For all v in $\mathbb{X}_2^s(\Omega)$, we have

$$\begin{aligned}\left|L(v)\right| &= \left|\int_\Omega f(x)v(x)\mathrm{d}x\right| \\ &\leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\ &\leq \|f\|_{L^2(\Omega)}\|v\|_{\mathbb{X}^2(\Omega)}.\end{aligned}$$

Therefore L is continuous.

Then, we conclude that, according to the Lax-Milgram theorem, the problem (2.28) admits unique solution u in $\mathbb{X}_2^s(\Omega)$.

2.4.2 The second method : Variational Method

Critical Point Theory and Variational Methods are crucial in the study of partial differential equations. Many problems can be solved in terms of the minimization of a functional.

In this section we use critical point theory to establish the existence of at least a minimizer of the variational problem, to solve problem (2.30).

Let U is Banach space and A an operator of U in U', and for f in U' we consider the equation

$$Au = f. \tag{2.32}$$

We consider the functional J of U in $\mathbb{R}$, let ∇J is gradient of J, such as $\nabla J(u) = Au - f$ for all u in U we say that u is a solution of (2.32) if only if u is a critical point of J that's to say

$$\nabla J(u) = 0.$$

Note that the bilinear form $a(\cdot,\cdot)$ is symmetric and positive, i.e.

$$\forall\, u, v \in \mathbb{X}_2^s(\Omega),\ a(u,v) = a(v,u) \text{ and } \forall\, v \in \mathbb{X}_2^s(\Omega),\ a(v,v) \geq 0,$$

so the problem (2.30) can be interpreted as a minimization problem.

$$J(u) = \inf_{v \in \mathbb{X}_2^s(\Omega)} J(v), \tag{2.33}$$

where the functional J is defined as

$$\forall v \in \mathbb{X}_2^s(\Omega),\ J(v) = \frac{C(N,s)}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x)-v(y))(v(x)-v(y))}{\left|x-y\right|^{N+2s}} \mathrm{d}x\mathrm{d}y - \int_\Omega f(x)v(x)\mathrm{d}x.$$

Definition 2.5 (Gateaux differentiability) *: Let U and V two Banach space and $A : U \longrightarrow V$ an operator*

i) If for u, v in U the quotient

$$\frac{A(u+tv)-A(u)}{t} \in V,$$

to a finite limit when $t \longrightarrow 0$, we note it $A'(u,v)$ and we call it the G-differential of A at point u in the direction v.

ii)If $A'(u,v)$ exists for all v in U, we say $A$is G-differential at the point u.
We note G-differential at the point u by $A'(u)$ so $A'(u).v = A'(u,v)$ with A' continuous linear of U in V.

Definition 2.6 *Let U is Banach space $J : U \longrightarrow \mathbb{R}$ a function.We say that:*
J is convex if:

$$\forall u, v \in \mathbb{X}_2^s(\Omega), \forall t \in [0,1] \qquad J(tu+(1-t)v) \leq tJ(u)+(1-t)J(v).$$

Definition 2.7 *Let U is Banach space $J : U \longrightarrow \mathbb{R}$ a function. We say that:*
J is lower semi-continuous if : for any u in V, and any sequence (u_m) in V that $u_m \rightharpoonup u$ weakly in V there holds:

$$J(u) \leq \liminf_{m \to +\infty} J(u_m).$$

Theorem 2.4 u in $\mathbb{X}_2^s(\Omega)$ is solution of the problem (2.28) if and only if u is solution of the minimization problem (2.33).

Proof.

- Let u is solution of the problem (2.28). We set $w = v - u$, then for all v in $\mathbb{X}_2^s(\Omega)$,

we have

$$\begin{aligned} J(v)=J(u+w) &= \tfrac{1}{2}a(u,u)+\tfrac{1}{2}a(w,w)+\tfrac{1}{2}a(w,u)+\tfrac{1}{2}a(u,w)-l(w)-l(u) \\ &= J(u)+\tfrac{1}{2}a(u,w)+\tfrac{1}{2}a(u,w)+\tfrac{1}{2}a(w,w)-l(w) \\ &= J(u)+a(u,w)+a(w,w)-l(w) \\ &\geq J(u). \end{aligned}$$

Therefore if u is a solution to the problem (2.28), then u is also solution to the problem (2.33).

