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The Locating Chromatic Number of Some Graphs

By

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Dedication

To **my parents** you planted faith in my heart and helped it grow with patience until it shone with light and knowledge.

To **my brothers**, who walked with me through weariness with love.

To **my bestie**, my forever childhood friend, we grew up, the world changed, but you stayed just the same.

To the **rare beloved ones**, who came like rain and nourished my days with their beautiful presence.

To every quiet prayer, every gentle hand that supported me from afar. This is my way of saying: I remember, and I am grateful.

Acknowledgments

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Abstract

A proper coloring of the vertices of a graph $G = (V, E)$ is an assignment of colors to the vertices such that no two adjacent vertices share the same color. The minimum number of colors needed for such a proper coloring is called the chromatic number, denoted by $\chi(G)$.

In this thesis, we focus on the concept of the locating chromatic number in graph theory, which is a relatively recent refinement of classical vertex coloring. The locating coloring is a proper coloring where any two vertices in the same color class are distinguished by their color codes. The minimum number of colors needed for such a proper coloring is called the locating chromatic number, denoted by $\chi_L(G)$.

In this work, we begin by introducing fundamental concepts in graph theory, then study the locating chromatic number by reviewing several existing results related to it. Finally, we present our contributions, providing new results on the locating chromatic number for families of prism graphs namely, the prism itself, the middle graph and the central graph.

Résumé

Une coloration propre des sommets d'un graphe $G = (V, E)$ est une attribution de couleurs aux sommets de telle sorte que deux sommets adjacents ne partagent pas la même couleur. Le nombre minimal de couleurs nécessaires pour une telle coloration est appelé le *nombre chromatique*, noté $\chi(G)$.

Dans ce mémoire, nous nous intéressons au concept de *nombre chromatique de localisation* en théorie des graphes, qui constitue un raffinement relativement récent de la coloration classique des sommets. Une coloration de localisation est une coloration propre dans laquelle toute paire de sommets appartenant à la même classe de couleur peut être distinguée par ses *codes de couleur*. Le nombre minimal de couleurs nécessaires pour une telle coloration est appelé le nombre chromatique de localisation, noté $\chi_L(G)$.

Dans ce travail, nous commençons par introduire les concepts fondamentaux de la théorie des graphes, puis nous étudions le nombre chromatique de localisation en passant en revue plusieurs résultats existants le concernant. Enfin, nous présentons nos propres contributions en apportant de nouveaux résultats sur le nombre chromatique de localisation pour certaines familles de graphes prismes, à savoir : le prisme lui-même, le graphe médian et le graphe central.

Introduction

Graph theory is considered a relatively modern branch of mathematics. It was first introduced in 1735 by the mathematician Leonhard Euler. Since then, this field has developed significantly and has become one of the fundamental pillars of modern mathematics, due to its ability to represent and analyze complex relationships between various entities. Graphs are used as an effective mathematical model in several fields, including computer science, biology, transportation, and network analysis, as they provide a comprehensive framework for understanding the internal structure of complex systems.

Among the fundamental topics in this theory is **graph coloring** according to certain rules and conditions, the most well-known of which is *proper coloring*, which requires that no two adjacent vertices share the same color. The minimum number of colors needed to achieve this is called the **chromatic number**, denoted by $\chi(G)$.

However, this classical concept is sometimes insufficient for fully distinguishing between the vertices of a graph in certain applications. This led to the introduction of a more refined concept known as **locating coloring**. This type of coloring not only satisfies the condition of being proper, but also requires that each vertex has a unique color code computed based on the distances from that vertex to each color class. It is given as follows: $C_\pi(v) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$, where S_i represents the set of vertices assigned the i-th color.

This concept was introduced for the first time by Chartrand and others in 2002. If each vertex receives a unique color code that differs from all others, then the coloring is

said to be a *locating coloring* using k colors. The smallest number of colors needed to achieve such a coloring is called the **locating chromatic number**, denoted by $\chi_L(G)$.

This type of coloring has proven to be effective in several applications, most notably in network localization and fault detection. However, computing $\chi_L(G)$ is considered a challenging problem due to its structural and computational complexity, which makes it a rich subject for research and study.

In this research, we aim to study the locating chromatic number, and we divide this thesis into three chapters:

Chapter One presents the basic concepts of graph theory along with the necessary definitions. It also introduces and defines all the notions and terminology used throughout this thesis.

Chapter Two focuses on the concepts of vertex coloring and locating coloring. It outlines the most important properties and previously established results related to these topics as found in the existing literature.

Chapter Three is the final and most significant chapter, in which we present our original contributions by determining the locating chromatic number $\chi_L(G)$ for certain special graphs. Specifically, the prism graph and its related families namely, the middle prism and central prism. For each of these graphs, we study and establish the corresponding locating chromatic number.

CHAPTER 1

General Notions about Graphs

In this chapter, we present fundamental concepts in graph theory, which form the foundation for understanding our topic. We will give a general overview of the main components of a graph, present different types of graphs, and explain key properties that will aid in analyzing and understanding the subject.

The definitions that are presented in this chapter can be found in more detail in [3],[4],[7],[8] and [12].

1.1 Basic Definitions in Graph Theory

Graphs

A **graph** is a pair $G = (V, E)$, consisting of a non-empty vertex set V and an edge set E. A **vertex** is a fundamental element of a graph, representing a point or node in the structure, and the set of all vertices is denoted by $V(G)$. The **order** of the graph, represented by $|V| = n(G)$, is the number of vertices in the graph. An **edge** is a connection between two vertices, and the set of all edges is denoted by $E(G)$. If an edge connects two vertices v_1 and v_2, it is denoted by $v_1v_2 = e$, where $e \in E$. The **size** of the graph, represented by $|E| = m(G)$, is the number of edges. Two vertices

$v_1, v_2 \in V$ are said to be **adjacent** if they share an edge. Similarly, two edges $e_1, e_2 \in E$ are *adjacent* if they share a common vertex. A vertex v is **incident** to an edge e if v is an endpoint of e. Conversely, e is *incident* to v if it is connected to it.

Example 1.1.1 *Consider the following graph G.*

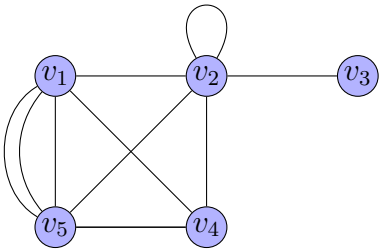


Figure 1.1.1 : The Graph G

- *a **loop** at vertex v_2 (an edge connecting v_2 to itself),*
- *and **multi-edges** between vertices v_1 and v_5 (more than one edge connecting the same pair of vertices).*

Simple Graph

A **simple graph** $G = (V, E)$ is a graph without **loops** (edges that connect a vertex to itself) and without **multi-edges** (more than one edge between the same couple of vertices).

For illustration of these definitions, we give an example in the following Figure

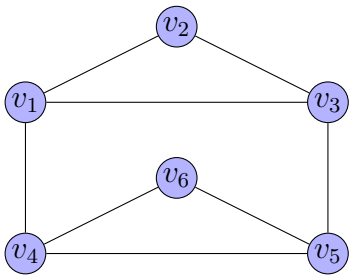


Figure 1.1.2 : Simple Graph

In this dissertation, all the graphs considered are simple and finite.

1.1.1 Partial Graphs and Subgraphs

Partial Graph

A **partial graph** of a graph $G = (V, E)$ is a graph $G' = (V', E')$ where $G' \subseteq G$ such that $V' = V$ and $E' \subseteq E$. That is, the partial graph contains all the vertices of the original graph with fewer edges.

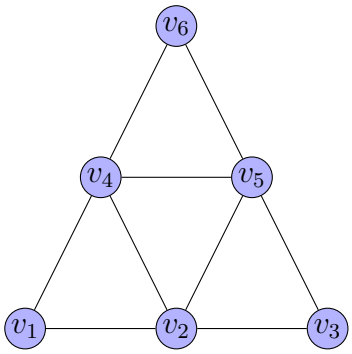


Figure 1.1.3 : Graph G

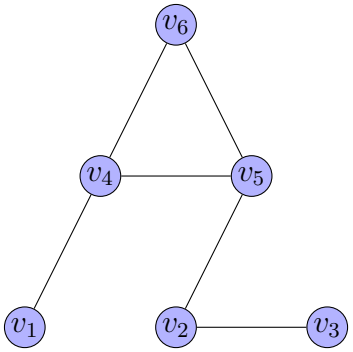


Figure 1.1.4 : Graph G_1 (Partial Graph of G)

Subgraph

A **subgraph** of a graph $G = (V, E)$ is a graph $G' = (V', E')$ where $G' \subseteq G$ such that $V' \subseteq V$ and $E' \subseteq E$. If E' contains an edge $e' \in E$ such that e' have both endpoints in V', we say G' is an **induced subgraph** (that is, each vertex in V' is connected to every vertex in V' by all the edges they share in the original graph).