- Conversely, let u minimizes J in $\mathbb{X}_2^s(\Omega)$. For all $t>0$ and v in $\mathbb{X}_2^s(\Omega)$ we have

$$J(u+tv)\geq J(u),$$

by substituting, we find :

$$\frac{1}{2}a(u,u)+\frac{t^2}{2}a(v,v)+\frac{t}{2}a(u,v)+\frac{t}{2}a(u,v)-l(v)-l(u)\geq J(u)$$

$$J(u)+\frac{t^2}{2}a(v,v)+\frac{t}{2}a(u,v)-l(v)\geq J(u),$$

by dividing by t we get:

$$a(u,v)-l(v)+\frac{t}{2}a(v,v)\geq 0,$$

by letting t tend to 0, we deduce that

$$a(u,v)-l(v)\geq 0, \tag{2.34}$$

if we replacing v by $-v$ we get

$$a(u,-v)-l(-v)\geq 0.$$

Since a is a bilinear from and L it is a linear form, we have

$$-a(u,v)+l(v)\geq 0,$$

so,

$$a(u,v)-l(v)\leq 0. \tag{2.35}$$

Then (2.34) and (2.35) gives us that:

$$a(u,v)-l(v)=0,$$

$$a(u,v)=l(v).$$

Therefore if u in $\mathbb{X}_2^s(\Omega)$ is a solution to the problem (2.33). therefore u in $\mathbb{X}_2^s(\Omega)$ is a solution to the problem (2.28).This finishes the proof.

■

Theorem 2.5 Let f in $L^2(\Omega)$, there exists a unique weak solution u in $\mathbb{X}_2^s(\Omega)$ for the minimization problem (2.33).

For the proof of the Theorem 2.5, we will need the following lemma:

Lemma 2.1 *Suppose V is a reflexive Banach space and $J : V \longrightarrow \mathbb{R}\cup\{+\infty\}$ is coercive, (sequentially) weakly lower semi-continuous and convex function on V, that is, suppose the following conditions are fulfilled:*

1. *J is convex.*
2. *J is coercive : $J(u)\longrightarrow +\infty$ as $|u|\longrightarrow +\infty$.*
3. *J is lower semicontinuous.*

Then there exists u in V such that

$$J(u)=\inf_{v\in V} J(v).$$

Proof of the Theorem 2.5. The proof of this theorem will be achieved in two steps.

Step 1 - Existence.

1. **Convexity of j.** Let u, v in $\mathbb{X}_2^s(\Omega)$ and t in $[0,1]$ we have:

$$\begin{aligned} J(tu+(1-t)v) &= \frac{C(N,s)}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(t(u(x)+(1-t)v(x))+(tu(y)+(1-t)v(y))\big)^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &\quad -\int_\Omega f(x)(tu(x)+(1-t)v(x))\mathrm{d}x \\ &= \frac{C(N,s)}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big(t(u(x)-u(y))+(1-t)(v(x)-v(y))\big)^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &\quad -t\int_\Omega f(x)u(x)\mathrm{d}x-(1-t)\int_\Omega f(x)v(x)\mathrm{d}x, \end{aligned}$$

since the function $x\mapsto x^2$ is a convex we obtain

$$\begin{aligned} J(tu+(1-t)v) &\leq t\left[\frac{C(N,s)}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big((u(x)-u(y))\big)^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y-\int_\Omega f(x)u(x)\mathrm{d}x\right] \\ &\quad +(1-t)\left[\frac{C(N,s)}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\big((v(x)-v(y))\big)^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y-\int_\Omega f(x)v(x)\mathrm{d}x\right] \\ &\leq tJ(u)+(1-t)J(v). \end{aligned}$$

Therefore J is convex.

2. **Coerciveness of j.**

$$J(u)=\frac{C(N,s)}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(u(x)-u(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y-\int_\Omega f(x)u(x)\mathrm{d}x$$

Using Cauchy-Schwartz inequality we get

$$\begin{aligned} J(u) &\geq \frac{C(N,s)}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y-\left(\int_\Omega |u(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega |f(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} \\ &\geq \frac{C(N,s)}{4}\|u\|^2_{\mathbb{X}_2^s(\Omega)}-C_1\|u\|_{L^2(\Omega)} \\ &\geq \frac{C(N,s)}{4}\|u\|^2_{\mathbb{X}_2^s(\Omega)}-C_2\|u\|_{\mathbb{X}_2^s(\Omega)}. \end{aligned}$$

By passing to the limit that we have:

$$\lim_{\|u(x)\|_{\mathbb{X}_2^s(\Omega)}\to+\infty} J(u)=+\infty.$$

Therefore J is coercive.