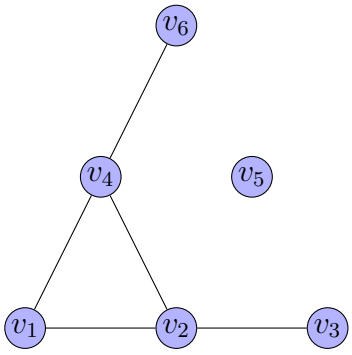


Figure 1.1.5 : Graph G_2 (Subgraph of Graph G)

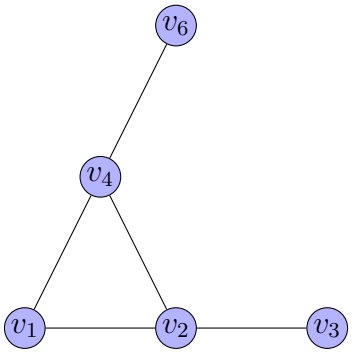


Figure 1.1.6 : Graph G_3 (Induced Subgraph of Graph G)

1.1.2 Neighborhood and Degree

- The **neighborhood** of a vertex v in a graph G, denoted by $N(v)$, is the set of all vertices adjacent to v; that is, there exists an edge $e \in E$ such that e connects v and $v' \in V$.

- The **open neighborhood** of v is defined as : $N(v) = \{u \in V \mid \{u, v\} \in E\}$.

- The **closed neighborhood** of v, denoted by $N[v]$, includes the vertex v itself in addition to its adjacent vertices : $N[v] = N(v) \cup \{v\}$.

- The **degree** of a vertex v, denoted $d(v)$, is the number of edges incident to v:

$$d(v) = |N(v)|$$

- An **isolated vertex** is a vertex v with no incident edges : $d(v) = 0$.

- A **pendant vertex** is a vertex v with exactly one incident edge : $d(v) = 1$.

Example 1.1.2 *As shown in (Figure 1.1.2):*

$$\begin{cases} N(v_1) = \{v_2, v_3, v_4\} \\ N(v_5) = \{v_3, v_4, v_6\} \end{cases}$$

$$\begin{cases} N[v_1] = \{v_1, v_2, v_3, v_4\} \\ N[v_5] = \{v_5, v_3, v_4, v_6\} \end{cases}$$

1.1.3 Distance

Let $G = (V, E)$ be a graph and u, v two vertices of G, then

- The **distance** between two vertices u and v in G denoted $d(u, v)$, is the length of a *shortest path* connecting them.

- For a subset S of V, the distance between u and the set S is given by

$$d(u, S) = \min\{d(u, x) : x \in S\}$$

Example 1.1.3 *As shown in (Figure 1.1.2):*

$$\begin{cases} d(v_1, v_2) = d(v_1, v_3) = d(v_1, v_4) = d(v_2, v_3) = d(v_3, v_5) = d(v_4, v_5) = d(v_4, v_6) = d(v_5, v_6) = 1 \\ d(v_1, v_5) = d(v_1, v_6) = d(v_2, v_4) = d(v_2, v_5) = d(v_3, v_4) = d(v_3, v_6) = 2 \\ d(v_2, v_6) = 3 \end{cases}$$

1.1.4 Some Graph Parameters

Stable

A **stable** set of a graph G is an induced subgraph of G such that $E' = \emptyset$, so no edges are in the set (that is for all $v, v' \in S$, v and v' are not adjacent). The order of the largest stable set of G is denoted by $\alpha(G)$ and is called the **stability number**.

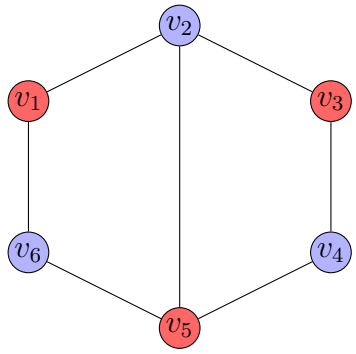


Figure 1.1.7 : Graph G with a Stable Set $S = \{v_1, v_3, v_5\}$

Clique

A **clique** in a graph G is an induced complete subgraph of G such that $|E'| = n'(n'-1)/2$ or for all $v, v' \in C$, v and v' are adjacent (i.e., every pair of vertices in the subgraph is connected by an edge, which makes it a complete subgraph). The order of the largest clique in G is denoted by $\omega(G)$ and is called the **clique number**.

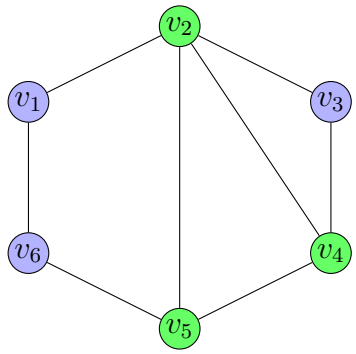


Figure 1.1.8 : Graph G with a Clique $C = \{v_2, v_4, v_5\}$

1.2 Fundamental Types of Graphs

In this section, we review the definitions of several well-known specific graphs.

Path

A **path** is a non-empty graph $P = (V, E)$, denoted by P_n, such as is a sequence of distinct vertices between v_1 and v_n, i.e., $P = (v_1, \ldots, v_n)$, such that

$$V = \{v_1, \ldots, v_n\} \quad \text{and} \quad E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}.$$

such that each edge $(v_i, v_{i+1}) \in E$ for all $1 \leq i < n$, and no vertex is repeated.

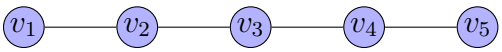


Figure 1.2.1 : Path Graph P_5

Cycle

A **cycle** is a closed path denoted by C_n, defined as a sequence of $n \geq 3$ vertices:

$$C = (v_1, v_2, \ldots, v_n, v_1)$$

such that $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$, where each edge $(v_i, v_{i+1}) \in E$ for all $1 \leq i < n$, and no vertex is repeated except for the first and last vertex v_1.

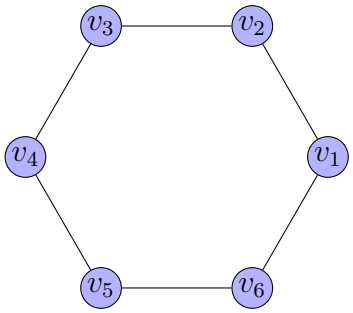
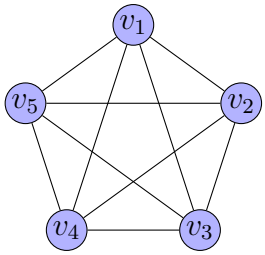


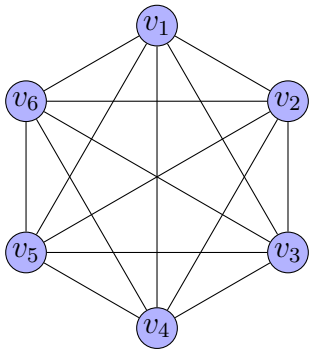
Figure 1.2.2 : Cycle Graph C_6

Complete Graph

A **complete graph** $K = (V, E)$, denoted by K_n, is a graph such that every pair of distinct vertices in V is connected by an edge, i.e., $\forall v, v' \in V, v \neq v' \Rightarrow v$ and v' are adjacent. This graph has n vertices and $m = \dfrac{n(n-1)}{2}$ edges.



(a) Complete Graph K_5



(b) Complete Graph K_6

Figure 1.2.3 : Complete Graphs K_5 and K_6 with vertices labeled as $v_1, v_2, \ldots$

The **complement** of a graph $G = (V, E)$ is a graph $\overline{G} = (V, \overline{E})$ such that G and $\overline{G}$ have the same set of vertices V, and for every pair of distinct vertices $v, v' \in V$:

$$(v, v') \in \overline{E} \quad \text{if and only if} \quad (v, v') \notin E$$

i.e., $\overline{E} = \{(v, v') \in V \times V \mid v \neq v' \text{ and } (v, v') \notin E\}$

Property: The union of a graph and its complement forms a complete graph:

$$E \cup \overline{E} = E(K_n)$$

where K_n is the complete graph on n vertices.