3. **Lower semi-continuous of j.** Let u in $\mathbb{X}_2^s(\Omega)$ and let (u_n) be sequence in $\mathbb{X}_2^s(\Omega)$ such as $u_n \rightharpoonup u$ weakly in $\mathbb{X}_2^s(\Omega)$

$$J(u_n) = \frac{C(N,s)}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y - \int_\Omega f(x) u_n(x) \mathrm{d}x.$$

In one hand, due to the l.s.c. of the norm, we have

$$\frac{C(N,s)}{4} \|u\|^2_{\mathbb{X}_2^s(\Omega)} \leq \frac{C(N,s)}{4} \liminf_{n\to+\infty} \|u_n\|^2_{\mathbb{X}_2^s(\Omega)}, \tag{2.36}$$

and in other hand, because f in $L^2(\Omega)$ and $u_n \rightharpoonup u$ weakly in $L^2(\Omega)$, we get

$$\lim_{n\to+\infty} \int_\Omega f(x) u_n(x) \mathrm{d}x = \int_\Omega f(x) u(x) \mathrm{d}x. \tag{2.37}$$

Combining (2.36) and (2.37) yields

$$J(u) \leq \liminf_{n\to+\infty} J(u_n).$$

Therefore J is l.s.c

Step 2 - Uniqueness. Let u_1, u_2 in $\mathbb{X}_2^s(\Omega)$ be solutions of (2.28) e.i.,

$$\begin{cases} (-\Delta)^s u_1 = f, & \text{in } \Omega \\ u_1 = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.38}$$

and

$$\begin{cases} (-\Delta)^s u_2 = f, & \text{in } \Omega \\ u_2 = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.39}$$

By combining (2.38) and (2.39) , and set $w = u_1 - u_2$ we have:

$$\begin{cases} (-\Delta)^s w = 0, & \text{in } \Omega \\ w = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.40}$$

The weak formulation of the equation (2.40) is :

$$\frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = 0.$$

I.e. $\frac{C(N,s)}{2}\|w\|_{\mathbb{X}_2^s(\Omega)} = 0 \quad \Rightarrow \quad \|w\|_{\mathbb{X}_2^s(\Omega)} \Rightarrow \quad w = 0 \quad \Rightarrow \quad u_1 = u_2.$

Therefore u is unique. Then, we conclude that, according to the Variational Method, the problem (2.33) admits unique solution u in $\mathbb{X}_2^s(\Omega)$,So the problem (2.28) admits unique solution u in $\mathbb{X}_2^s(\Omega)$.

CHAPTER 3

Fractional p-Laplacian

3.1 Weak formulation

Let s in $]0,1[$, p in $]1,+\infty[$and $\Omega \subset \mathbb{R}^N$, N in $\mathbb{N}$, be a Lipschitz domain. In this section, We prove Existence and uniqueness of the weak solution for the following fractions problems

$$\begin{cases} (-\Delta)_p^s u = f & \text{in } \Omega \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.1}$$

where f in $L^2(\Omega)$ and we define the p-Laplacian factionary by:

$$\begin{aligned} (-\Delta)_p^s u &= 2\lim_{\epsilon \to 0} \int_{\mathbb{R}^N / B_\epsilon} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \mathrm{d}y \\ &= 2PV \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \mathrm{d}y, \end{aligned} \tag{3.2}$$

where P.V. stands for the Cauchy principal value.

We define the fractional Sobolev space $\mathbb{X}_p^s(\Omega)$ where the solutions of (3.1) live as follow

$$\mathbb{X}_p^s(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : \ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

Definition 3.1 *Let s in $]0,1[$ and p in $]1,+\infty[$. Then*

$$(-\Delta)_p^s : W^{s,p}(\mathbb{R}^N) \to (W^{s,p}(\mathbb{R}^N))'$$

And for all u and v in $W^{s,p}(\mathbb{R}^N)$, we have:

$$\left\langle (-\Delta)_p^s u, v \right\rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y.$$

Proof. we've got the integral in the definition of $-(\Delta)_p^s$ because u in $W^{s,p}(\mathbb{R}^N)$ exist , so

$$(\Delta)_p^s u(x) = \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \mathrm{d}x,$$

$\forall u, v$ in $W^{s,p}(\mathbb{R}^N)$ and according to Fubini theorem we have:

$$\begin{aligned} \left\langle (-\Delta)_p^s u, v \right\rangle &= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \mathrm{d}y\, v(x)\mathrm{d}x \\ &= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} v(x)\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.3}$$

We notice

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} v(x)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} v(y)\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{3.4}$$

by changing the role of x and y , we get:

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} v(x)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} v(x)\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{3.5}$$

so By adding (3.4)and (3.5), we find:

$$\begin{aligned} &2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} v(x)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y, \end{aligned}$$

therefore,

$$\left\langle (-\Delta)_p^s u, v \right\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y.$$