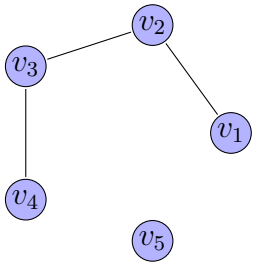


Figure 1.2.4 : Graph G

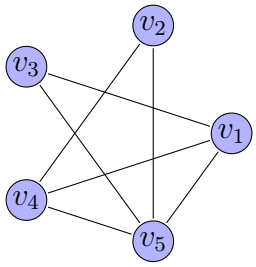


Figure 1.2.5 : Complement Graph $\overline{G}$

Bipartite Graph

A graph $G = (V, E)$ is called **bipartite** if there exists a partition of the vertex set V into two disjoint subsets S_1 and S_2 such that:

$$E \subseteq \{(u, v) \mid u \in S_1, v \in S_2\}$$

That is, no edges exist between any two vertices within the same subset.

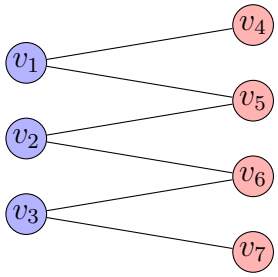


Figure 1.2.6 : Incomplete Bipartite Graph

Complete Bipartite Graph

A graph $K_{p,q}$ is called **complete bipartite** if every vertex in S_1 is connected to every vertex in S_2, meaning:

$$E = \{(u, v) \mid u \in S_1, v \in S_2\}$$

where $|S_1| = p$ and $|S_2| = q$

$$\text{such that:} |V_{K_{p,q}}| = p + q, |E_{K_{p,q}}| = pq.$$

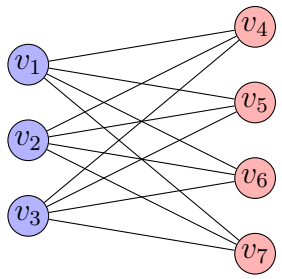


Figure 1.2.7 : Complete Bipartite Graph $K_{3,4}$

Tree

A **tree** is a connected, acyclic graph. Formally, a tree is a graph $T = (X, E)$ that satisfies the following equivalent conditions:

- T is a connected graph with no cycles.
- T is an acyclic graph where any two vertices are connected by exactly one simple path.
- T is a connected graph with $|E| = |V| - 1$.

If a tree has n vertices, it has exactly $n - 1$ edges.

A **binary tree** is defined as a tree in which there is exactly one vertex of degree two, namely the root vertex x_0, and each of the remaining vertices is of degree one or three.

A **complete binary tree** is a binary tree in which all leaves are on the same level, or equivalently, all leaves have the same distance from the root vertex x_0. We denote a complete binary tree with diameter $2k$ by $BT(k)$, where $k \geq 1$.

A **non-complete binary tree** is a binary tree in which some of the leaves are located at different distances from the root vertex x_0. We denote a non-complete binary tree with diameter $2k$ by $BTnc(k)$, where $k \geq 1$.

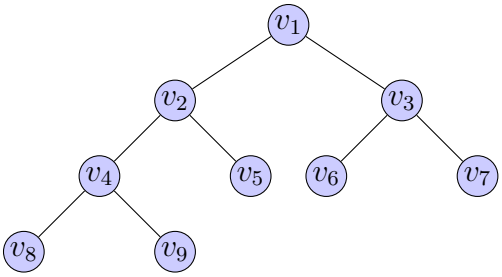


Figure 1.2.8 : Tree Graph

Regular Graph

A **regular graph** is a simple graph in which every vertex has the same degree r. Formally, a graph $G = (V, E)$ is **r-regular** if:

$$\forall v \in V, \quad d(v) = r$$

A **3-regular graph** is called a **cubic graph**.

The complete graph K_n is a regular graph of degree $n - 1$.

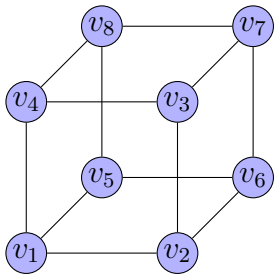
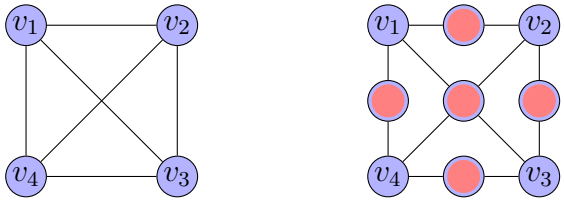


Figure 1.2.9 : Cube Graph - A 3-Regular Graph

1.3 Graph Operations

Subdivision

A **subdivision** of a graph $G = (V, E)$ is a graph obtained by adding new vertices along the edges of G. These newly added vertices each have a degree of 2, ensuring that the graph maintains its connectivity without introducing cycles.



(a) Original Graph (b) Subdivision

Figure 1.3.1 : Subdivision of Graph

Middle Graph:

The **middle graph** of a graph G, denoted by $M(G)$, is defined as follows: $M(G) = (V_{M(G)}, E_{M(G)})$, where each edge of the original graph G is subdivided by a new vertex. Then : $V_{M(G)} = V(G) \cup E(G) = \{v_i \mid 1 \leq i \leq n\} \cup \{v'_i \mid 1 \leq i \leq m\}$ Here, each v'_i represents the edge e_i of the original graph G.

Two vertices in $M(G)$ are adjacent if and only if one of the following holds:

- They correspond to two adjacent edges in G.
- One corresponds to a vertex $v \in V(G)$ and the other to an edge $e \in E(G)$ incident to v in G.

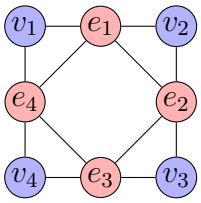


Figure 1.3.2 : Middle Graph of C_4

Central Graph

The **central graph** of a graph G, denoted by $C(G)$, is defined as follows : $C(G) = (V_{C(G)}, E_{C(G)})$, where each edge of the original graph G is subdivided by a new vertex. Then : $V_{C(G)} = V(G) \cup E(G) = \{v_i \mid 1 \leq i \leq n\} \cup \{v'_i \mid 1 \leq i \leq m\}$ Here, each v'_i represents the edge e_i of the original graph G. Two vertices in $C(G)$ are adjacent if and only if one of the following holds:

- They correspond to two non-adjacent vertices in G.
- One corresponds to a vertex $v \in V(G)$ and the other to an edge $e \in E(G)$ incident to v.

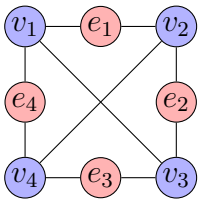


Figure 1.3.3 : Central Graph of C_4

Power Graph

A **power graph** of a simple graph G, denoted by G^k, is defined as the graph that has the same vertex set as G, and where two distinct vertices are adjacent in G^k if and only if their distance in G is at most k. Formally: $E(G^k) = \{uv : 1 \leq d_G(u, v) \leq k\}$. where $d_G(u, v)$ denotes the distance between vertices u and v in the original graph G.

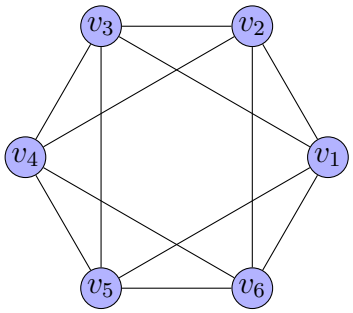
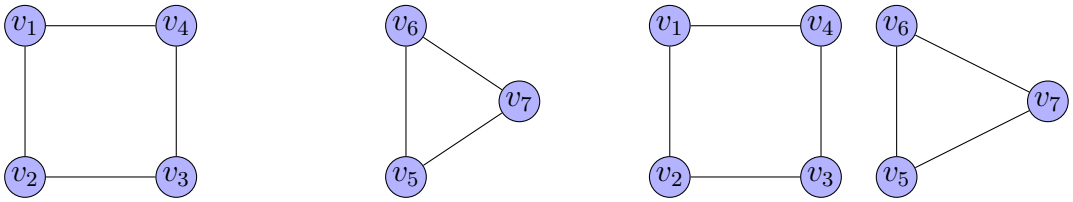


Figure 1.3.4 : C_6^2 Power Graph of C_6

1.4 Union, Join Graphs and Cartesian Product

Union

The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.



(a) Graph C_4 (b) Graph C_3 (c) Union $C_4 \cup C_3$

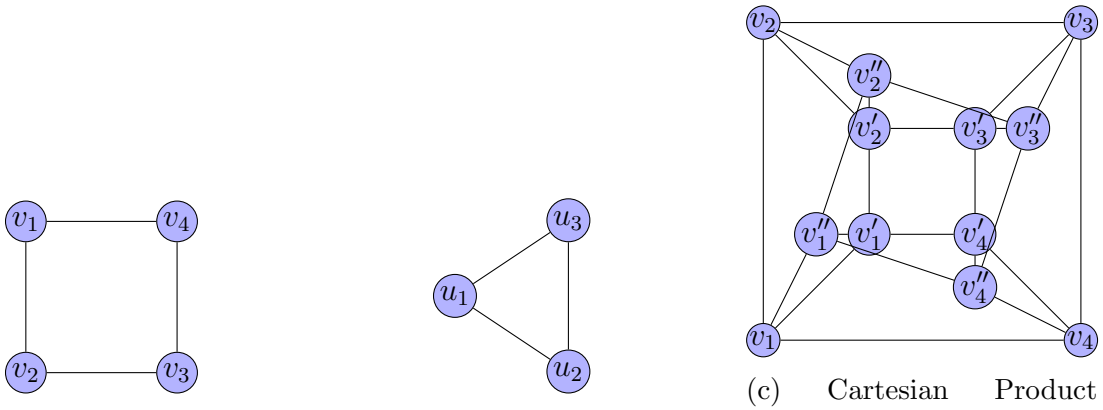
Figure 1.4.1 : Union of two graphs C_4 and C_3

Cartesian Product

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. The **Cartesian product** of G and H, denoted by $G \square H$, is the graph defined by:

$$V_{G \square H} = V_G \times V_H,$$

$$E_{G \square H} = (E_G \times V_H) \cup (V_G \times E_H).$$

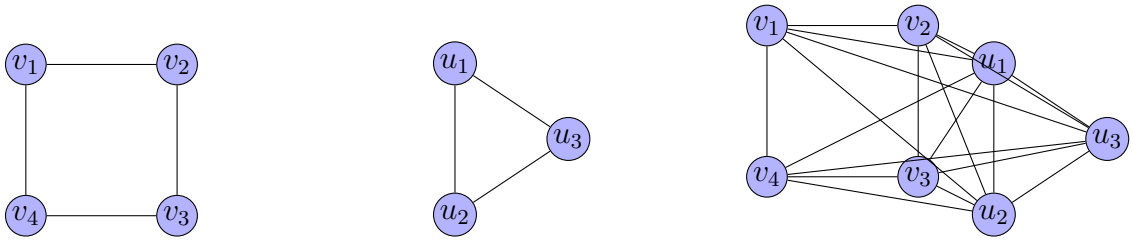


(a) Graph C_4 (b) Graph C_3 (c) Cartesian Product $C_4 \square C_3$

Figure 1.4.2 : Cartesian Product of Graphs C_4 and C_3

Join Graphs

The **join** $G + H$ of the graphs G and H is obtained from the graph union $G \cup H$ by adding an edge between each vertex of G and each vertex of H.



(a) Graph C_4 (b) Graph C_3 (c) Join $C_4 + C_3$

Figure 1.4.3 : Join of the graphs C_4 and C_3

CHAPTER 2

Chromatic Number and Locating Chromatic Number

The locating coloring of a graph is an important extension of the traditional graph coloring concept. It not only requires that adjacent vertices have different colors but also takes into account the distances between each vertex and the color classes in the graph. This makes locating coloring especially useful in practical applications that require precise identification and differentiation of vertices, such as in networks and cryptography. In this chapter, we present the fundamental concepts of locating coloring and explain how it differs from traditional coloring. We also provide several examples and key results related to this concept.

2.1 Graph Coloring

Graph coloring is one of the most interesting topics in graph theory due to its rich theoretical results and numerous practical applications. The first problem in graph coloring emerged in 1852 is the famous Four Color Problem posed by Francis Guthrie, who asked: **" Is it possible to color the countries of a geographical map using**

only four colors such that no two countries sharing a common boundary have the same color? "

This problem remained unsolved for more than a century and was eventually proven in 1976 by two American researchers, Kenneth Appel and Wolfgang Haken, with the assistance of a computer.

Among the different types of graph coloring, vertex coloring is the most well-known and widely studied. It involves finding the minimum number of colors needed to color the vertices of a simple graph so that no two adjacent vertices share the same color.

The deffnitions that are presented in this section can be found in([6] [9],[10] , [11] and [12])

Definition 2.1.1 *A **vertex coloring** of a graph G means assigning a color to each vertex of G.*

Definition 2.1.2 *A **proper vertex coloring** of a graph G can be considered as a function $c : V(G) \to \mathbb{N}$(where $\mathbb{N}$ is the set of positive integers) such that $c(u) \neq c(v)$ if u and v are adjacent in G.*

A k-coloring of a graph G is a vertex coloring of G using k colors. A graph G is said to be k-colorable if it can be properly colored using k colors. If V_i, $(1 \leq i \leq k)$ is the set of vertices in G colored by the color i, then each non-empty set V_i is called a **color class** and the non-empty elements of $V_1, V_2, \ldots, V_k$ produce a partition of $V(G)$. Note that the adjacent vertices of G have different colors by c. So, each non-empty color class V_i, $(1 \leq i \leq k)$ is an independent set of $V(G)$.

Definition 2.1.3 *The minimum positive integer k for which G is k-colorable is called the **chromatic number** of G and is denoted by $\chi(G)$. In other words, the chromatic number of G can be considered as the cardinality of a minimum partition of V into independent sets.*

Example 2.1.1 *Proper Coloring of Cycle*

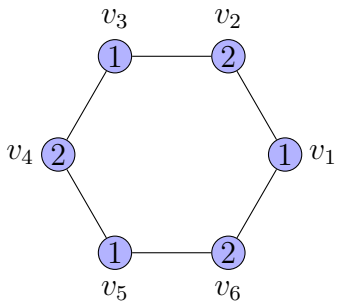


Figure 2.1.1 : Proper coloring of C_6 , $\chi(C_6) = 2$

2.2 Some Results for the Chromatic Number

Theorem 2.2.1 *According to the previous results, let G be a graph of order n. Then, the chromatic number $\chi(G)$ satisfies*

$$1 \leq \chi(G) \leq n.$$

Theorem 2.2.2 ([13]) *If $H \subseteq G$, then the chromatic number satisfies*

$$\chi(H) \leq \chi(G).$$

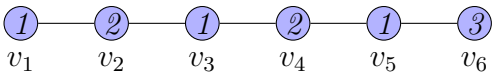
2.3 Locating Chromatic Number

In this section, we focus on the definition of the locating-chromatic number and review some known results related to this parameter.

Let G be a connected graph and c be a proper k-coloring of G. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be an ordered partition of $V(G)$ into the resulting color classes S_i where $i = 1, \ldots, k$.

Definition 2.3.1 *The **color code** of a vertex v with respect to Π is defined as the ordered k-tuple $c_\Pi(v) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$ where $d(v, S_i)$ is the distance between the vertex v and the color class S_i.*

Example 2.3.1 *Coloring and color code of Path P_6*



We obtain a partition $\pi = \{S_1, S_2, S_3\}$ of $V(P_6)$, where $S_1 = \{v_1, v_3, v_5\}$, $S_2 = \{v_2, v_4\}$, and $S_3 = \{v_6\}$

The color code of each vertex is: $c_\Pi(v_1) = (d(v_1, S_1) = 0, d(v_1, S_2) = 1, d(v_1, S_3) = 5) = (0, 1, 5)$, $c_\Pi(v_3) = (0, 1, 3)$, $c_\Pi(v_5) = (0, 1, 1)$, $c_\Pi(v_2) = (1, 0, 4)$, $c_\Pi(v_4) = (1, 0, 2)$, $c_\Pi(v_6) = (1, 2, 0)$. Observe that distinct vertices of G have distinct color codes.