■

Proposition 3.1 $\forall u, v \ in \ \mathbb{W}_0^{s,p}(\Omega)$

$$\left\langle (-\Delta)_p^s u, v \right\rangle \leq [u]_{s,p}^{p-1}[v]_{s,p}$$

Proof.

$$\begin{aligned}
\left\langle (-\Delta)_p^s u, v \right\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\
&\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-1}(v(x)-v(y))}{|x-y|^{(N+sp)(\frac{p-1}{p})}|x-y|^{(N+sp)(\frac{1}{p})}}\mathrm{d}x\mathrm{d}y \\
&\leq \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{(N+sp)}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{(N+sp)}}\right)^{\frac{1}{p}} \\
&= [u]_{s,p}^{p-1}[v]_{s,p}.
\end{aligned}$$

■

Proposition 3.2 *Let* $s \ in \ ]0,1[$ *and* $p \ in \ ]1,+\infty[$. *So:*

$$(-\Delta)_p^s) : W_0^{s,p}(\mathbb{R}^N) \longrightarrow (W_0^{s,p}(\mathbb{R}^N))',$$

is continuous.

Proof. Let $u_n \longrightarrow u$ in $W_0^{s,p}(\mathbb{R}^N)$ let's show that $(-\Delta)_p^s u_n \longrightarrow (-\Delta)_p^s u$ in $(W_0^{s,p}(\mathbb{R}^N))'$ .Indeed, in accordance with the proposal (3.1), we have:

$$\|(-\Delta)_p^s u_n - (-\Delta)_p^s u\|_{W_0^{s,p}(\mathbb{R}^N)} \leq [u_n - u]_{s,p}^{p-1},$$

therefore, it is enough to demonstrate that:$[u_n - u]_{s,p}^{p-1} \longrightarrow 0$; we have $u_n \longrightarrow u$ in $W_0^{s,p}(\mathbb{R}^N)$ so $u_n \longrightarrow u$ in $L^p(\mathbb{R}^N)$ so for a subsequence, we have: $\exists g \in L^p(\mathbb{R}^N)$ such that:

$$u_n(x) \longrightarrow u(x) \quad \text{a.e and } |u_n| \leq g(x),$$

based on the Dominated Convergence Theorem, we achieve the desired outcome. ■

We prove that the problem (3.1) has a solution using the Minty-Browder theorem.

3.1.1 Minty-Browder

Theorem 3.1 (Minty-Browder theorem) *Let V Banach space and V' its topological dual, and let A of V in V' is an operator bounded, continuous, coercive and monotone, so A is surjective*

Definition 3.2 (strictly monotonic operator) *Let u, v in $\mathbb{X}_p^s(\Omega)$ and $u \neq v$*

$$\left\langle (-\Delta)_p^s u - (-\Delta)_p^s v, u - v \right\rangle > 0.$$

Definition 3.3 (coercive operator) *Let V Banach space*

$$\lim_{\|u\|_v \to +\infty} \frac{\langle Au, u \rangle}{\|u\|_v} = +\infty.$$

Definition 3.4 (hemicontinuous operator) *An operator A in $v \longrightarrow w$ is said to be hemicontinuous on point u of v if:*

$$\forall v \in V, \forall \{\lambda_n\} \subset \mathbb{R}, \lambda_n \longrightarrow 0 \Longrightarrow A(u + \lambda_n v) \Longrightarrow Au \quad \textit{weakly in } w$$

1. **monotone of operator $(-\Delta)_p^s$.**

$$\left\langle (-\Delta)_p^s u - (-\Delta)_p^s v, u - v \right\rangle \geq \left([u]_{s,p}^{p-1} - [v]_{s,p}^{p-1} \right) \left([u]_{s,p} - [v]_{s,p} \right).$$

It is clear that: $\left([u]_{s,p}^{p-1} - [v]_{s,p}^{p-1} \right) \left([u]_{s,p} - [v]_{s,p} \right) \geq 0$, it is sufficient to prove that: $\left([u]_{s,p}^{p-1} - [v]_{s,p}^{p-1} \right) \left([u]_{s,p} - [v]_{s,p} \right) \neq 0$.