Definition 2.3.2 *A **locating coloring** of a graph G is a coloring in which each vertex has a distinct color code from the other vertices. This means that the color code of each vertex v is unique when compared to all other vertices in the graph.*

Definition 2.3.3 *The **locating chromatic number** of a graph G, denoted by $\chi_L(G)$, is the minimum number of colors that allows a locating coloring which distinguishes all the vertices in the graph using these colors.*

Example 2.3.2 *The locating chromatic number of this graph*

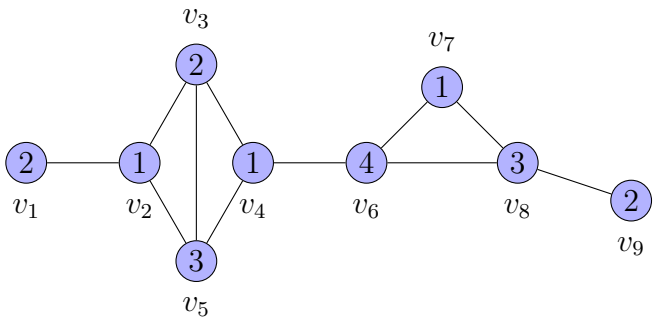


Figure 2.3.1 : Proper locating coloring of G , $\chi_L(G) = 4$

We obtain a partition $S = \{S_1, S_2, S_3, S_4\}$ of $V(G)$ where $S_1 = \{v_2, v_4, v_7\}, S_2 = \{v_1, v_3, v_9\}, S_3 = \{v_5, v_8\}, S_4 = \{v_6\}$, The color code of each vertex is $c_\pi(v_1) = (1, 0, 2, 4)$, $c_\pi(v_2) = (0, 1, 1, 3)$, $c_\pi(v_3) = (1, 0, 1, 2)$, $c_\pi(v_4) = (0, 1, 1, 1)$,$c_\pi(v_5) = (1, 1, 0, 2)$,$c_\pi(v_6) = (1, 2, 1, 0)$,$c_\pi(v_7) = (0, 2, 1, 1)$,$c_\pi(v_8) = (1, 1, 0, 1)$,$c_\pi(v_9) = (2, 0, 1, 2)$. The distinct vertex of G have distinct color codes.

2.4 Some Results for Locating Chromatic Number

The determination of the locating chromatic number of a graph is a complex topic, primarily because no general formula exists for its computation. Despite this complexity, several researchers have investigated this parameter and obtained interesting results for certain families of graphs. In this section, we present some of these results in the following theorems.

Theorem 2.4.1 ([5]) *Let G be a connected graph of order n. Then, the locating chromatic number $\chi_L(G)$ satisfies*

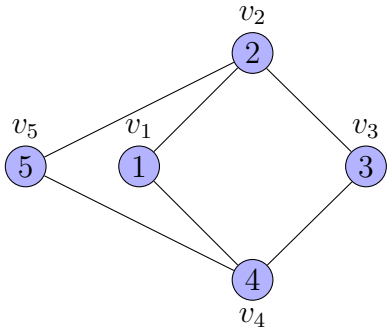
$$\chi(G) \leq \chi_L(G) \leq n.$$

Theorem 2.4.2 ([5]) *The locating chromatic number of a complete graph K_n is $\chi_L(K_n) = n$*

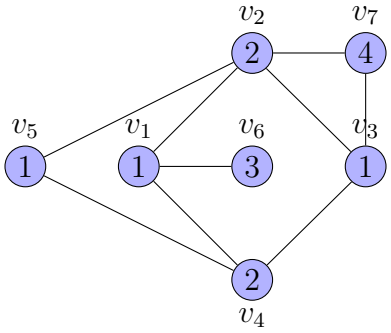
Remark 2.4.1 ([5]) *For any graph G, $\omega(G) \leq \chi_L(G)$.*

Remark 2.4.2 *The locating chromatic number does not necessarily behave in a predictable way under subgraph inclusion. In particular, there are cases where a subgraph H of a graph G satisfies $H \subseteq G$ and yet $\chi_L(H) > \chi_L(G)$.*

Example 2.4.1 *This example shows this clearly.*



(a) Graph H with $\chi_L(H) = 5$



(b) Graph G with $\chi_L(G) = 4$

Theorem 2.4.3 ([7]) *The locating chromatic number of a path graph P_n where $n \geq 3$ is $\chi_L(P_n) = 3$*

Theorem 2.4.4 ([7]) *The locating chromatic number of a cycle graph C_n where $n \geq 3$ is*

$$\chi_L(C_n) = \begin{cases} 3 & \text{if } n \text{ is odd,} \\ 4 & \text{if } n \text{ is even.} \end{cases}$$

Theorem 2.4.5 ([12]) *Let $BT(k)$ be a complete binary tree of height k, where $k = 3, 4,$ or 5. Then the locating chromatic number of $BT(k)$ is $\chi_L(BT(k)) = 4$.*

Theorem 2.4.6 ([7]) *Let P_n be the path of order n and P_n^m be the m-th power of P_n. Then:*

- $\chi_L(P_n^m) = n$ *when* $m = n - 1$.
- $\chi_L(P_n^m) = m + 2$ *when* $m < n - 1$.

Theorem 2.4.7 ([7]) *Let C_n be the path of order n and C_n^m be the m-th power of C_n. Then:*

- $\chi_L(C_n^m) = n$ *when* $n \leq 2m + 2$.
- $\chi_L(C_n^m) \geq m + 3$ *when* $n \geq 2m + 3$.

- $m+3 \leq \chi_L(C_n^m) \leq m+t+2 \qquad$ *when* $m = 2t$ *or* $m = 2t+1$.
- $\chi_L(C_n^m) = m+t+2 \qquad m = 2t$ *and* $n = 2(m+1)+m$.

Theorem 2.4.8 ([2]) *Let P_m and P_n be two path graphs with $n \geq m \geq 2$. Then the locating chromatic number of their Cartesian product satisfies:*

$$\chi_L(P_m \square P_n) = 4.$$

Example 2.4.2 *Proper locating coloring of cartesian product of two Path*

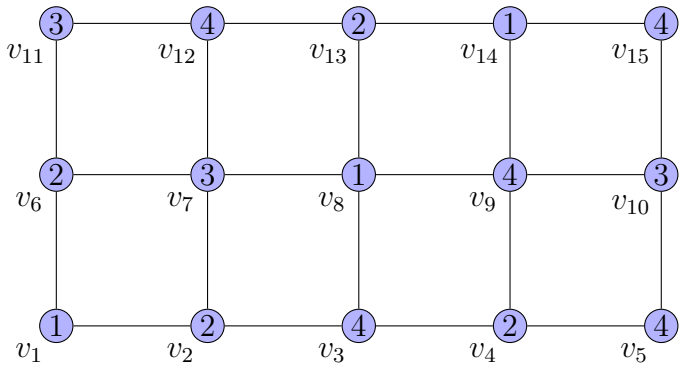


Figure 2.4.2 : Proper locating coloring of $P_5 \square P_3$, $\chi(P_5 \square P_3) = 4$

Theorem 2.4.9 ([2]) *Let $m \geq 3$ and $n \geq 2$ be two positive integers. Then the locating chromatic number of the Cartesian product $K_m \square P_n$ is given by*

$$\chi_L(K_m \square P_n) = \begin{cases} m+2 & \text{if } m \leq n-2, \\ m+1 & \text{if } m \geq n-1. \end{cases}$$

Example 2.4.3 *Proper locating coloring of cartesian product*

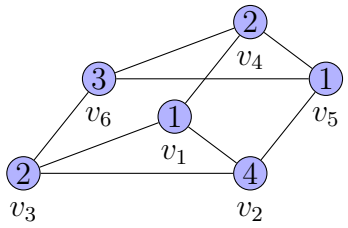


Figure 2.4.3 : Proper locating coloring of $K_3 \square P_2$, $\chi(K_3 \square P_2) = 4$

Theorem 2.4.10 ([2]) *Let $m \geq 2$ and $n \geq 3$ be two positive integers such that $m \leq n$. Define $m_0 := \max\{k \in \mathbb{N} \,|\, k(k-1) - 1 \leq n\}$. Then*

$$\chi_L(K_m \square K_n) = \begin{cases} n+1 & \text{if } m \leq m_0 - 1, \\ n+2 & \text{if } m_0 + 1 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor. \end{cases}$$

CHAPTER 3

Locating Chromatic Number of Prism Graph Families

The main contribution of this chapter is the determination of the locating-chromatic number for prism graphs families. This chapter is organized into four sections, each focusing on the locating-chromatic number of a different graph associated with prism graphs: the prism graph itself and its central, middle, and total graph[1].

3.1 The Locating Chromatic Number of the Prism Graph

This section is devoted to determine the locating-chromatic number of the prism graph Y_n.

The Cartesian product of a cycle C_n and a path P_2 is said to be the prism graph and is denoted by Y_n. Denote the vertices of the first cycle by $\{v_i,\ i = 1, 2, \ldots, n\}$ and the vertices of the second cycle by $\{u_i,\ i = 1, 2, \ldots, n\}$.

Example 3.1.1 *The prism graph Y_8 is the following graph.*

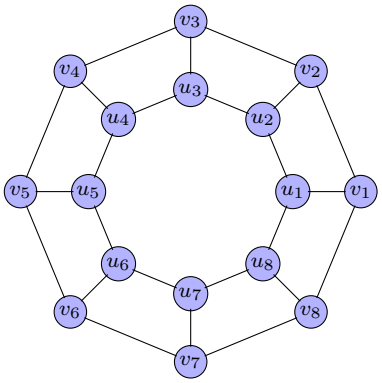


Figure 3.1.1 : Prism Graphs Y_8

Theorem 3.1.1 *The locating chromatic number of the prism graph Y_n , where $n \geq 3$, is given by $\chi_L(Y_n) = 4$, except when $n = 4$, in which case $\chi_L(Y_4) = 5$.*

Proof. Since Y_n contains C_4, it follows that $\chi_L(Y_n) \geq 3$. For a contradiction, assume c be a locating 3-coloring of Y_n. Observe that at least 4-cycle must be colored using 3 colors. In such a case, two vertices on the cycle must share the same color and, as well as the same color code, a contradiction. Therefore $\chi_L(Y_n) > 3$.

To construct the upper bound, let c be a vertex coloring of the prism graph using 4 colors as follows:

Case 1: If n is odd

$$c(v_i) = \begin{cases} 1 & \text{if } i \text{ is odd,} \\ 2 & \text{if } i \text{ is even,} \\ 4 & \text{if } i = n. \end{cases}$$

$$c(u_i) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 2 & \text{if } i \text{ is odd,} \\ 3 & \text{if } i = n. \end{cases}$$

Case 2: If n is even

We consider the following two cases:

if k=(n/2) odd

$$c(v_i) = \begin{cases} 1 & \text{if } i \in \{1, \dots, k-2\}, \text{ where } i \quad \text{is odd} \\ 2 & \text{if } i \in \{2, \dots, k-1\}, \text{ where } i \quad \text{is even} \\ 3 & \text{if } i \in \{k, \dots, n-1\}, \text{ where } i \quad \text{is odd} \\ 4 & \text{if } i \in \{k, \dots, n\}, \text{ where } i \quad \text{is even} \end{cases}$$

$$c(u_i) = \begin{cases} 1 & \text{if } i \in \{2, \dots, k-3\}, \text{ where } i \quad \text{is even and } i = k \\ 2 & \text{if } i \in \{3, \dots, k-2\}, \text{ where } i \quad \text{is odd and } i = n \\ 3 & \text{if } i \in \{k+1, \dots, n-2\}, \text{ where } i \quad \text{is even and } i = 1 \\ 4 & \text{if } i \in \{k+2, \dots, n-1\}, \text{ where } i \quad \text{is odd and } i = k-1 \end{cases}$$

if k=(n/2) even

$$c(v_i) = \begin{cases} 1 & \text{if } i \in \{1, \dots, k-1\}, \text{ where } i \text{ is odd} \\ 2 & \text{if } i \in \{2, \dots, k\}, \text{ where } i \text{ is even} \\ 3 & \text{if } i \in \{k+1, \dots, n-1\}, \text{ where } i \text{ is odd} \\ 4 & \text{if } i \in \{k+2, \dots, n\}, \text{ where } i \text{ is even} \end{cases}$$

$$c(u_i) = \begin{cases} 1 & \text{if } i \in \{2, \dots, k-2\}, \text{ where } i \quad \text{is even or } i = k+1 \\ 2 & \text{if } i \in \{3, \dots, k-1\}, \text{ where } i \quad \text{is odd or } i = n \\ 3 & \text{if } i \in \{k+2, \dots, n-2\}, \text{ where } i \quad \text{is even or } i = 1 \\ 4 & \text{if } i \in \{k+3, \dots, n-1\}, \text{ where } i \quad \text{is odd or } i = k \end{cases}$$

We obtain a partition $\pi = \{S_1, S_2, S_3, S_4\}$ of $V(Y_n)$ as follows, when if n is odd:

$S_1 = \{v_1, v_3, \dots\} \cup \{u_2, u_4, \dots\}$, $S_2 = \{u_1, u_3, \dots\} \cup \{v_2, v_4, \dots\}$, $S_3 = \{u_n\}$, $S_4 = \{v_n\}$

Now, observe that for any distinct pair of vertices x, y of $V(Y_n)$, if $d(x, v_n) = d(y, v_n)$ or $d(x, u_n) = d(y, u_n)$, then $c(x) \neq c(y)$ since x and y belong to different color classes.

Consequently, $d(x, S_3) \neq d(y, S_3)$ and $d(x, S_4) \neq d(y, S_4)$ This implies that even if x and y belong to the same color class, their color codes differ. Hence, this coloring c is indeed a locating 4-coloring of Y_n.

We also obtain a partition $\pi = \{S_1, S_2, S_3, S_4\}$ of $V(Y_n)$ for the case when n is even. Depending on the parity of $k = n/2$, we define the sets as follows:

- If k is odd, then:

  $S_1 = \{v_1, v_3, \ldots, v_{k-2}\} \cup \{u_2, u_4, \ldots, u_{k-3}\} \cup \{u_k\}$, $S_2\{v_2, v_4, \ldots, v_{k-1}\} \cup \{u_3, u_5, \ldots, u_{k-2}\} \cup \{u_n\}$, $S_3 = \{v_k, v_{k+2}, \ldots, v_{n-1}\} \cup \{u_{k+1}, u_{k+3}, \ldots, u_{n-2}\} \cup \{u_1\}$, $S_4 = \{v_{k+1}, v_{k+3}, \ldots, v_n\} \cup \{u_{k+2}, u_{k+4}, \ldots, u_{n-1}\} \cup \{u_{k-1}\}$

- If k is even, then:

  $S_1 = \{v_1, v_3, \ldots, v_{k-1}\} \cup \{u_2, u_4, \ldots, u_{k-2}\} \cup \{u_{k+1}\}$, $S_2 = \{v_2, v_4, \ldots, v_k\} \cup \{u_3, u_5, \ldots, u_{k-1}\} \cup \{u_n\}$, $S_3 = \{v_{k+1}, v_{k+3}, \ldots, v_{n-1}\} \cup \{u_{k+2}, u_{k+4}, \ldots, u_{n-2}\} \cup \{u_1\}$, $S_4 = \{v_{k+2}, v_{k+4}, \ldots, v_n\} \cup \{u_{k+3}, u_{k+5}, \ldots, u_{n-1}\} \cup \{u_k\}$.

Now, observe that there exist two 4-cycles in Y_n that are colored using four distinct colors. Moreover, for every pair of vertices $x, y \in V(Y_n)$ such that $i \geq k$ and $c(x) = c(y) \in \{3, 4\}$, if $d(x, S_1) = d(y, S_1)$, then it must be that $d(x, S_2) \neq d(y, S_2)$, similarly if $d(x, S_2) = d(y, S_2)$, then $d(x, S_1) \neq d(y, S_1)$. In the same way, for any pair of vertices $x, y \in V(Y_n)$ with $i < k$ and $c(x) = c(y) \in \{1, 2\}$, if $d(x, S_3) = d(y, S_3)$, then $d(x, S_4) \neq d(y, S_4)$, and if $d(x, S_4) = d(y, S_4)$, then $d(x, S_3) \neq d(y, S_3)$.

As a consequence, the coloring c can be considered a locating 4-coloring of the prism graph. ■

3.2 The Locating Chromatic Number of the Central Graph of Prism Graph

This section is devoted to determining the locating-chromatic number of the central graph of the prism graph, $C(Y_n)$

Let Y_n be the prism graph with vertex set $V(Y_n) = \{v_i, u_i \mid 1 \leq i \leq n\}$, and edge set $E(Y_n) = \{v_iv_{i+1},\ u_iu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{v_nv_1,\ u_nu_1\} \cup \{v_iu_i \mid 1 \leq i \leq n\}$.