We impose

$$\left([u]_{s,p}^{p-1} - [v]_{s,p}^{p-1} \right) \left([u]_{s,p} - [v]_{s,p} \right) = 0,$$

so according to the theorem of Poincaré we have:

$$\left(\|u\|_{s,p}^{p-1} - \|v\|_{s,p}^{p-1} \right) \left(\|u\|_{s,p} - \|v\|_{s,p} \right) = 0,$$

i.e $\|u\|_{W^{s,p}(\Omega)} - \|v\|_{W^{s,p}(\Omega)} = 0$, or $\|u\|_{W^{s,p}(\Omega)}^{p-1} - \|v\|_{W^{s,p}(\Omega)}^{p-1} = 0$, so $\|u\|_{W^{s,p}(\Omega)} = \|v\|_{W^{s,p}(\Omega)}$ which is absurd with the fact that $u \neq v$, Therefore $(-\Delta)_p^s$ is monotone.

2. $(-\Delta)_p^s$ **is coercive.**

$$\begin{aligned}\frac{\left\langle (-\Delta)_p^s u, u\right\rangle}{\|u\|_{W^{s,p}(\Omega)}} &= \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))^2}{|x-y|^{N+sp}}}{\|u\|_{W^{s,p}(\Omega)}} \\ &= \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p}}{|x-y|^{N+sp}}}{\|u\|_{W^{s,p}(\Omega)}} \\ &= \frac{\|u\|^p_{W^{s,p}(\Omega)}}{\|u\|_{W^{s,p}(\Omega)}} \\ &= \|u\|^{p-1}_{W^{s,p}(\Omega)},\end{aligned}$$

so, by passing to the limit that we have:

$$\lim_{\|u\|_{W^{s,p}(\Omega)}\to+\infty} \frac{\left\langle (-\Delta)_p^s u, u\right\rangle}{\|u\|_{W^{s,p}(\Omega)}} = \lim_{\|u\|_{W^{s,p}(\Omega)}\to+\infty} \|u\|^{p-1}_{W^{s,p}(\Omega)} = +\infty.$$

Therefore $(-\Delta)_p^s$ is coercive.

3. **Operator**$(-\Delta)_p^s$ **is semi-continuous.** Let u in $W_0^{s,p}(\mathbb{R}^N)$, v in $W_0^{s,p}(\mathbb{R}^N)$ and $\{\lambda_n\}\subset\mathbb{R}^N$ such as $\lambda_n \to 0$. Let's show that

$$(-\Delta)_p^s(u+\lambda_n) \rightharpoonup (-\Delta)_p^s u.$$

Let w in $W_0^{s,p}(\mathbb{R}^N)$ We have:

$$\begin{aligned}&\left\langle (-\Delta)_p^s(u+\lambda_n), w\right\rangle \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|(u+\lambda_n v)(x)-(u+\lambda_n v)(y)\right|^{p-2}\big((u+\lambda_n v)(x)-(u+\lambda_n v)(y)\big)(w(x)-w(y)}{|x-y|^{N+sp}}.\end{aligned}$$

By continuity, we obtain:

$$\frac{\left|(u+\lambda_n v)(x)-(u+\lambda_n v)(y)\right|^{p-2}\big((u+\lambda_n v)(x)-(u+\lambda_n v)(y)\big)}{|x-y|^{N+sp}} \longrightarrow \frac{\left|u(x)-u(y)\right|^{p-2}\big(u(x)-u(y)\big)}{|x-y|^{N+sp}}.$$

And as $\{\lambda_n\}$ is convergent so it is bounded with in $\mathbb{R}, \exists c>0\ |\lambda_n|\le c$.

Based on the lemme (1.1), we find:

$$\left|\frac{|(u+\lambda_n v)(x)-(u+\lambda_n v)(y)|^{p-2}\big((u+\lambda_n v)(x)-(u+\lambda_n v)(y)\big)\big(w(x)-w(y)\big)}{|x-y|^{N+sp}}\right|$$
$$\leq max(1,2^{p-2})\Big(\big|u(x)-u(y)\big|^{p-1}(w(x)-x(y))+c\big|v(x)-v(y)\big|^{p-1}(w(x)-w(y))\Big) \in W^{s,p'}(\Omega).$$

Using dominated convergence yields

$$\lim_{n\to+\infty}\Big\langle(-\Delta)_p^s(u+\lambda_n v),w\Big\rangle=\Big\langle\lim_{n\to+\infty}(-\Delta)_p^s(u+\lambda_n v),w\Big\rangle=\Big\langle(-\Delta)_p^s u,w\Big\rangle,$$

so $(-\Delta)_p^s$ is semi-continuous.

Consequently the problem (3.1) admits unique solution $u \in \mathbb{X}_p^s(\Omega)$ according to Minty-Browder theorem.

CONCLUSION

In this master's dissertation, we are interested in the study of some elliptic problems in the context of fractional Sobolev spaces. We have first briefly presented some definitions and important properties of fractional Sobolev spaces, then we have we have established the existence of the solution for the p-Laplacian fractional problem.

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