The **central graph** of Y_n, denoted by $C(Y_n)$, is given by subdividing each edge of Y_n only once, Then connecting all the middle vertices (representing subdivided edges) to their incident original vertices in Y_n, and any two original vertices that are not adjacent in Y_n are also connected . Thus, the vertex set of $C(Y_n)$is : $V(C(Y_n)) = V(Y_n) \cup \{v'_i, u'_i, v''_i \mid 1 \leq i \leq n\}$ where:

- v'_i denotes the middle vertex on edge v_iv_{i+1} for $1 \leq i \leq n-1$, and v'_n on edge v_nv_1.
- u'_i denotes the middle vertex on edge u_iu_{i+1} for $1 \leq i \leq n-1$, and u'_n on edge u_nu_1.
- v''_i denotes the middle vertex on edge v_iu_i for all $1 \leq i \leq n$.

Example 3.2.1 *Central graph of prism graph for n=4*

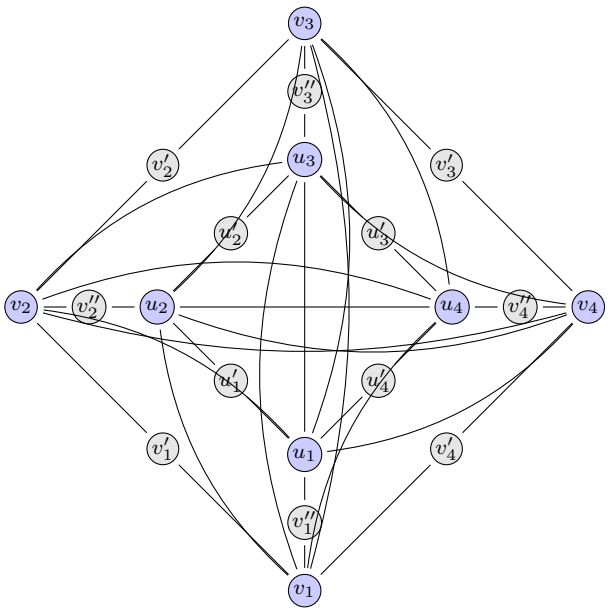


Figure 3.2.1 : Central Graph of Prism Graphs $C(Y_4)$

Observation :

For even n, there exists an edge between every vertex v_i such that i is even. Similarly, there exists an edge between every vertex u_j such that j is odd. Moreover, there is an edge between every vertex v_i and every vertex u_j. Therefore, we obtain the join of two cliques of size $n/2$, resulting in an n-clique.
If we repeat the same construction using vertices v_i with odd indices and u_j with even indices, we obtain another n-clique.

For odd n, repeating the same process leads to the construction of at least cliques of size $n-1$, i.e. two $(n-1)$-cliques.

Theorem 3.2.1 *The locating chromatic number of the central graph of prism graph $C(Y_n)$, where $n \geq 3$, is given by*

$$\chi_L(C(Y_n)) = n + 1$$

Proof. For n even, since there exist two distinct n-clique in $C(Y_n)$, there must exist vertices $x, y \in C(Y_n)$ such that $c(x) = c(y)$. Consequently we get $c_\pi(x) = c_\pi(y)$, meaning x and y share the same color code. Therefore $\chi_L(C(Y_n)) > n$.

For odd n, Since the graph contains at least two distinct $(n-1)$-cliques, we can apply the same argument as in the even case to conclude that $\chi_L(C(Y_n)) > n-1$.

Now, suppose $\chi_L(C(Y_n)) = n$, and c be a locating n-coloring of $C(Y_n)$. Let S_i, for $i \in \{1, \ldots, n\}$ be the color classes induced by c. Since $|V(Y_n)| = 2n$, there must be at least two vertices of Y_n that receive the same color under c.

Now, let x, y be two distinct vertices of the prism graph with $c(x) = c(y) = k$ where $k \in \{1, \ldots, n\}$. Then, for every $i \in \{1, \ldots, n\} \backslash k$, we have $d(x, S_i) = d(y, S_i) = 1$, since x and y are both adjacent to all vertices in each color class. Therefore, their color codes satisfy $c_\pi(x) = c_\pi(y)$, which contradicts the assumption that c is a locating coloring. Therefore, $\chi_L(C(Y_n)) > n$.

Now, we construct a locating coloring $c : V(C(Y_n)) \to \{1, \ldots, n\}$ of the central graph of prism graph as follows

$c(v_i) = c(u_i) = i \quad \text{for} \quad i \in \{1, \ldots, n\}$,

$c(v_i'') = i - 2 \quad \text{for} \quad i \in \{3, \ldots, n+2\}$,

$c(v_i') = n + 1 \quad \text{for} \quad i \in \{1, \ldots, n\}$,

$c(u_i') = i - 2 \quad \text{for} \quad i \in \{3, \ldots, n+2\}$.

By using the coloring c, we obtain the following color codes of

$$c_\pi(v_i) = \begin{cases} 0 \quad \text{for} \quad i^{th} \quad \text{component} \quad i \in \{1, ..., n\} \\ 1 \quad \text{Otherwise} \end{cases}$$

$$c_\pi(u_i) = \begin{cases} 0 \quad \text{for} \quad i^{th} \quad \text{component} \quad i \in \{1, ..., n\} \\ 2 \quad \text{for} \quad (n+1)^{th} \quad \text{component} \\ 1 \quad \text{Otherwise} \end{cases}$$

$$c_\pi(v_i'') = \begin{cases} 0 \quad \text{for} \quad j^{th} \quad \text{component} \quad j \equiv i + 2(\mod n) \\ 1 \quad \text{for} \quad i^{th} \quad \text{component} \\ 2 \quad \text{Otherwise} \end{cases}$$

$$c_\pi(v_i') = \begin{cases} 0 \quad \text{for} \quad (n+1)^{th} \quad \text{component} \\ 1 \quad \text{for} \quad j^{th} \quad \text{component} \quad j \equiv i \text{ or } i + 1(\mod n) \\ 2 \quad \text{Otherwise} \end{cases}$$

$$c_\pi(u_i') = \begin{cases} 0 \quad \text{for} \quad j^{th} \quad \text{component} \quad j \equiv i + 2(\mod n) \\ 1 \quad \text{for} \quad j^{th} \quad \text{component} \quad j \equiv i \text{ or } i + 1(\mod n) \\ 3 \quad \text{for} \quad (n+1)^{th} \quad \text{component} \\ 2 \quad \text{Otherwise} \end{cases}$$

As a result, this coloring c is indeed a locating $(n+1)$-coloring of central graph of prism graph. ■

3.3 The Locating Chromatic Number of the Middle Graph of Prism Graph

This section focuses on determining the locating-chromatic number of the middle graph $M(Y_n)$ of the prism graph.

Let Y_n be the prism graph with vertex set $V(Y_n) = \{v_i, u_i \mid 1 \leq i \leq n\}$, and edge set $E(Y_n) = \{v_iv_{i+1},\ u_iu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{v_nv_1,\ u_nu_1\} \cup \{v_iu_i \mid 1 \leq i \leq n\}$.

The **middle graph** of Y_n, denoted by $M(Y_n)$, is given by subdividing each edge only once, then connecting all the middle vertices of adjacent edges and the older incident vertex and edge of Y_n. Thus, the vertex set of $M(Y_n)$ is: $V(M(Y_n)) = V(Y_n) \cup \{v'_i, u'_i, v''_i \mid 1 \leq i \leq n\}$, where:

- v'_i denotes the middle vertex on edge v_iv_{i+1} for $1 \leq i \leq n-1$, and v'_n on edge v_nv_1.
- u'_i denotes the middle vertex on edge u_iu_{i+1} for $1 \leq i \leq n-1$, and u'_n on edge u_nu_1.
- v''_i denotes the middle vertex on edge v_iu_i for all $1 \leq i \leq n$.

Example 3.3.1 *Middle of Prism Graph for* $n = 5$

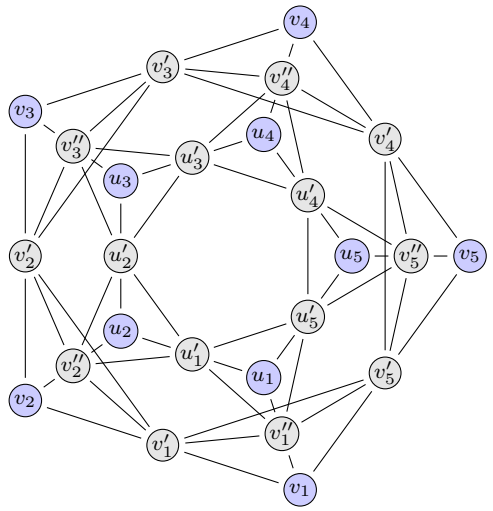


Figure 3.3.1 : Middle Graph of Prism Graph $M(Y_5)$

Theorem 3.3.1 *The locating chromatic number of the middle graph of prism graph $M(Y_n)$, where $n \geq 3$, is given by $\chi_L(M(Y_n)) = 6$.*

Proof. Since $M(Y_n)$ contains 4-clique, $\chi_L(M(Y_n)) \geq 4$. Now assume c be a locating 4-coloring of $M(Y_n)$. Observe that $M(Y_n)$ contains at least two distinct 4-cliques, so certainly two vertices of $M(Y_n)$ must have the same color, as well as the same color code. Thus, $\chi_L(M(Y_n)) > 4$.

Now, suppose that we have locating coloring using 5 colors.Now, suppose that, and let c be a locating 5-coloring of $M(Y_n)$. Let S_i, for $i \in \{1, \ldots, 5\}$, denote the color classes induced by c. So, one of the following cases hold.

Case one If one of the colors appears exactly once.

Let $x \in V(M(Y_n))$ be the unique vertex such that $c(x) = a$ where $a \in \{1, \ldots, 5\}$. Due to the structure of the middle graph of prism graph, there exist at least 9 vertices $y \in V(M(Y_n))\backslash\{x\}$ such that $d(x, y) = 2$. Among these vertices, at least two say z and t receive the same color, i.e., $c(z) = c(t)$. Since each vertex in $V(M(Y_n))$ belongs to a 4-clique, this implies that $c_\pi(z) = c_\pi(t)$, leading to a contradiction.

Case two Each color appears more than two times.

Clearly, for all $x, y \in S_i$, where $i \in \{1, \ldots, 5\}$, we have $d(x, y) = 2$. Otherwise, the situation would be similar to Case 1.

Now, consider the cliques $v_i^{'}v_{i+1}v_{i+1}^{'}v_{i+1}^{''}$ and $u_i^{'}v_{i+1}^{''}u_{i+1}^{'}u_{i+1}$, $i \in \{1, \ldots, n\}$. These cliques requires five different colors. As a result two vertices $x, y \in \{v_i^{''} \mid i \in \{1, \ldots, n\}\}$ must share the same color. Therefore, they will have the same color code, which is a contradiction.

Now, define the coloring function $c : V(M(Y_n)) \longrightarrow \{1, \ldots, 6\}$ as follows:

Case 1: If n is odd

$$c(v_i) = \begin{cases} 1 & \text{if } i \in \{[\frac{n}{2}] + 1, \ldots, n\} \\ 2 & \text{if } i \in \{1, \ldots, [\frac{n}{2}]\} \end{cases}$$

$$c(u_i) = \begin{cases} 1 & \text{if } i \in \{[\frac{n}{2}]+1, \ldots, n\} \\ 2 & \text{if } i \in \{1, \ldots, [\frac{n}{2}]\} \end{cases}$$

$$c(v'_i) = \begin{cases} 3 & \text{if } i \in \{1, \ldots, n-2\}, \text{and } i \text{ is odd} \\ 4 & \text{if } i \in \{2, \ldots, n-1\}, \text{and } i \text{ is even} \\ 5 & \text{if } i = n \end{cases}$$

$$c(u'_i) = \begin{cases} 3 & \text{if } i \in \{2, \ldots, n-1\}, \text{and } i \text{ even} \\ 4 & \text{if } i \in \{1, \ldots, n-2\}, \text{and } i \text{ odd} \\ 6 & \text{if } i = n \end{cases}$$

$$c(v''_i) = \begin{cases} 1 & \text{if } i \in \{1, \ldots, [\frac{n}{2}]\} \\ 2 & \text{if } i \in \{[\frac{n}{2}]+1, \ldots, n\} \end{cases}$$

Case 2: If n is even:

$$c(v_i) = \begin{cases} 1 & \text{if } i \in \{2, \ldots, [\frac{n}{2}]\}, \\ 2 & \text{if } i \in \{[\frac{n}{2}]+1, \ldots, n\}, \\ 5 & \text{if } i = n. \end{cases}$$

$$c(u_i) = \begin{cases} 1 & \text{if } i \in \{2, \ldots, [\frac{n}{2}]\}, \\ 2 & \text{if } i \in \{[\frac{n}{2}]+1, \ldots, n\}, \\ 6 & \text{if } i = n. \end{cases}$$

$$c(v'_i) = c(u'_i) = \begin{cases} 3 & \text{if } i \in \{1, \ldots, n\}, \text{ and } i \text{ odd} \\ 4 & \text{if } i \in \{1, \ldots, n\}, \text{ and } i \text{ even.} \end{cases}$$

$$c(v''_i) = \begin{cases} 1 & \text{if } i \in \{[\frac{n}{2}]+1, \ldots, n\} \\ 2 & \text{if } i \in \{1, \ldots, [\frac{n}{2}]\} \end{cases}$$

Using the coloring c for the middle graph of the prism graph $M(Y_n)$, we obtain the partition of vertices of six color classes $\pi = \{S_1, S_2, S_3, S_4, S_5, S_6\}$

if n odd we have the partition

$S_1 = \{v_{k+1}, v_{k+2}, \ldots, v_n\} \cup \{u_{k+1}, u_{k+2}, \ldots, u_n\} \cup \{v''_1, v''_2, \ldots, v''_k\}$, $S_2 = \{v_1, v_2, \ldots, v_k\} \cup \{u_1, u_2, \ldots, u_k\} \cup \{v''_{k+1}, v''_{k+2}, \ldots, v''_n\}$, $S_3 = \{v'_1, v'_3, \ldots, v'_{n-2}\} \cup \{u'_2, u'_4, \ldots, u'_{n-1}\}$, $S_4 = \{v'_2, v'_4, \ldots, v'_{n-1}\} \cup \{u'_1, u'_3, \ldots, u'_{n-2}\}$, $S_5 = \{v'_n\}$, $S_6 = \{u'_n\}$.

if n even we have the partition

$S_1 = \{v_1, v_2, \ldots, v_k\} \cup \{u_1, u_2, \ldots, u_k\} \cup \{v''_{k+1}, v''_{k+2}, \ldots, v''_n\}$, $S_2 = \{v_{k+1}, v_{k+2}, \ldots, v_{n-1}\} \cup \{u_{k+1}, u_{k+2}, \ldots, u_{n-1}\} \cup \{v''_1, v''_2, \ldots, v''_k\}$, $S_3 = \{v'_1, v'_3, \ldots, v'_{n-1}\} \cup \{u'_1, u'_3, \ldots, u'_{n-1}\}$, $S_4 = \{v'_2, v'_4, \ldots, v'_n\} \cup \{u'_2, u'_4, \ldots, u'_n\}$, $S_5 = \{v_n\}$, $S_6 = \{u_n\}$.

Observe that for all distinct x, y belongs to $V(M(Y_n)) \backslash \{v_n, u_n\}$ if $c(x) = c(y)$ then $d(x, S_5) \neq d(y, S_5)$ or $d(x, S_6) \neq d(y, S_6)$ and consequently $c_\pi(x) \neq c_\pi(y)$. Therefore all vertices of $M(Y_n)$ have distinct color codes. ■

Conclusion

In this dissertation, we studied the locating chromatic number in graphs. Finding $\chi_L(G)$ is considered a difficult task because there is no general formula that applies to all types of graphs. After introducing the basic concepts used in this study, we presented several results related to this topic.

Our main contribution was to determine the exact value of the locating chromatic number for the prism graph and its related graphs (middle, total, and central). We also suggested special coloring methods that satisfy the required distinguishing conditions for these graphs.

However, it is important to note that coloring prism-type graphs is quite challenging due to their special structure. When we analyzed the prism graph and its family more closely, we noticed that they have a very regular and symmetrical structure. Even though this was the main difficulty we faced, it also helped us find the solution.

These results add to the theory of graph coloring and open up new ways to deal with similar complex cases.

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